THE KING'S SCHOOL

2011
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Disclaimer

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Use the table of standard integrals to show that

$$\int_0^1 \frac{dx}{\sqrt{4 + x^2}} = \ln\left(\frac{1 + \sqrt{5}}{2}\right)$$ **2**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}}\, dx$ **2**

(b) Find $\displaystyle\int \frac{2x + 3}{x + 1}\, dx$ **2**

(c) By using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 + \cos\theta}\, d\theta$$ **3**

(d) Let $u_n = \displaystyle\int_1^e x(\ln x)^n\, dx, \quad n = 0, 1, 2, \ldots$

(i) Use integration by parts to show that

$$2u_n = e^2 - n\,u_{n-1}, \quad n = 1, 2, 3, \ldots$$ **4**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_1^e x(\ln x)^2\, dx$ **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $z = 1 - i$ and $w = -1 + 2i$

(i) Find $|zw|$ **2**

(ii) Show that $z + iw = \overline{w}$ **2**

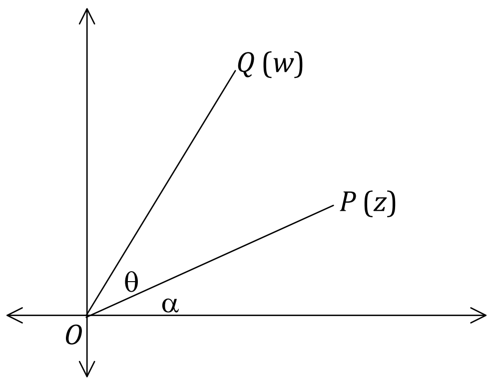
(iii) Find $\arg\left(\dfrac{12}{z + w}\right)$ **2**

(b) Let $z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

(i) Express z in modulus-argument form. **1**

(ii) Simplify $1 + z^2 + z^4 + \ldots + z^{20} + z^{22}$ **3**

(c) Let $z = \cos\alpha + i\sin\alpha$, $0 < \alpha < \dfrac{\pi}{4}$



On the Argand diagram the point P represents the complex number z and Q represents the complex number w.

$OP = OQ$ and $\angle QOP = \theta$, $0 < \theta \leq \dfrac{\pi}{4}$

(i) Write down the complex number w. **1**

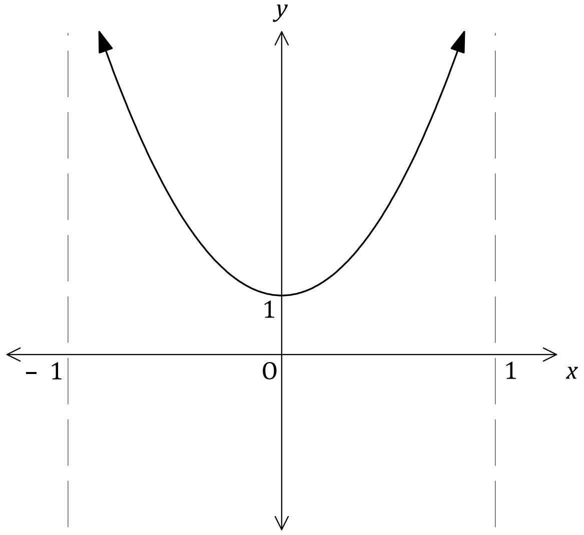
(ii) Show that $z\overline{w} = \cos\theta - i\sin\theta$ **2**

(iii) By considering $\Delta\, OPQ$, or otherwise, deduce that $\cos\dfrac{\theta}{2} = -\dfrac{Im(z\overline{w})}{|z - w|}$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The diagram shows the sketch of $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

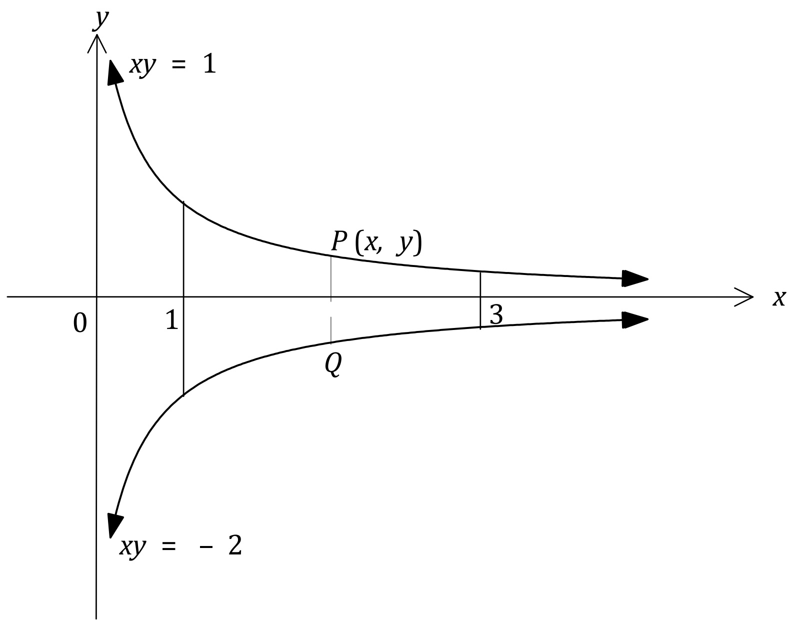
Draw sketches of the graphs of the following:

(i) $y = f^{-1}(x)\,, \quad y \le 0$ **1**

(ii) $y = \dfrac{x}{\sqrt{x^2 - x^4}}$ **2**

Question 3 continues on the next page

(b)



The enclosed region between $xy = 1$ and $xy = -2$ from $x = 1$ to $x = 3$ is the base of a solid.

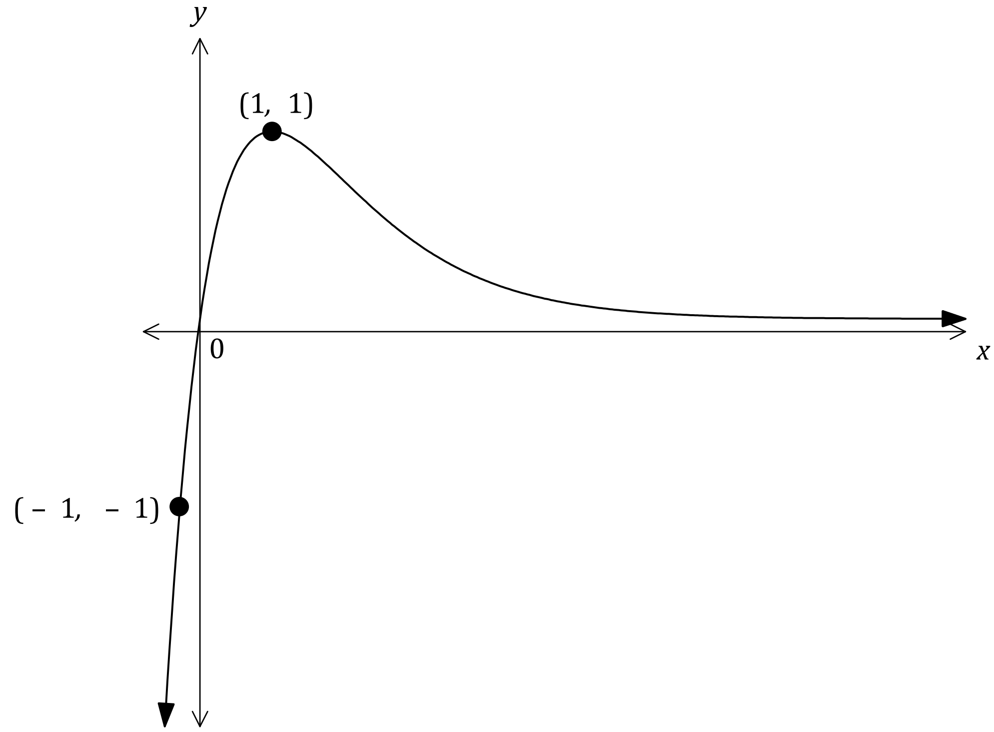
Cross-sections of the solid perpendicular to this base and the *x* axis are equilateral triangles.

(i) Show that the area of the smallest cross-section is $\dfrac{\sqrt{3}}{4}$ **1**

(ii) Find the volume of the solid. **3**

Question 3 continues on the next page

(c)



The sketch shows the graph of $y = f(x)$.

Draw separate sketches of the following:

(i) $y = f'(x)$ **2**

(ii) $y = \ln f(x)$ **2**

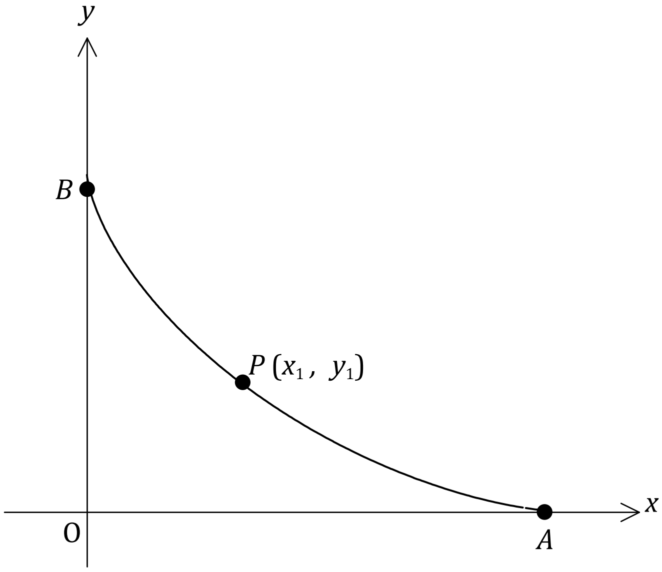
(iii) $y = f(x + |x|)$ **2**

(iv) $y = \cos^{-1} f(x)$ **2**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The sketch shows the graph of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, $x \geq 0$, $y \geq 0$, meeting the axes at A, B

(i) Find the coordinates of A **1**

(ii) Show that $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$, $x \neq 0$ **2**

(iii) Prove that the equation of the tangent at $P(x_1, y_1)$ is

$y = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} x + 4y_1^{\frac{1}{3}}$, $x_1 \neq 0$ **2**

(iv) The tangent at $P(x_1, y_1)$ meets the x axis at X and the y axis at Y.

Find the length of XY. **2**

Question 4 continues on the next page

(b) Suppose $\alpha,\ \beta,\ \gamma,\ \delta$ are the four roots of the polynomial equation

$P(x) = x^4 - Ax - 1 = 0$ where A is real.

(i) By considering $P''(x)$, or otherwise, show that at most two of the roots are real. **3**

(ii) Prove that the polynomial equation with the four roots $\alpha^2,\ \beta^2,\ \gamma^2,\ \delta^2$ is $x^4 - 2x^2 - A^2x + 1 = 0$ **3**

(iii) By considering another suitable polynomial equation, or otherwise, prove that

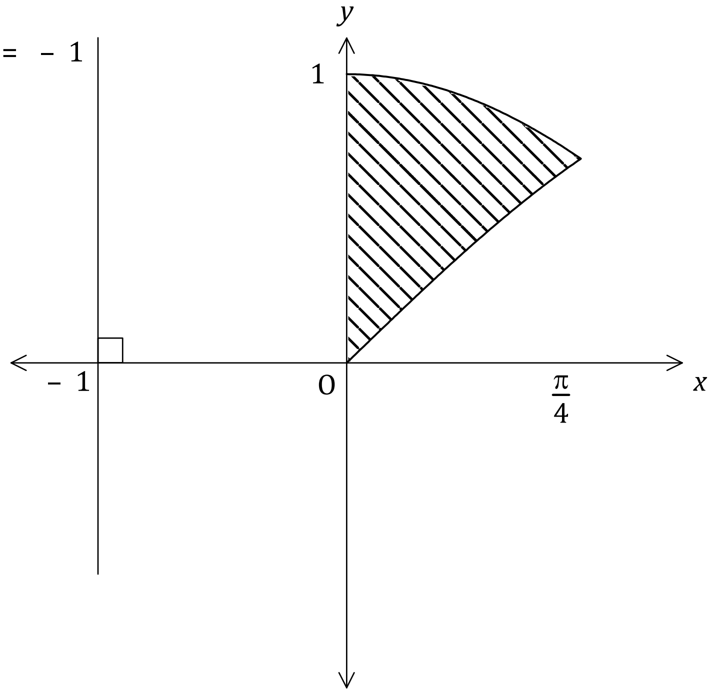
$$\left(\alpha^2 + 1\right)\left(\beta^2 + 1\right)\left(\gamma^2 + 1\right)\left(\delta^2 + 1\right) = A^2$$ **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the derivative of $x(\sin x + \cos x) + \cos x - \sin x$ **1**

(ii)



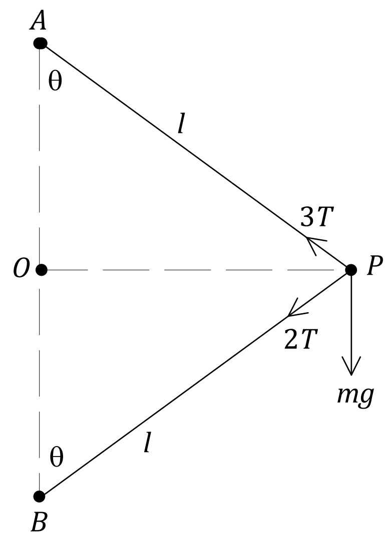
The diagram shows the region bounded by the curves $y = \sin x$ and $y = \cos x$ and the y axis.

The region is revolved about the line $x = -1$

Use the method of cylindrical shells to find the volume of the solid of revolution. **4**

Question 5 continues on the next page

(b)



The ends of a string of length $2l$ are attached to two points A and B where B is vertically below A.

A particle P of mass m is attached at the mid-point of the string and rotates about O, vertically below A, at constant angular speed ω.

The tensions in the strings PA and PB are $3T$ and $2T$, respectively.

g is the acceleration due to gravity.

Let $\angle PAO = \theta$

(i) Show that $T\cos\theta = mg$ **2**

(ii) Show that $m\omega^2 l = 5T$ **2**

(iii) Prove that $\omega > \sqrt{\frac{5g}{l}}$ **2**

(c) S and S' are the two foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$

e is the eccentricity of the ellipse.

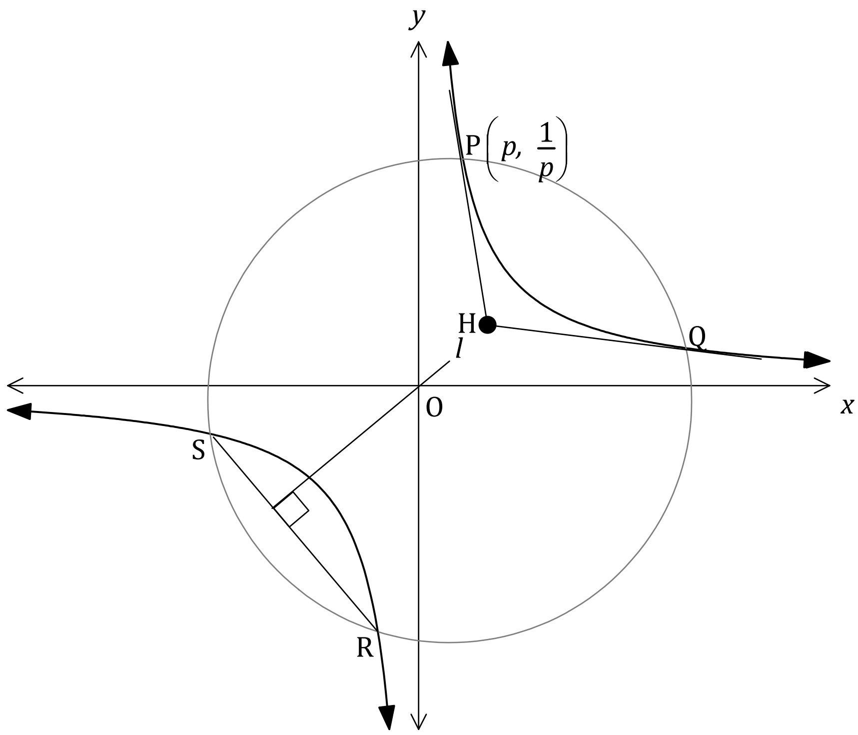
(i) Find S and S' **1**

(ii) $P(x_1, y_1)$ is any point on the ellipse. Prove that $PS + PS' = 9e$ **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the hyperbola $xy = 1$ and the circle $x^2 + y^2 + Ax + By + C = 0$ meeting at the points P, Q, R, S.

Let $x = t,\ y = \frac{1}{t}$ be the parametric equations of the hyperbola.

The parameters at P, Q, R, S are *p, q, r, s* respectively so that P = $\left(p, \frac{1}{p}\right)$

(i) Prove that $pqrs = 1$ **2**

(ii) Show that the equation of the line *l* passing through O (0, 0) and perpendicular to RS is $y = rsx$ **2**

(iii) Prove that the equation of the tangent at $P\left(p, \frac{1}{p}\right)$ on the hyperbola is

$x + p^2y = 2p$ **2**

(iv) Prove that the line *l* passes through the point of intersection H of the tangents at P and Q. **3**

Question 6 continues on the next page

(b) (i) Express $\dfrac{7n + 3}{n(n + 1)(n + 3)}$ in the form $\dfrac{a}{n} + \dfrac{b}{n + 1} + \dfrac{c}{n + 3}$ **3**

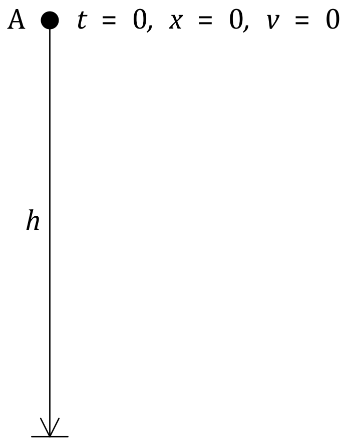
(ii) Let $S_n = \sum_{1}^{n} \left(\dfrac{a}{n} + \dfrac{b}{n+1} + \dfrac{c}{n+3} \right) = \sum_{1}^{n} \dfrac{a}{n} + \sum_{1}^{n} \dfrac{b}{n+1} + \sum_{1}^{n} \dfrac{c}{n+3}$

Prove that $S_n < \dfrac{7}{2}$ **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

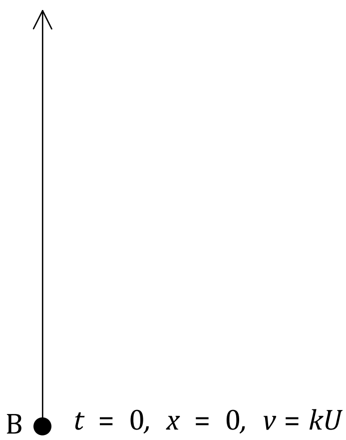
(a)



A particle A of mass m falls vertically from rest at a height h above the ground. It is subject to a resistance mk$v$ where v is its speed and k is a positive constant. g is the acceleration due to gravity.

(i) Using the vertical axis as in the diagram, show that $\ddot{x} = k(U - v)$ where U is the terminal speed of the particle. **2**

(ii)



At the same time as particle A falls from rest another particle B of mass m is projected upward in the same vertical line as particle A with a speed of kU, where U is the terminal speed of the particle.

Using the vertical axis as in the diagram, show that

$$\frac{dv}{dx} = \frac{-k\,(U + v)}{v}$$ **1**

Question 7 continues on the next page

(iii) Prove that the greatest possible height reached by particle B is

$$H = U\left(1 - \frac{1}{k}\ln(1 + k)\right)$$ **3**

(iv) Prove by using the equation of motion in (i) that the speed of particle A at any time t is given by $v = U\left(1 - e^{-kt}\right)$ **3**

(v) Similarly, it can be shown that for particle B, $v = kUe^{-kt} - U\left(1 - e^{-kt}\right)$

[DO NOT PROVE THIS]

The two particles collide after time T when particle A has fallen a distance X.

Given that $h <$ H, prove that $T = \frac{1}{k}\ln\left(\frac{U}{U - h}\right)$ **3**

(b) A point moves in the xy plane so that its coordinates at any time $t \geq 0$ are given by $x = 1 + \sin t$, $y = -\cos 2t$

Show that the point moves on a parabola and sketch the path of its motion. **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $J_n = \int_2^4 \frac{dx}{x\sqrt{x^{2n} - 1}}$, $n = 1, 2, 3, \ldots$

(i) Show that $J_n < \frac{\ln 2}{\sqrt{2^{2n} - 1}}$ **2**

(ii) Show that $J_n > \frac{2^n - 1}{n \,.\, 4^n}$ **2**

(iii) Show that $\frac{1}{x\sqrt{x^{2n} - 1}} = \frac{x^{-n-1}}{\sqrt{1 - x^{-2n}}}$ **1**

(iv) Find J_n **3**

(b) A sequence $u_1, u_2, u_3, \ldots$ is given by

$u_1 = 0$, $u_2 = 1$ and $u_n = (n-1)(u_{n-1} + u_{n-2})$, $n \geq 3$

(i) Find u_4 **1**

(ii) Let $A_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$

Show that $A_{n-2} - A_{n-1} = (-1)^n \frac{1}{(n-1)!}$ **1**

(iii) Prove by induction for $n \geq 3$ that $u_n = n!\,A_n$ **5**

End of Examination Paper

Student Number

THE KING'S SCHOOL

2011
Higher School Certificate
Trial Examination

Mathematics Extension 2

Question (Marks)	Complex Numbers	Functions	Harder Extension 1	Integration	Conics	Mechanics
1 (15)				*a-d* /15		
2 (15)	*a-c* /15					
3 (15)		*a, c* /11		*b* /14		
4 (15)		*a, b* /15				
5 (15)				*a* / 5	*c* / 4	*b* / 6
6 (15)		*b* / 6			*a* / 9	
7 (15)		*b* / 3				*a* /12
8 (15)			*b* / 7	*a* / 8		
Total (120)	**/15**	**/35**	**/ 7**	**/32**	**/13**	**/18**

TKS EXTENSION 2 TRIAL H.S.C. 2011 SOLUTIONS

Question 1

(a) (i) $I = \left[\ln(x + \sqrt{x^2+4})\right]_0^1$

$= \ln(1+\sqrt{5}) - \ln 2 = \ln\left(\frac{1+\sqrt{5}}{2}\right)$

(ii) Put $u = \tan x$ $\quad x=0, u=0$

$\frac{du}{dx} = \sec^2 x \quad x=\frac{\pi}{4}, u=1$

$\therefore I = \int_0^1 \frac{du}{\sqrt{4+u^2}} = \ln\left(\frac{1+\sqrt{5}}{2}\right)$

(b) $I = \int \frac{2(x+1)+1}{x+1}\,dx = \int 2 + \frac{1}{x+1}\,dx = 2x + \ln(x+1) + c$

(c) $t = \tan\frac{\theta}{2}$ $\quad \theta=0, t=0$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\theta/2 = \frac{1}{2}(1+t^2) \quad \theta=\frac{\pi}{2}, t=1$

$\therefore I = \int_0^1 \frac{2\,dt}{2(1+t^2)+1-t^2}$

$= \int_0^1 \frac{2}{3+t^2}\,dt$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3\sqrt{3}}$ or $\frac{\sqrt{3}\pi}{9}$

(d) (i) $u = (\ln x)^n$, $\frac{dv}{dx} = x$

$\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}$, $v = \frac{x^2}{2}$

$\therefore\ u_n = \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$

$= \frac{e^2}{2} - \frac{n}{2}u_{n-1}$

ie. $2u_n = e^2 - n\,u_{n-1}$

(ii) $2u_2 = e^2 - 2u_1$

$= e^2 - (e^2 - u_0)$

$= u_0 = \int_1^e x\,dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$

$\Rightarrow \int_1^e x(\ln x)^2\,dx = \frac{1}{4}(e^2 - 1)$

Question 2

(a) (i) $|zw| = |z||w| = \sqrt{2}\sqrt{5} = \sqrt{10}$

(ii) $z + iw = 1 - i + i(-1+2i)$

$= 1 - i - i - 2 = -1 - 2i = \bar{w}$

(iii) $z + w = i$

$\therefore \arg\left(\frac{12}{i}\right) = \arg 12 - \arg i = -\frac{\pi}{2}$

(b) (i) $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

(ii) $1 + z^2 + \dots + z^{22} = \frac{(z^2)^{12} - 1}{z^2 - 1} = \frac{z^{24} - 1}{z^2 - 1} = \frac{\cos 4\pi + i\sin 4\pi - 1}{z^2 - 1}$

$= 0$

(c) (i) $w = \cos(\alpha + \theta) + i\sin(\alpha + \theta)$

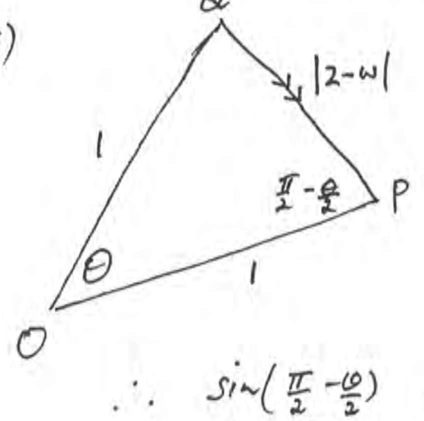
(ii) $z\bar{w} = (\cos\alpha + i\sin\alpha)(\cos(\alpha+\theta) - i\sin(\alpha+\theta))$

$= \cos\alpha\cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta) + i(\sin\alpha\cos(\alpha+\theta) - \cos\alpha\sin(\alpha+\theta))$

$= \cos(\alpha+\theta-\alpha) - i\sin(\alpha+\theta-\alpha)$

$= \cos\theta - i\sin\theta$ [ALTERNATIVES]

(iii)



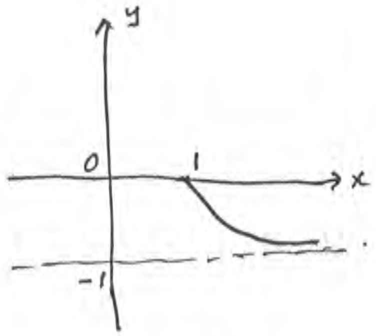
ie. $\vec{QP} = z - w$

$\therefore \frac{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{1} = \frac{\sin\theta}{|z-w|}$

ie. $\cos\frac{\theta}{2} = -\frac{\operatorname{Im}(z\bar{w})}{|z-w|}$, from (ii)

Question 3

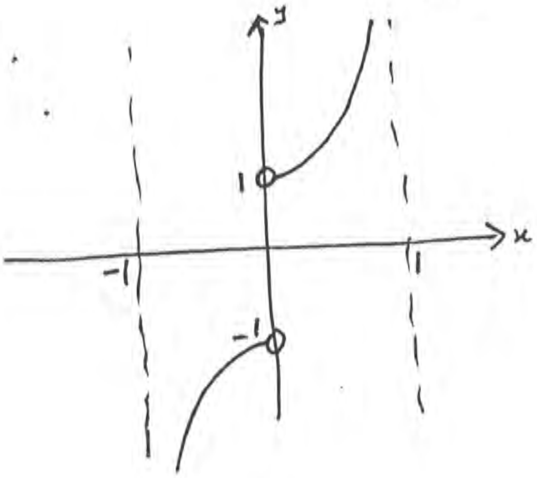
(a) (i)



(ii) $y = \dfrac{x}{\sqrt{x^2}\sqrt{1-x^2}} = \dfrac{x}{x\sqrt{1-x^2}}$ if $x>0$

$= \dfrac{1}{\sqrt{1-x^2}}$

& $y = -\dfrac{1}{\sqrt{1-x^2}}$ if $x<0$



(b) (i) least occurs at $x=3$

$\therefore$ Area $= \frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\right)^2 \sin 60^\circ = \frac{\sqrt{3}}{4}$

(ii) $PQ = \frac{1}{x} + \frac{2}{x} = \frac{3}{x}$

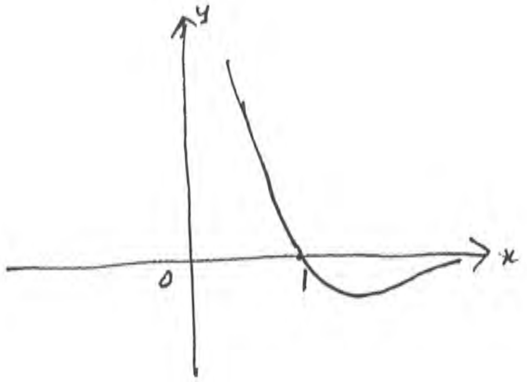
$\therefore$ Area cross-section $= \frac{1}{2}\cdot\frac{9}{x^2}\cdot\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{4x^2}$

$\therefore$ Volume $= \frac{9\sqrt{3}}{4}\int_1^3 \frac{1}{x^2}\,dx = -\frac{9\sqrt{3}}{4}\left[\frac{1}{x}\right]_1^3$

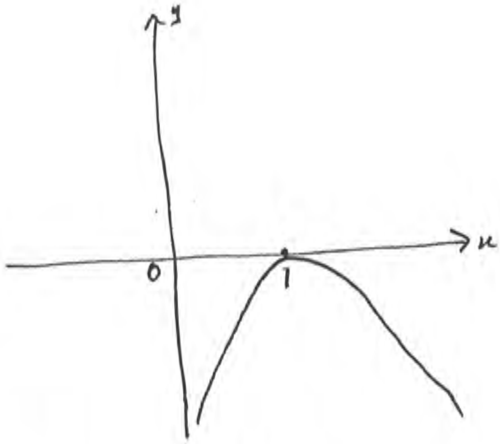
$= -\frac{9\sqrt{3}}{4}\left(\frac{1}{3}-1\right)$

$= \frac{3\sqrt{3}}{2}$

(c) (i)

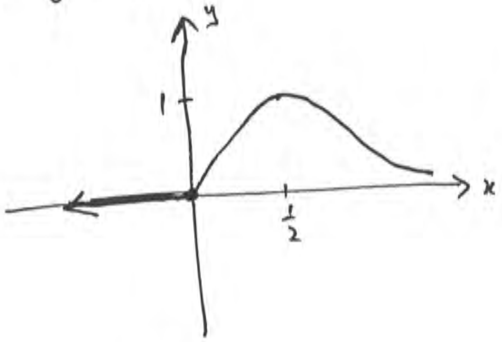


(ii)

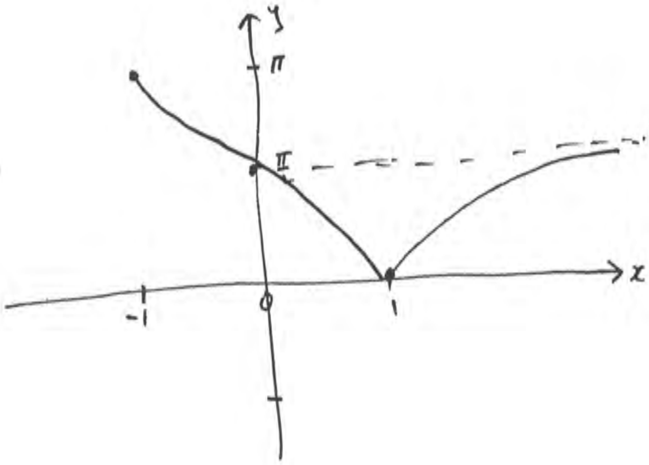


(iii) If $x \geq 0$, $y = f(2x)$

If $x \leq 0$, $y = f(0)$



(iv)



Question 4

(a) (i) at A, $y=0$ $\therefore$ $x = 4^{3/2} = 8$ ie $A=(8,0)$

(ii) $\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

(iii) Tangent is $y - y_1 = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} (x - x_1)$

ie $y = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} x + y_1^{\frac{1}{3}} x_1^{\frac{2}{3}} + y_1$

$= -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} x + y_1^{\frac{1}{3}} \left(x_1^{2/3} + y_1^{2/3}\right)$

ie $y = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}} x + 4 y_1^{\frac{1}{3}}$

(iv) At Y, $x=0$ $\Rightarrow y = 4 y_1^{\frac{1}{3}}$

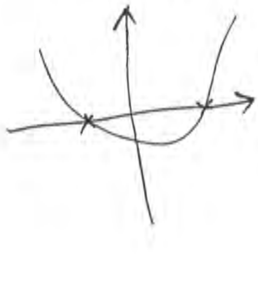
$\therefore$, on symmetry, at X, $x = 4 x_1^{\frac{1}{3}}$

$\therefore XY = 4\sqrt{x_1^{2/3} + y_1^{2/3}} = 4\sqrt{4} = 8$

(b) (i) $P'(x) = 4x^3 - A$

$P''(x) = 12x^2 \geq 0 \quad \forall x$

$\Rightarrow$ curve is concave upward $\forall x$

$\Rightarrow$  "at best"

ie. at most 2 real roots

(ii) Put $x = \alpha^2 \Rightarrow \alpha = \sqrt{x}$

$\therefore$ poly is $(\sqrt{x})^4 - A\sqrt{x} - 1 = 0$

or $A\sqrt{x} = x^2 - 1$

$\therefore A^2 x = x^4 - 2x^2 + 1$

i.e. $x^4 - 2x^2 - A^2 x + 1 = 0$

(iii) Put $x = \alpha^2 + 1 \Rightarrow \alpha^2 = x - 1$

$\therefore$ from (ii), eqn with roots $\alpha^2+1, \ldots, \delta^2+1$ is

$$(x-1)^4 - 2(x-1)^2 - A^2(x-1) + 1 = 0$$

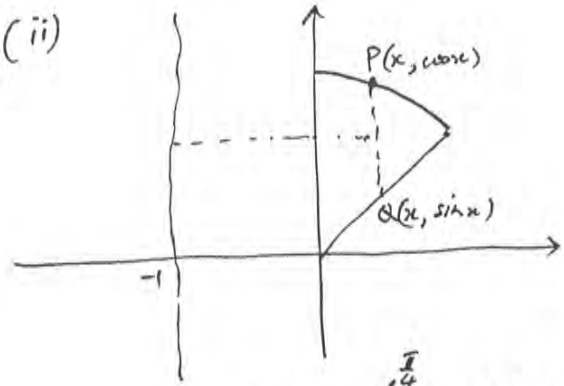
& $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)$ is the product of the roots

$$= 1 - 2 + A^2 + 1 = A^2$$

Question 5

(a) (i) $x(\cos x - \sin x) + \sin x + \cos x - \sin x - \cos x = x(\cos x - \sin x)$

(ii)



$\therefore \delta V \simeq \pi\left[(x+\delta x+1)^2 - (x+1)^2\right](\cos x - \sin x)$

$\simeq \pi\left(2(x+1)\,\delta x\right)(\cos x - \sin x)$

$\therefore V = 2\pi \int_0^{\frac{\pi}{4}} (x+1)(\cos x - \sin x)\, dx$

$= 2\pi \int_0^{\frac{\pi}{4}} x(\cos x - \sin x) + \cos x - \sin x \; dx$

$= 2\pi \left[x(\sin x + \cos x) + \cos x - \sin x + \sin x + \cos x \right]_0^{\pi/4}$ from (i)

$= 2\pi \left[x(\sin x + \cos x) + 2\cos x \right]_0^{\pi/4}$

$= 2\pi \left[\frac{\pi}{4} \cdot \frac{2}{\sqrt{2}} + \sqrt{2} - 2 \right]$ will do

$= 2\pi \left(\frac{\sqrt{2}\pi}{4} + \sqrt{2} - 2 \right)$

(b) (i) Resolving vertically at P,

$mg + 2T\cos\theta = 3T\cos\theta$

$\therefore T\cos\theta = mg$

(ii) Resolving in direction PO,

$m\omega^2(OP) = 3T\sin\theta + 2T\sin\theta$ where $OP = l\sin\theta$

$\therefore m\omega^2 l = 5T$

(iii) From (i), (ii), $\frac{m\omega^2 l}{mg} = \frac{5T}{T\cos\theta}$

$\therefore \omega^2 l = \frac{5g}{\cos\theta}$ or $\omega^2 = \frac{5g}{l\cos\theta} > \frac{5g}{l}$ since $0 < \cos\theta < 1$

$\therefore \omega > \sqrt{\frac{5g}{l}}$

(c) (i) $9 = 5 + c^2 \quad \therefore c^2 = 4, \; c = 2$

$\therefore S = (2, 0) \;, \; S'(-2, 0)$

(ii) $c = ae \quad \therefore e = \frac{2}{3}$

directrices are $x = \pm \frac{3}{\frac{2}{3}} = \pm \frac{9}{2}$

$\therefore PS + PS' = ePD + ePD'$

$= e\left(\frac{9}{2} - x_1 + x_1 + \frac{9}{2} \right)$

$= 9e$

Question 6

(a) (i) Any point on $xy=1$ is $(t, \frac{1}{t})$

$\therefore$ at P, Q, R, S $\quad t^2 + \frac{1}{t^2} + At + \frac{B}{t} + C = 0$

or $t^4 + At^3 + Ct^2 + Bt + 1 = 0$

has the roots p, q, r, s

$\therefore pqrs = 1$, product of roots

(ii) $\text{grad } RS = \dfrac{\frac{1}{r} - \frac{1}{s}}{r - s} = \dfrac{s - r}{rs(r-s)} = -\dfrac{1}{rs}$

$\therefore$ grad of l is rs

$\Rightarrow l$ is $y = rsx$

(iii) $y = \frac{1}{x}$ $\quad \therefore \frac{dy}{dx} = -\frac{1}{x^2} = -\frac{1}{p^2}$ at P

$\therefore$ tangent at P is $y - \frac{1}{p} = -\frac{1}{p^2}(x - p)$

or $p^2 y - p = -x + p$

ie. $x + p^2 y = 2p$

(iv) Tangent at Q is $x + q^2 y = 2q$

$\therefore$ at H, $(p^2 - q^2)y = 2(p - q)$

$\therefore y = \dfrac{2}{p+q}$, $x = 2p - p^2 \cdot \dfrac{2}{p+q} = \dfrac{2pq}{p+q}$

ie $H = \left(\dfrac{2pq}{p+q}, \dfrac{2}{p+q}\right)$

If H is on l then $\dfrac{2}{p+q} = rs \cdot \dfrac{2pq}{p+q}$

ie $1 = rspq$ is true from (i)

(b) (i) Put $u_n = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+3}$

$\therefore \; A(n+1)(n+3) + Bn(n+3) + Cn(n+1) \equiv 7n+3$

$n=0 \Rightarrow 3A = 3, \quad A = 1$

$n=-1 \Rightarrow -2B = -4, \quad B = 2$

$\therefore \; 1+2+C=0, \quad C=-3$

ie $u_n = \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+3}$

(ii) $S_n = \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) + 2\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right) - 3\left(\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n+3}\right)$

$= \left(\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n}\right)(1+2-3) + 1 + \frac{1}{2} + \frac{1}{3} + 2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{n+1}\right)$

$- 3\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$

$= 0 + 3\frac{1}{2} - \frac{1}{n+1} - \frac{3}{n+2} - \frac{3}{n+3}$

$= \frac{7}{2} - \left(\frac{1}{n+1} + \frac{3}{n+2} + \frac{3}{n+3}\right)$

$< \frac{7}{2}$ since $n = 1, 2, \ldots$

Question 7

(a) (i) $m\ddot{x} = mg - mkv$

$\therefore \ddot{x} = g - kv \Rightarrow g - kU = 0, \quad U = \frac{g}{k}$

$\therefore \ddot{x} = k\left(\frac{g}{k} - v\right) = k(U - v)$

(ii) $m\ddot{x} = -mg - mkv$

$\therefore \ddot{x} = -k(U + v)$

$\therefore v\frac{dv}{dx} = -k(U + v)$

or $\frac{dv}{dx} = -k\frac{(U+v)}{v}$

(iii) From (ii), $\frac{dx}{dv} = -\frac{1}{k} \cdot \frac{v}{U+v}$

$= -\frac{1}{k}\left(1 - \frac{U}{U+v}\right)$

$\therefore [x]_0^H = -\frac{1}{k}\left[v - U\ln(U+v)\right]_{kU}^{0}$

$\therefore H = -\frac{1}{k}\left(-U\ln U - kU + U\ln U(1+k)\right)$

$= -\frac{1}{k}\left(U\ln(1+k) - kU\right)$

ie $H = U\left(1 - \frac{1}{k}\ln(1+k)\right)$

(iv) $\frac{dv}{dt} = k(U - v)$

$\therefore \frac{dt}{dv} = \frac{1}{k} \cdot \frac{1}{U-v}$

$\Rightarrow kt = -\left[\ln(U-v)\right]_0^v = -\left(\ln(U-v) - \ln U\right)$

$\therefore \ln\left(\frac{U-v}{U}\right) = -kt$

$\therefore \frac{U-v}{U} = e^{-kt}$

$\Rightarrow v = U(1 - e^{-kt})$

(v) From (iii) $X = \int_0^T U(1-e^{-kt})\,dt$

and ∴ from particle B equation,

$$h - X = \int_0^T kUe^{-kt} - U(1-e^{-kt})\,dt$$

$$= -U\left[e^{-kt}\right]_0^T - X$$

$$\therefore h = -U(e^{-kT} - 1)$$

$$\therefore e^{-kT} = \frac{U-h}{U}$$

$$\text{or } e^{kT} = \frac{U}{U-h}$$

$$\therefore kT = \ln\left(\frac{U}{U-h}\right)$$

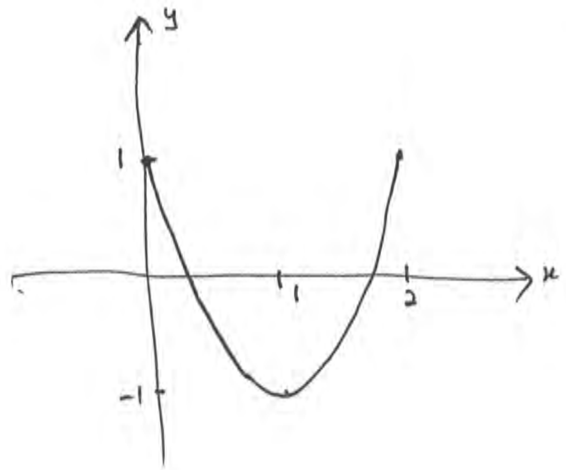
$$\therefore T = \frac{1}{k}\ln\left(\frac{U}{U-h}\right)$$

(b) $y = -\cos 2t = 2\sin^2 t - 1 \Rightarrow y = 2(x-1)^2 - 1$, a parabola

Now $y + 1 = 2(x-1)^2$ where $0 \le x \le 2$

and $-1 \le y \le 1$

∴



Question 8

(a) (i) Since $2 \le x \le 4$, then $\sqrt{2^{2n}-1} < \sqrt{x^{2n}-1}$, $2 < x \le 4$

$$\Rightarrow \frac{1}{\sqrt{x^{2n}-1}} < \frac{1}{\sqrt{2^{2n}-1}}, \quad 2 < x \le 4$$

$$\therefore J_n < \int_2^4 \frac{dx}{x\sqrt{2^{2n}-1}}$$

$$= \frac{1}{\sqrt{2^{2n}-1}} \int_2^4 \frac{1}{x}\,dx$$

$$= \frac{1}{\sqrt{2^{2n}-1}} \left[\ln x\right]_2^4 = \frac{\ln 2}{\sqrt{2^{2n}-1}}$$

(ii) $$J_n > \int_2^4 \frac{dx}{x\sqrt{x^{2n}}}$$

$$= \int_2^4 \frac{dx}{x^{n+1}} = -\frac{1}{n}\left[\frac{1}{x^n}\right]_2^4 = \frac{1}{n}\left(\frac{1}{2^n} - \frac{1}{4^n}\right)$$

$$= \frac{1}{n}\left(\frac{2^n-1}{4^n}\right) = \frac{2^n-1}{n.4^n}$$

(iii) $$\frac{1}{x\sqrt{x^{2n}-1}} = \frac{x^{-n-1}}{x^{-n}\sqrt{x^{2n}-1}} = \frac{x^{-n-1}}{\sqrt{x^{-2n}(x^{2n}-1)}}$$

$$= \frac{x^{-n-1}}{\sqrt{1-x^{-2n}}}$$

(iv) From (iii), put $u = x^{-n}$; $x = 2, u = 2^{-n}$

$\frac{du}{dx} = -n x^{-n-1}$; $x = 4, u = 4^{-n}$

$$\therefore J_n = -\frac{1}{n}\int_{2^{-n}}^{4^{-n}} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{n}\left[\cos^{-1} u\right]_{2^{-n}}^{4^{-n}} = \frac{1}{n}\left(\cos^{-1} 4^{-n} - \cos^{-1} 2^{-n}\right)$$

or, of course, $-\frac{1}{n}\left(\sin^{-1} 4^{-n} - \sin^{-1} 2^{-n}\right)$

(b) (i) $u_3 = 2(1+0) = 2$

$\therefore u_4 = 3(2+1) = 9$

(ii) $A_{n-2} - A_{n-1} = \left(\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-1}\frac{1}{(n-2)!}\right) - \left(\frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-1}\frac{1}{(n-2)!} + (-1)^{n-1}\frac{1}{(n-1)!}\right)$

$= -(-1)^{n-1}\frac{1}{(n-1)!} = (-1)^n \frac{1}{(n-1)!}$

(iii) $u_3 = 3!\left(\frac{1}{2!} - \frac{1}{3!}\right) = 3 - 1 = 2$

$u_4 = 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 12 - 4 + 1 = 9$

$\therefore$ from (i), $u_n = n!\,A_n$ for $n = 3, 4$

$\therefore$ Assume $u_n = n!\,A_n$ for consecutive integers $n-2, n-1$

Then $u_n = (n-1)(u_{n-1} + u_{n-2})$

$= (n-1)\left((n-1)!\,A_{n-1} + (n-2)!\,A_{n-2}\right)$ using the assumption

$= (n-1)(n-2)!\left((n-1)A_{n-1} + A_{n-2}\right)$

$= (n-1)!\left(n A_{n-1} + A_{n-2} - A_{n-1}\right)$

$= (n-1)!\left(n A_{n-1} + (-1)^n \frac{1}{(n-1)!}\right)$ from (ii)

$= n!\left(A_n - (-1)^n \frac{1}{n!}\right) + (-1)^n$

$= n!\,A_n - (-1)^n + (-1)^n$

$= n!\,A_n$

$\therefore$ by induction $u_n = n!\,A_n$, $n \geq 3$