2015

Year 12 Mathematics Extension 2

Trial HSC Examination

Teacher Setting Paper: Mr P Mirrington
Head of Department: Mrs M Hill

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used (graphics calculators are not allowed)
- A table of standard integrals is provided at the back of this paper
- For Section I record your answers on the multiple choice answer sheet provided
- For Section II, answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
- For Section II show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2 – 5

10 Marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 13

90 Marks

- Attempt Questions 11 – 16
- Each Question is worth 15 marks
- Allow about 2 hours and 45 minutes for this section

This examination paper does not necessarily reflect the content or format of the Higher School Certificate Examination in this subject

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet to record your answers for Questions 1–10.

QUESTION 1

The eccentricity of the hyperbola $x^2 - 4y^2 = 1$ is:

(A) $\frac{4}{5}$

(B) $\frac{2}{\sqrt{5}}$

(C) $\frac{\sqrt{5}}{2}$

(D) $\frac{5}{4}$

QUESTION 2

The *acute* angle between the asymptotes of the hyperbola $x^2 - \frac{y^2}{3} = 1$ is:

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

QUESTION 3

P, Q and R are three consecutive terms in an arithmetic progression.

Which of the following is the simplification of $\frac{\sin(P+R)}{\cos Q}$?

(A) $2\sin Q$

(B) $2\cos Q$

(C) $\sin 2Q$

(D) $\cos 2Q$

QUESTION 4

How many asymptotes does the graph of the curve $y = \dfrac{x-1}{x^2-1}$ have?

(A) 1

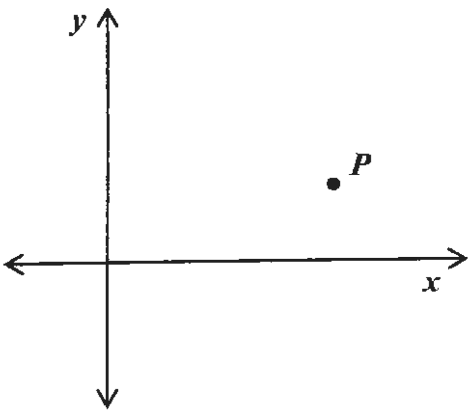
(B) 2

(C) 3

(D) 4

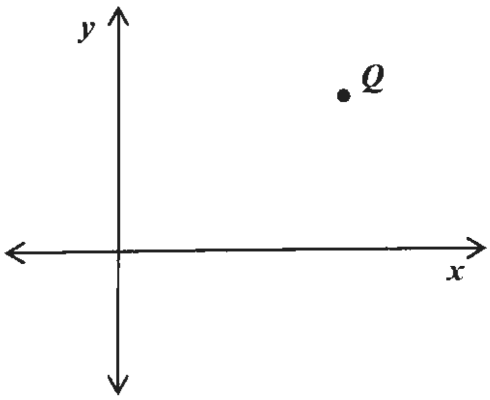
QUESTION 5

On the Argand diagram below, P represents the complex number z.

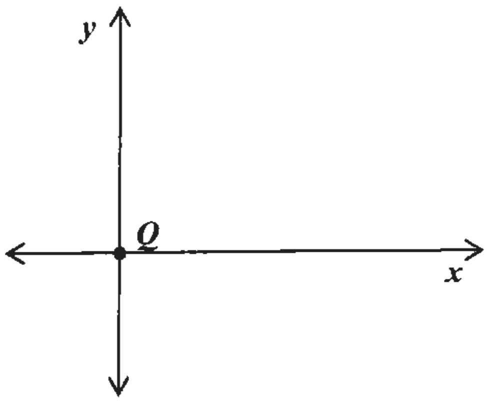


Which of the following Argand diagrams shows the point Q representing $z - \overline{z}$?

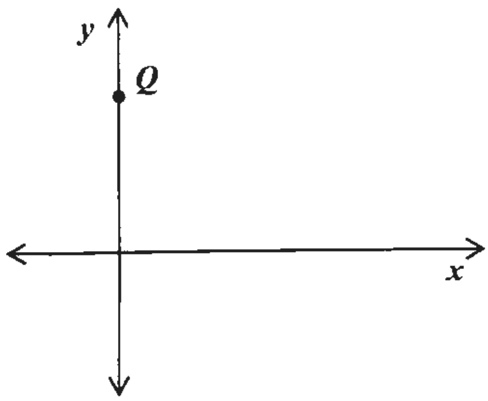
(A)



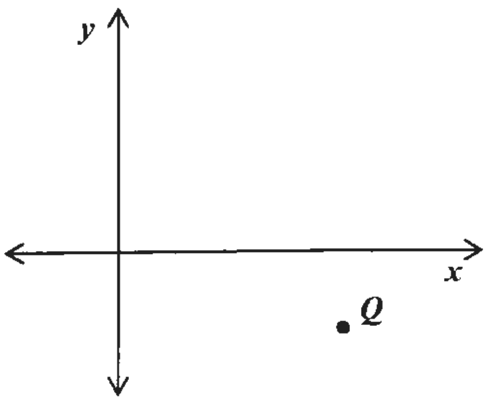
(B)



(C)



(D)



QUESTION 6

The equation $x^3 + 3x - 2 = 0$ has roots α, β and γ. The value of $\alpha^3 + \beta^3 + \gamma^3$ is:

(A) 2

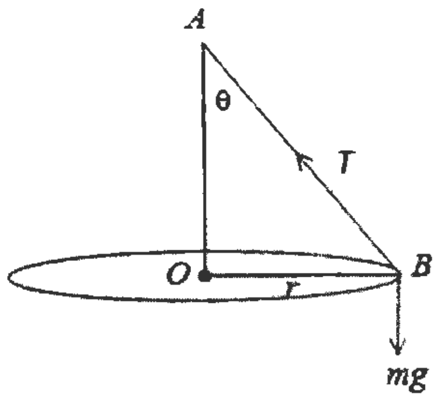
(B) 6

(C) -2

(D) -6

QUESTION 7

A particle of mass m at B is attached to a string AB that is fixed at A. The particle rotates in a horizontal circle with a radius of r. Let T be the tension in the string and $\angle BAO = \theta$.



Which of the following statements is *not* correct:

(A) $T > mg$

(B) $T = \dfrac{m\omega^2 r}{\sin\theta}$

(C) $r = \dfrac{g\tan\theta}{\omega^2}$

(D) $\omega = \sqrt{\dfrac{g\cot\theta}{r}}$

QUESTION 8

Consider the rectangular hyperbola $xy = 16$. If this hyperbola was rotated 45 degrees in a clockwise direction, its equation would be:

(A) $x^2 - y^2 = 8$

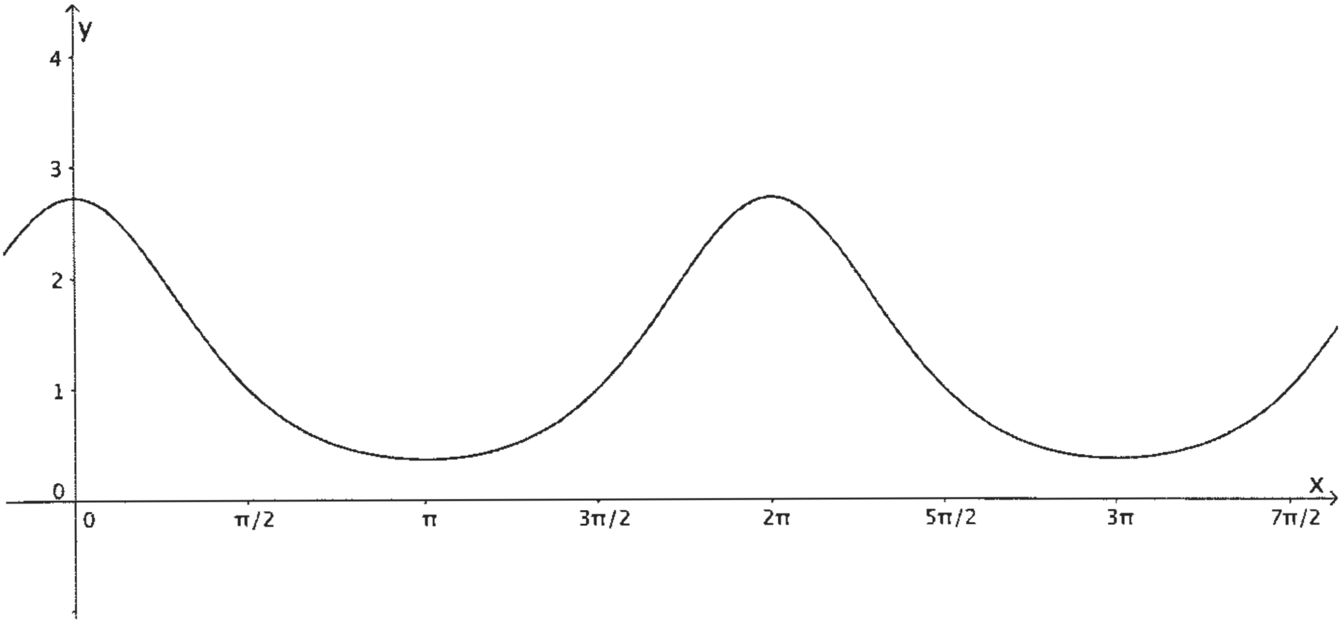
(B) $x^2 - y^2 = 16$

(C) $x^2 - y^2 = 16\sqrt{2}$

(D) $x^2 - y^2 = 32$

QUESTION 9

Which function best represents the graph below?



(A) $y = e\cos x$

(B) $y = e + 1 - \cos x$

(C) $y = \dfrac{e}{\cos x}$

(D) $y = e^{\cos x}$

QUESTION 10

When $y^2 + xy = 1$ is differentiated with respect to x, the correct expression for the derivative is:

(A) $\dfrac{dy}{dx} = \dfrac{1}{1+2y}$

(B) $\dfrac{dy}{dx} = \dfrac{1-y}{1+2y}$

(C) $\dfrac{dy}{dx} = \dfrac{-y}{x+2y}$

(D) $\dfrac{dy}{dx} = \dfrac{1-y}{x+2y}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

QUESTION 11 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 1 - 3i$ and $\omega = 2 + i$.

Find the value of $\dfrac{z}{\omega}$ in the form $a + ib$, where a and b are real numbers: **2**

(b) (i) If $z = x+iy$, write down expressions for $z\overline{z}$ and $z+\overline{z}$. **1**

(ii) Sketch the region on the Argand diagram described by: $z\overline{z} \le 3(z+\overline{z})$ **2**

(c) Factorise $P(x) = x^3 + x^2 + 4x + 4$ into its three linear factors. **2**

(d) Find these indefinite integrals:

(i) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ **2**

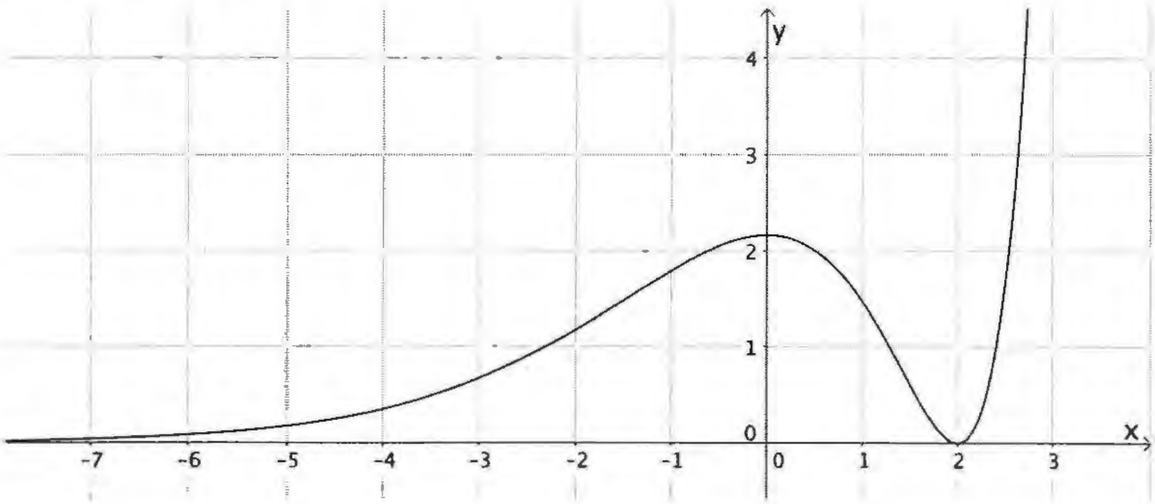
(ii) $\displaystyle\int \frac{1}{x^2+2x-3}\,dx$ **2**

(e) (i) Find all pairs of integers a and b such that $(a + ib)^2 = 7 + 24i$ **2**

(ii) Hence, or otherwise, solve: $z^2 + 3iz - (4 + 6i) = 0$ **2**

QUESTION 12 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) The function $f(x) = 4(x-2)^2 e^{x-2}$ has stationary points at $x = 0$ and $x = 2$ as shown in the diagram below.



Draw separate half-page sketches of the following graphs. In each case, label any asymptote and the coordinates of any turning points.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = f(|x|)$ **1**

(b) (i) Use the result $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ to show that

$$\sin A + \sin B = 2\sin\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)$$ **2**

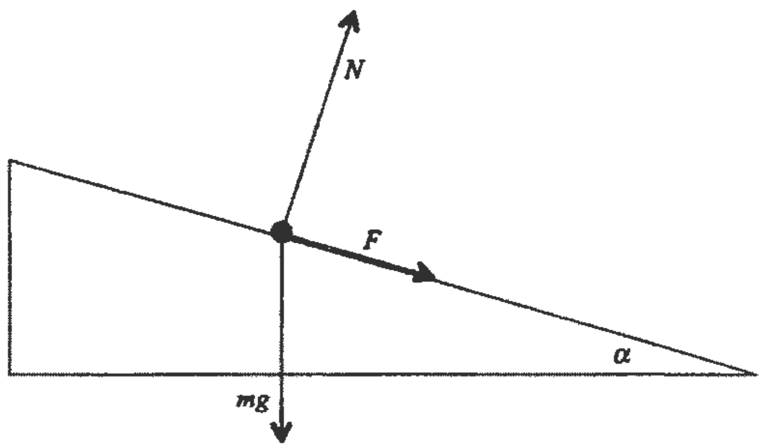
(ii) Hence find the general solution to the equation: $\sin 2x + \sin 4x = \sin 6x$ **3**

(iii) Use part (i) to evaluate $\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx$ **3**

(c) If $A(x)$ and $B(x)$ are *odd* polynomial functions show that the product $P(x) = A(x) \times B(x)$ is an *even* polynomial function. **2**

QUESTION 13 (15 Marks) Use a SEPARATE writing booklet. **Marks**

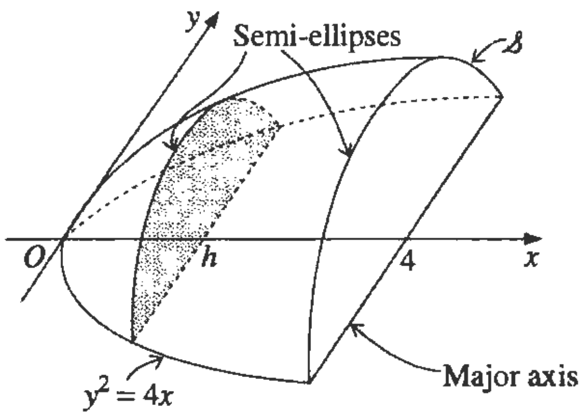
(a) A car of mass 800kg moves with constant speed in a horizontal circle of radius 250m on a track that is banked at an angle of $\alpha = 7^0$ to the horizontal. The forces acting on the car are the weight force *mg*, a normal reaction force *N* to the road and a friction force *F* acting down the slope as shown. The acceleration due to gravity is 9.8 ms^{-2}.



(i) Show that the optimum speed for the car (when the friction force *F* acting on the car is zero) is about 62 km/hr. **2**

(ii) If the car is travelling at a constant speed of 108km/hr, find the size of the friction force *F* acting on the car? (Give your answer to the nearest Newton) **3**

(b)



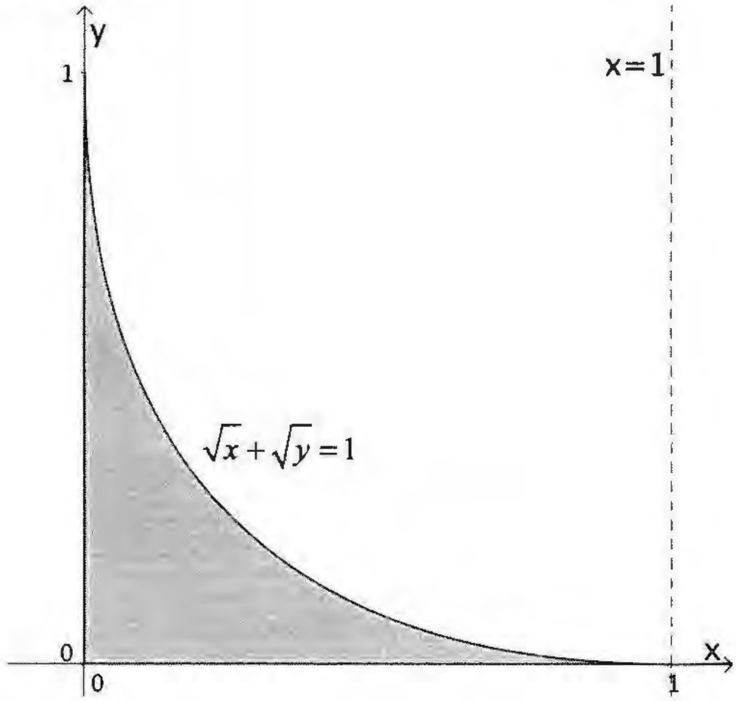
The base of a solid S is the region in the *xy* plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$, and each cross section perpendicular to the x axis is a semi-ellipse with the minor axis one-half of the major axis.

(i) Show that the area of the semi-ellipse at $x = h$ is πh.
(You may assume that the area of an ellipse with semi-axes a and b is πab.) **2**

(ii) Find the volume of the solid S. **2**

QUESTION 13 is continued on the next page.

(c) The region bounded by the curve $\sqrt{x}+\sqrt{y}=1$ and the x axis between x = 0 and x = 1 is rotated about the line $x = 1$.



(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by: **2**

$$V = 2\pi\int_0^1 1 - x^2 + 2x\sqrt{x} - 2\sqrt{x}\,dx$$

(ii) Hence find the value of V in simplest form **1**

(d) The equation $x^3 + 2x^2 + 3x + 4 = 0$ has roots α, β and γ.

Find the monic cubic equation with roots α^2, β^2 and γ^2. **3**

QUESTION 14 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) The complex number ω is given by $\omega = 2(\cos\theta + i\sin\theta)$, where $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

(i) Express each of ω^2 and $\frac{1}{\omega}$ in modulus/argument form. **2**

(ii) On an Argand diagram, the points Q and R represent the complex numbers ω^2 and $\frac{1}{\omega}$ respectively. If the points Q, O and R are collinear, find ω in the form $a + ib$, where a and b are real numbers. **2**

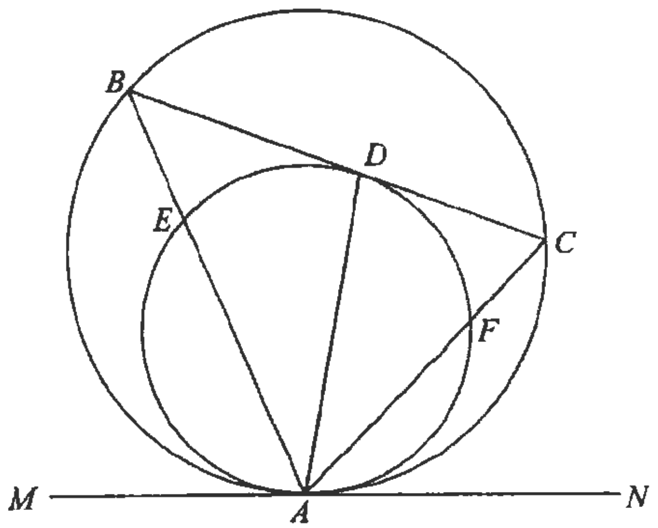
(b) Let $I_n = \int_0^a (1+x^2)^n \, dx$, $n = 1, 2, 3, \ldots..$

(i) Use integration by parts to show that $I_n = \frac{a(1+a^2)^n}{2n+1} + \frac{2n}{2n+1} I_{n-1}$ **3**

(ii) Show that $I_1 = \frac{3a+a^3}{3}$ **2**

(iii) Hence, or otherwise, evaluate $\int_0^1 (1+x^2)^3 \, dx$ **2**

(c)



In the diagram, MAN is the common tangent to two circles touching internally at A. B and C are two points on the larger circle such that BC is a tangent to the smaller circle with point of contact D. AB and AC cut the smaller circle at E and F respectively. Copy the diagram. Show that AD bisects $\angle BAC$. **4**

QUESTION 15 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m kg is dropped from rest in a medium where the resistance to the motion has magnitude $\frac{1}{40}mv^2$ when the speed of the particle is v ms^{-1}. After t seconds the particle has fallen x metres. The acceleration due to gravity is 10 ms^{-2}.

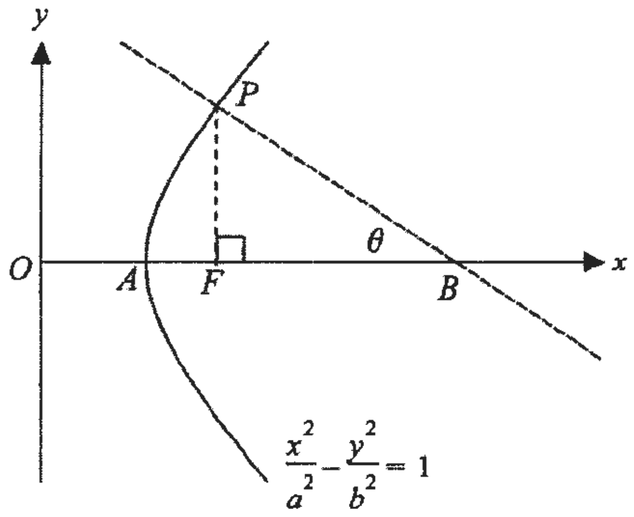
(i) Explain why $\ddot{x} = \dfrac{1}{40}\left(400 - v^2\right)$. **1**

(ii) Find an expression for t in terms of v by integration. **2**

(iii) Show that $v = 20\left(1 - \dfrac{2}{1+e^t}\right)$. **1**

(b) Consider the polynomial $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ over the complex field. Given that $P(1-i) = 0$, find all four solutions to $P(x)=0$. **2**

(c) In the diagram below, F is a focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with eccentricity e.



This branch of the hyperbola cuts the x axis at A where $AF = h$. P is the point on the hyperbola vertically above F and the normal at P cuts the x axis at B making an acute angle θ with the x axis.

(i) Show that $\tan\theta = \dfrac{1}{e}$ **3**

(ii) Show that $PF = h(e+1)$ **1**

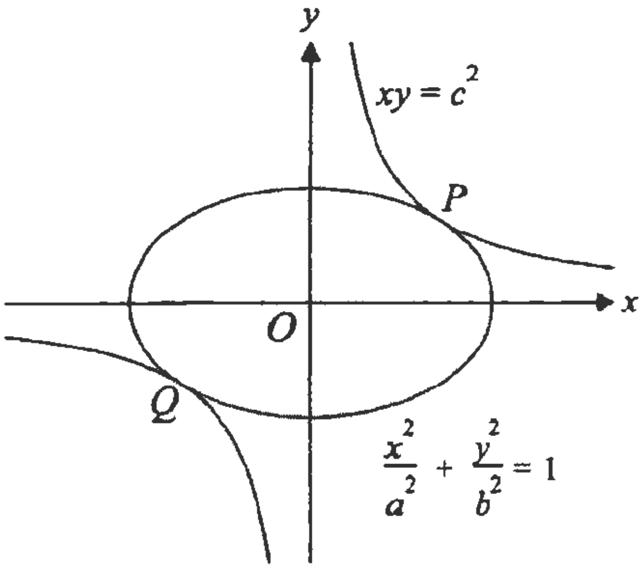
A bowl is formed by rotating the hyperbola above through one revolution about the x axis. The bowl is then placed on a horizontal table with point A on the table.
A particle P of mass m is set in motion around the inside of the bowl, travelling with constant angular velocity ω in a horizontal circle with centre F.

(iii) Show that $\omega^2 = \dfrac{g}{he(e+1)}$ **3**

(iv) N is the normal reaction force between the particle P and the bowl. Show that if the hyperbola used to form the bowl is a rectangular hyperbola, then $N = mg\sqrt{\frac{3}{2}}$. **2**

QUESTION 16 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) The hyperbola $xy = c^2, c > 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$ at points P and Q, where $P\left(cp, \frac{c}{p}\right)$ lies in the first quadrant.



$\left(ct, \frac{c}{t}\right)$ is the general parametrisation of $xy = c^2$, $t \neq 0$

(i) Explain why the equation $(bc)^2 t^4 - (ab)^2 t^2 + (ca)^2 = 0$ has roots $p, p, -p, -p$ where $p > 0$. **2**

(ii) Deduce that $p = \frac{a}{c\sqrt{2}}$ and $ab = 2c^2$. **2**

(iii) Show that if $S\left(c\sqrt{2}, c\sqrt{2}\right)$ and S' are the foci of the hyperbola $xy = c^2$, then the quadrilateral with vertices P, S, Q and S' has area $2c(a-b)$. **2**

(b) If a > 0 and b > 0, prove that:

(i) $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$. **2**

(ii) $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{(a+b)^2}$. **2**

QUESTION 16 is continued on the next page.

Marks

(c) Let $f(x) = \dfrac{x^n}{e^x}$, where $n > 1$.

(i) Show that $f'(x) = \dfrac{x^{n-1}(n-x)}{e^x}$ **1**

(ii) Show that the graph of $y = f(x)$ has a maximum turning point at $\left(n, \dfrac{n^n}{e^n}\right)$, and hence sketch the graph for $x \geq 0$. (Don't attempt to find any points of inflexion). **2**

(iii) Explain, by considering the graph of $y = f(x)$ for $x > n$, why $\dfrac{x^n}{e^x} < \dfrac{n^n}{e^n}$ for $x > n$. **1**

(iv) Deduce from part (iii), using the substitution $x = n + 1$, that $\left(1 + \dfrac{1}{n}\right)^n < e$. **1**

End of examination

1. $\frac{x^2}{1^2} - \frac{y^2}{(1/2)^2} = 1$ $\quad a=1 \quad b=\frac{1}{2}$

$$b^2 = a^2(e^2-1)$$
$$\frac{1}{4} = 1(e^2-1)$$
$$e^2 = \frac{5}{4} \qquad e = \frac{\sqrt{5}}{2}$$

C

2. $\frac{x^2}{1} - \frac{y^2}{3} = 1$ $\quad a=1 \quad b=\sqrt{3}$

assymptotes $y = \pm \frac{b}{a} x$

$y = \pm \sqrt{3}\, x$

$$\tan\theta = \left| \frac{m_1 - m_2}{1+m_1m_2} \right|$$
$$= \left| \frac{\sqrt{3} - -\sqrt{3}}{1-3} \right|$$
$$= \sqrt{3}$$
$$\theta = \pi/3.$$

A

3. P, Q, R in AP

$\therefore Q = \frac{P+R}{2}$

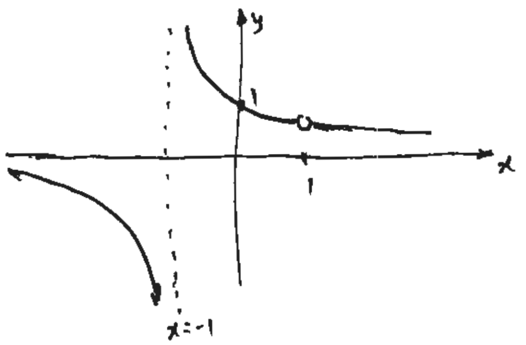
$P+R = 2Q$

$$\frac{\sin(P+R)}{\cos Q} = \frac{\sin 2Q}{\cos Q}$$
$$= \frac{2\sin Q\cos Q}{\cos Q}$$
$$= 2\sin Q$$

A

4. $y = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)}$

$$= \frac{1}{x+1}$$



2 assymptotes.

B

5. $z - \bar{z} = (x+iy) - (x+iy)$

$= 2iy$

C

$\therefore$ Q has x coord = zero

y coord = 2 × y coord of P.

6. $x^3 + 3x - 2 = 0$

$$\left.\begin{aligned} \alpha^3 + 3\alpha - 2 &= 0 \\ \beta^3 + 3\beta - 2 &= 0 \\ \gamma^3 + 3\gamma - 2 &= 0 \end{aligned}\right\} \text{ add } \Rightarrow \alpha^3+\beta^3+\gamma^3 + 3(\alpha+\beta+\gamma) - 6 = 0$$

$\alpha^3+\beta^3+\gamma^3 + 3 \times 0 = 6$

$\alpha^3+\beta^3+\gamma^3 = 6$

B

7. $T\cos\theta = mg$

$T = \frac{mg}{\cos\theta}$

$\therefore T > mg$ since $\cos\theta < 1$ ✓

$T\sin\theta = m\omega^2 r \Rightarrow T = \frac{m\omega^2}{\sin\theta}$ ✓

$\therefore \tan\theta = \frac{\omega^2 r}{g}$

$r = \frac{g\tan\theta}{\omega^2}$ ✓

$\omega^2 = \frac{g\tan\theta}{r}$.

$\therefore$ D is not correct.

D

8. $xy = 16 = c^2 = \frac{a^2}{2}$

$\therefore a^2 = 32$

$x^2 - y^2 = 32$

D

9. $\cos 0 = 1$

$\cos \pi = -1$ eliminates A and B

$\cos \frac{\pi}{2} = 0$ eliminates C

D

10. $y^2 + xy = 1$

$$2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx}(2y + x) + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{2y+x}$$

C

QUESTION 11

(a) $z = 1 - 3i$, $w = 2 + i$

$$\frac{z}{w} = \frac{1-3i}{2+i} \times \frac{2-i}{2-i}$$
$$= \frac{2 - i - 6i + 3i^2}{2^2 - i^2}$$
$$= \frac{2 - 7i - 3}{4 - -1}$$
$$= \frac{-1 - 7i}{5}$$
$$= -\frac{1}{5} - \frac{7i}{5}$$

(b) (i) $z = x + iy$

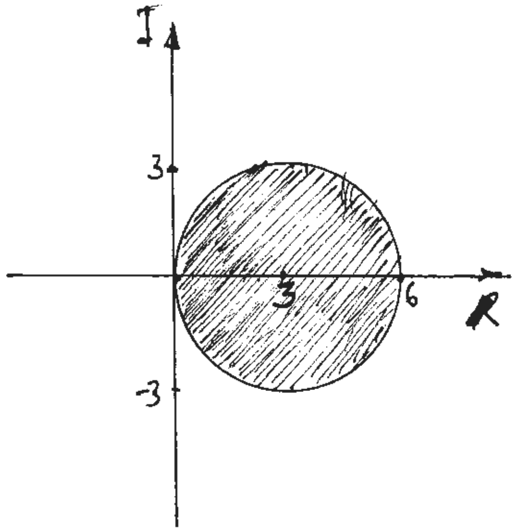
$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 - i^2y^2$$
$$= x^2 + y^2$$

$$z + \bar{z} = x + iy + x - iy$$
$$= 2x.$$

(ii) $z\bar{z} \leq 3(z + \bar{z})$

$$x^2 + y^2 \leq 3(2x)$$
$$x^2 - 6x + y^2 \leq 0$$
$$(x-3)^2 + y^2 \leq 9$$

circle centre $(3,0)$ radius 3.



(c) $P(x) = x^3 + x^2 + 4x + 4$

$P(-1) = -1 + 1 - 4 + 4 = 0$

$\therefore$ $(x+1)$ is a factor.

$P(x) = (x+1)(x^2+4)$ by inspection (or division)

$= (x+1)(x - 2i)(x + 2i)$

QUESTION 11 (continued)

(d) (i) $\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$

$= \int \frac{1}{u^2+2^2}\,du$

$u = x+1$

$\frac{du}{dx} = 1$

$du = dx$.

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$

(ii) $\int \frac{1}{x^2+2x-3}\,dx = \int \frac{1}{(x+3)(x-1)}\,dx$

Let $\frac{1}{(x+3)(x-1)} = \frac{a}{(x+3)} + \frac{b}{(x-1)}$

$1 = a(x-1) + b(x+3)$

$$\left.\begin{array}{ll} x=1 \Rightarrow & 1 = 4b \\ & b = \frac{1}{4} \\ x=-3 \Rightarrow & 1 = -4a \\ & a = -\frac{1}{4} \end{array}\right\} \quad \therefore \frac{1}{(x+3)(x-1)} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)$$

$\therefore \int \frac{1}{x^2+2x-3}\,dx = \frac{1}{4}\int \frac{1}{x-1} - \frac{1}{x+3}\,dx$

$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+3) +$

$= \frac{1}{4}\ln\left(\frac{x-1}{x+3}\right) + c$

QUESTION 11 (continued)

(e) (i) $(a+ib)^2 = 7+24i$

$a^2-b^2+2abi = 7+24i$

equating real & imaginary parts:

$a^2-b^2=7$ and $2ab=24$

$ab=12$

$a=\frac{12}{b}$

$\left(\frac{12}{b}\right)^2-b^2=7$ substitute

$144-b^4=7b^2$

$b^4+7b^2-144=0$

$(b^2+16)(b^2-9)=0$

$b=\pm 3 \quad a=\pm 4$

Solutions: $a=4 \quad b=3$ OR $a=-4 \quad b=-3$.

(ii) $z^2+3iz-(4+6i)=0$

$z=\frac{-3i \pm \sqrt{(3i)^2 - {}^-4(4+6i)}}{2}$

$=\frac{-3i \pm \sqrt{-9+16+24i}}{2}$

$=\frac{-3i \pm \sqrt{7+24i}}{2}$

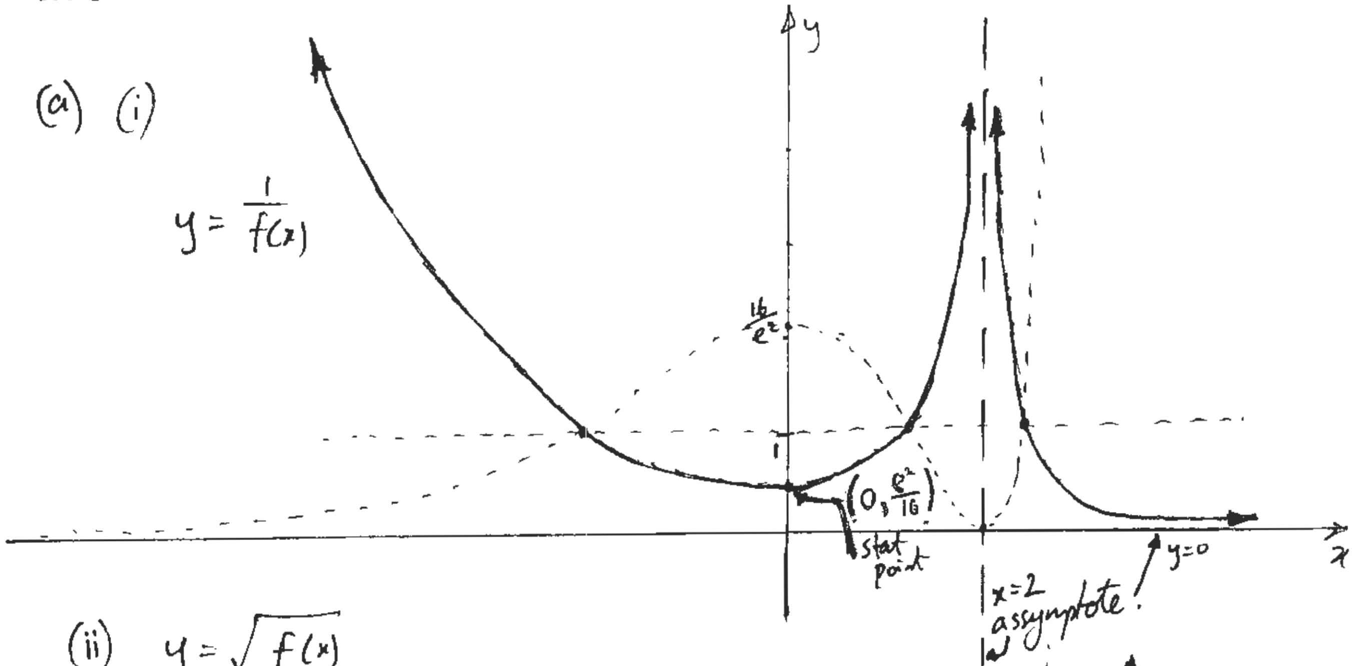
$=\frac{-3i \pm (4+3i)}{2}$ from part (i)

$=\frac{-3i+4+3i}{2}$ OR $\frac{-3i-4-3i}{2}$

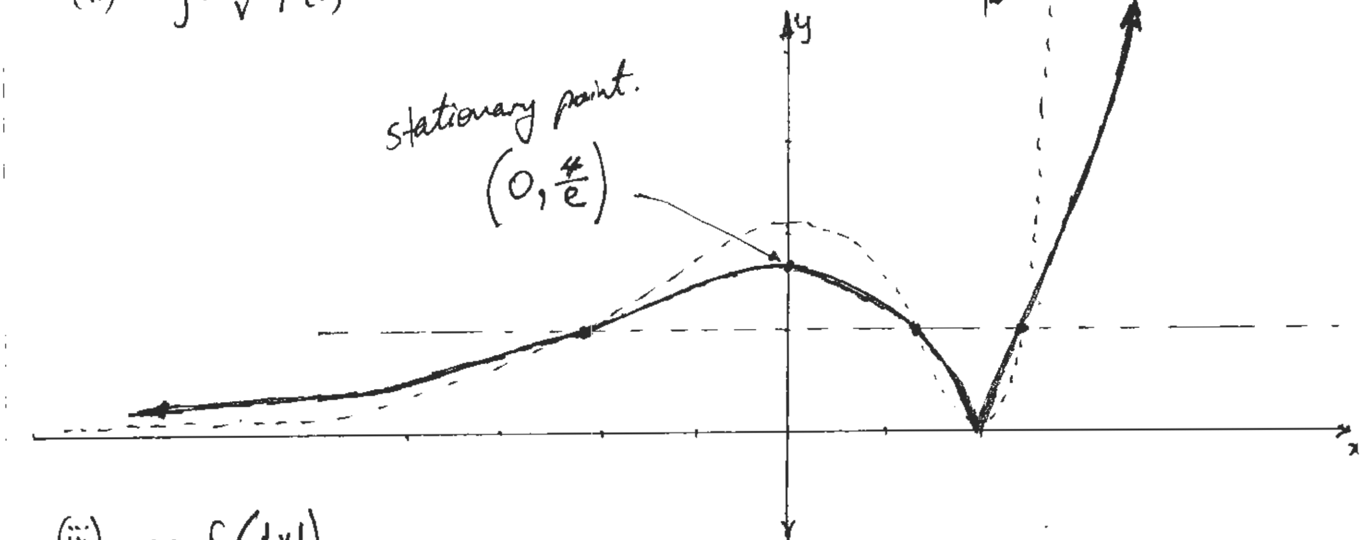
$=2$ OR $-2-3i$

QUESTION 12

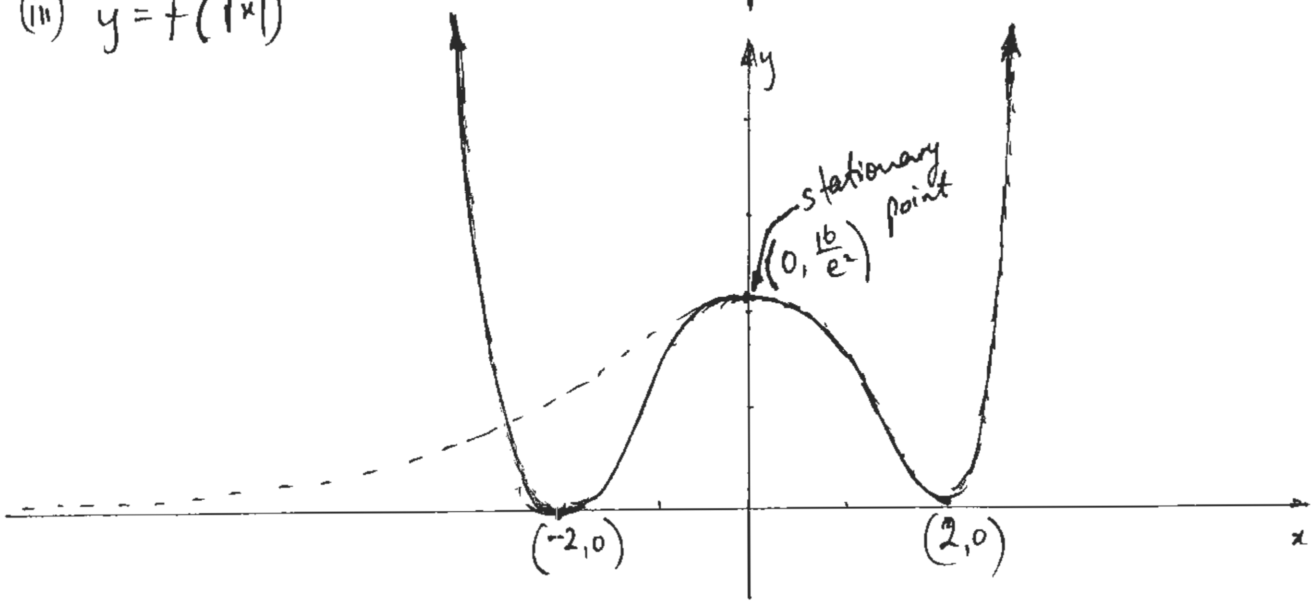
(a) (i)



(ii) $y = \sqrt{f(x)}$



(iii) $y = f(|x|)$



QUESTION 12 (continued)

(b) (i) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{—} ①$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \quad \text{—} ②$$

① + ② $\quad \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta \quad$ — ③

Let $A = \alpha+\beta$

$B = \alpha - \beta$

$\therefore 2\alpha = A+B \qquad 2\beta = A-B$

$\alpha = \dfrac{A+B}{2} \qquad \beta = \dfrac{A-B}{2}$

substitute into ③ $\quad \sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$.

(ii)
$$\sin 2x + \sin 4x = \sin 6x$$
$$2\sin\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) = \sin 6x \,.$$

using (i) with
$A = 2x$
$B = 4x$.

$$2\sin 3x\cos(-x) = \sin 6x$$
$$2\sin 3x\cos x = 2\sin 3x\cos 3x$$
$$2\sin 3x(\cos x - \cos 3x) = 0$$

$\sin 3x = 0 \quad$ OR $\quad \cos 3x = \cos x$

$3x = \pi n \quad$ or $\quad 3x = 2\pi n \pm x$

$x = \dfrac{\pi n}{3} \quad$ OR $\quad 2x = 2\pi n \quad$ OR $\quad 4x = 2\pi n \qquad n = \pm 1, \pm 2, \pm 3, \ldots$

$x = \dfrac{\pi n}{3} \quad$ OR $\quad x = \dfrac{\pi n}{2}$

(iii)
$$\int_0^{\pi/4} \sin 5x\cos 3x\,dx = \frac{1}{2}\int_0^{\pi/4} 2\sin\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)dx$$
$$= \frac{1}{2}\int_0^{\pi/4} \sin 8x + \sin 2x\,dx$$
$$= -\frac{1}{2}\left[\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right]_0^{\pi/4}$$
$$= -\frac{1}{2}\left(\frac{1}{8} + 0 - \left(\frac{1}{8} + \frac{1}{2}\right)\right) = \frac{1}{4}$$

$\dfrac{A+B}{2} = 5x$

$A+B = 10x$

$\dfrac{A-B}{2} = 3x$

$A-B = 6x$

$A = 8x,\ B = 2x$

QUESTION 12 (continued)

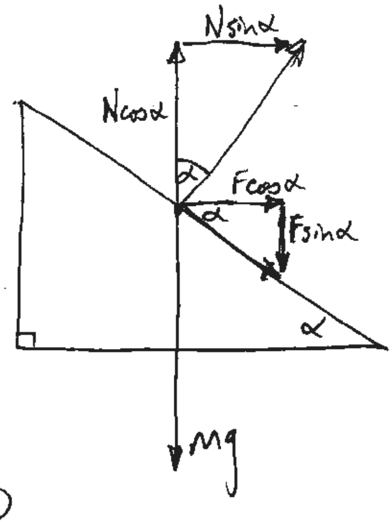
(c) $\left.\begin{aligned} A(-x) &= -A(x) \\ B(-x) &= -B(x) \end{aligned}\right\}$ since odd functions.

$P(x) = A(x) \times B(x)$

$$\begin{aligned} P(-x) &= A(-x) \times B(-x) \\ &= -A(x) \times -B(x) \\ &= A(x) \times B(x) \\ &= P(x) \end{aligned}$$

$\therefore$ $P(x)$ is even.

QUESTION 13



(a) (i) <u>Vertically</u>

$$N\cos\alpha - F\sin\alpha - mg = 0$$

$$N\cos\alpha = F\sin\alpha + mg \quad \text{—— (1)}$$

<u>Radially</u>

$$F\cos\alpha + N\sin\alpha = \frac{mv^2}{r}$$

$$N\sin\alpha = \frac{mv^2}{r} - F\cos\alpha \quad \text{—— (2)}$$

If $F = 0$

$$N\cos\alpha = mg \quad \text{—— (3)}$$
$$N\sin\alpha = \frac{mv^2}{r} \quad \text{—— (4)}$$

(4) ÷ (3) $\tan\alpha = \dfrac{v^2}{gr}$ $\quad \therefore v^2 = gr\tan\alpha$

$= 9.8 \times 250 \times \tan 7°$

$v = \sqrt{9.8 \times 250 \times \tan 7°}$ m/s

$\approx \sqrt{300.8}$ m/s

≈ 17.344 m/s

$= 17.344 \times 3.6$ km/h

≈ 62.4 km/h as required.

(ii) (2) ÷ (1) gives

$$\tan\alpha = \frac{\frac{mv^2}{r} - F\cos\alpha}{F\sin\alpha + mg}$$

$$F\frac{\sin^2\alpha}{\cos\alpha} + mg\frac{\sin\alpha}{\cos\alpha} = \frac{mv^2}{r} - F\cos\alpha$$

$$F\sin^2\alpha + mg\sin\alpha = \frac{mv^2}{r}\cos\alpha - F\cos^2\alpha$$

$$F(\sin^2\alpha + \cos^2\alpha) = \frac{mv^2\cos\alpha}{r} - mg\sin\alpha$$

108 km/hr = 30 m/s

$$F = \frac{mv^2\cos\alpha}{r} - mg\sin\alpha$$

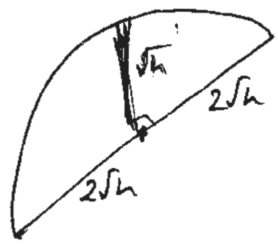
$$= \frac{800 \times 30^2\cos 7}{250} - 800 \times 9.8 \times \sin 7°$$

$$= 2858.53\ldots - 955.45\ldots = 1903 \text{ Newtons (nearest newton)}$$

QUESTION 13 (continued)

(b) (i) when $x = h$ $\quad y^2 = 4h$

$$y = \pm 2\sqrt{h}$$



$\therefore$ the base of the semi-ellipse is $4\sqrt{h}$

$\therefore$ The area of the semi-ellipse $= \frac{1}{2}\pi ab$

$$= \frac{1}{2}\pi(2\sqrt{h})(\sqrt{h})$$
$$= \frac{1}{2}\times 2\pi(\sqrt{h})^2$$
$$= \pi h \text{ (as required).}$$

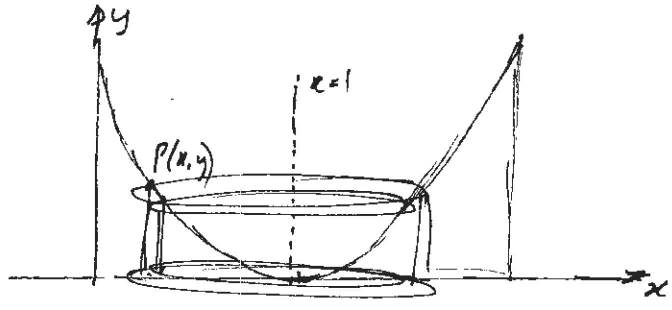
(ii) Volume of solid $= \sum_{h=0}^{4} \text{Area} \times \delta h$

$$= \int_0^4 \pi h \, dh$$
$$= \pi\left[\frac{h^2}{2}\right]_0^4$$
$$= \pi\left(\frac{16}{2} - 0\right)$$
$$= 8\pi \text{ units}^3.$$

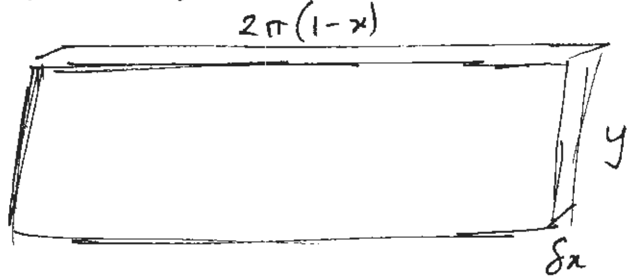
(c) over page ....

QUESTION 13 (continued)

(c) (i)



The cylindrical shell opens out to form a thin rectangular prism as shown below:



$$\delta V = 2\pi(1-x)\,y\,\delta x$$
$$= 2\pi(1-x)(1-\sqrt{x})^2\,\delta x$$

$$\sqrt{y} = 1-\sqrt{x}$$
$$y = (1-\sqrt{x})^2$$

$$V = 2\pi\int_0^1 (1-x)(1-\sqrt{x})^2\,dx$$

$$= 2\pi\int_0^1 (1-x)(1-2\sqrt{x}+x)\,dx$$

$$= 2\pi\int_0^1 1-2\sqrt{x}+\cancel{x}-\cancel{x}+2x\sqrt{x}-x^2\,dx$$

$$= 2\pi\int_0^1 1-x^2+2x\sqrt{x}-2\sqrt{x}\,dx$$ as required.

(ii) $$V = 2\pi\left[x-x^3/3+\frac{2x^{5/2}}{5/2}-\frac{2x^{3/2}}{3/2}\right]_0^1$$

$$= 2\pi\left[x-\frac{x^3}{3}+\frac{4x^{5/2}}{5}-\frac{4x^{3/2}}{3}\right]_0^1$$

$$= 2\pi\left(1-\frac{1}{3}+\frac{4}{5}-\frac{4}{3}\right)$$

$$= \frac{4\pi}{15}\text{ units}^3.$$

QUESTION 13 (continued)

(d) $P(x) = x^3 + 2x^2 + 3x + 4 = 0$ roots α, β, γ

$P(\sqrt{x}) = 0$ will have roots $\alpha^2, \beta^2, \gamma^2$

$(\sqrt{x})^3 + 2(\sqrt{x})^2 + 3(\sqrt{x}) + 4 = 0$

$x\sqrt{x} + 2x + 3\sqrt{x} + 4 = 0$

$$\sqrt{x}(x+3) = -2x-4$$
$$x(x+3)^2 = (-2)^2(x+2)^2$$
$$x(x^2+6x+9) = 4(x^2+4x+4)$$
$$x^3 + 6x^2 + 9x = 4x^2 + 16x + 16$$
$$x^3 + 2x^2 - 7x - 16 = 0$$

$\therefore$ The cubic equation with roots $\alpha^2, \beta^2, \gamma^2$

is $x^3 + 2x^2 - 7x - 16 = 0$

QUESTION 14

(a) $w = 2(\cos\theta + i\sin\theta) = 2\,\text{cis}\,\theta \qquad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}.$

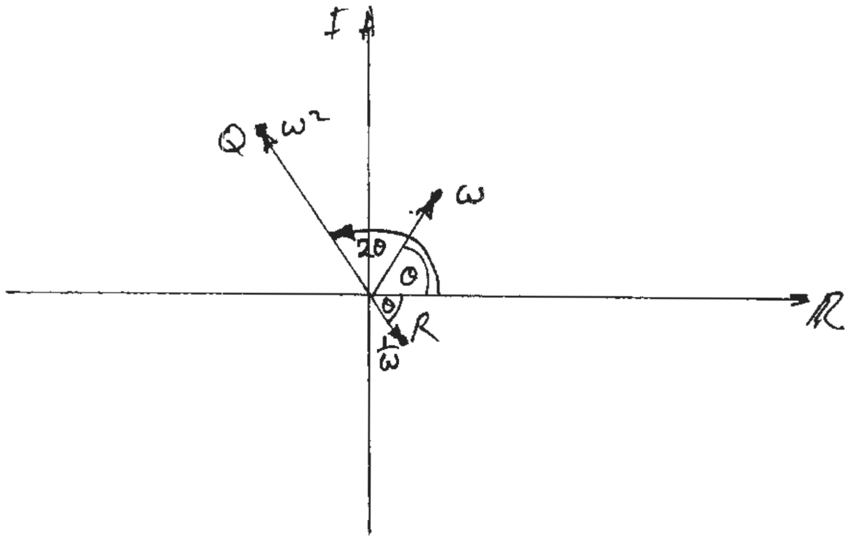
(i) $w^2 = 4\,\text{cis}\,2\theta$

$= 4(\cos 2\theta + i\sin 2\theta)$

$\frac{1}{w} = \frac{1}{2}\,\text{cis}(-\theta)$

$= \frac{1}{2}(\cos(-\theta) + i\sin(-\theta))$

(ii)



$2\theta + \theta = \pi$

$\theta = \frac{\pi}{3}$

$w = 2\cos\frac{\pi}{3} + 2i\sin\frac{\pi}{3}$

$= 2 \times \frac{1}{2} + i \times 2\frac{\sqrt{3}}{2}$

$w = 1 + i\sqrt{3}$

$= a + ib \qquad$ where $a = 1$, $b = \sqrt{3}$

QUESTION 14 (continued)

(b) $$I_n = \int_0^a (1+x^2)^n \, dx \qquad n = 1, 2, 3, \ldots$$

(i) $$I_n = \left[x(1+x^2)^n\right]_0^a - \int_0^a 2x^2 n (1+x^2)^{n-1} dx$$

$u = (1+x^2)^n$ $\qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = n(1+x^2)^{n-1} \times 2x$ $\qquad v = x$

$= 2xn(1+x^2)^{n-1}$

$$= \left(a(1+a^2)^n - 0\right) - 2n\int_0^a x^2(1+x^2)^{n-1} dx$$

$$= a(1+a^2)^n - 2n\int_0^a (1+x^2-1)(1+x^2)^{n-1} dx$$

$$= a(1+a^2)^n - 2n\int_0^a (1+x^2)^n - (1+x^2)^{n-1} dx$$

$$= a(1+a^2)^n - 2n I_n + 2n I_{n-1}$$

$$\therefore \quad I_n(1+2n) = a(1+a^2)^n + 2n I_{n-1}$$

$$I_n = \frac{a(1+a^2)^n}{2n+1} + \frac{2n}{2n+1} I_{n-1}$$

(ii) $$I_1 = \frac{a(1+a^2)^1}{2(1)+1} + \frac{2(1)}{2(1)+1} I_0$$

$$= \frac{a(1+a^2)}{3} + \frac{2}{3} I_0$$

$$= \frac{a+a^3}{3} + \frac{2a}{3}$$

$$= \frac{3a+a^3}{3}$$

$I_0 = \int_0^a 1 \, dx$

$= [x]_0^a$

$= a$

(iii) $$\int_0^1 (1+x^2)^3 dx = I_3$$

where $a = 1$ and $I_n = \frac{2^n}{2n+1} + \frac{2n}{2n+1} I_{n-1}$

$$= \frac{8}{7} + \frac{6}{7} I_2$$

$$= \frac{8}{7} + \frac{6}{7}\left(\frac{4}{5} + \frac{4}{5} I_1\right)$$

and $I_1 = \frac{3a+a^3}{3}$

$= \frac{4}{3}$

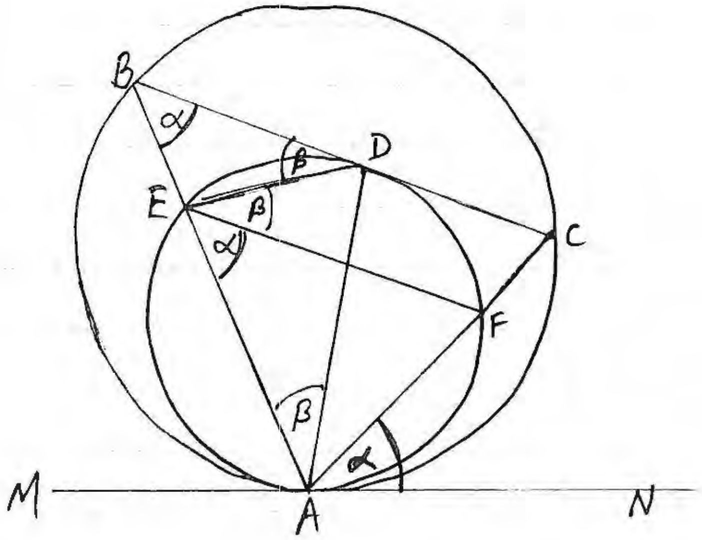
$$= \frac{8}{7} + \frac{24}{35} + \frac{6}{7} \times \frac{4}{5} \times \frac{4}{3}$$

$$= \frac{8}{7} + \frac{24}{35} + \frac{32}{35}$$

$$= 2\frac{26}{35}$$

QUESTION 14 (continued)

(c)



RTP AD bisects $\angle BAC$

that is RTP $\angle BAD = \angle FAD$.

Construct EF and ED.

Let $\alpha = \angle FEA = \angle FAN$ (angle between tangent and chord equals the angle in the alternate segment)

then $\alpha = \angle CBA = \angle CAN$ (" " " " " " ")

$\therefore$ $BC \parallel EF$ (corresponding angles $\angle CBA = \angle FEA$ are equal so parallel lines exist).

Let $\angle DEF = \beta$

Then $\angle BDE = \angle DEF = \beta$ (alternate angles in $\parallel$ lines are equal)

$\angle BAD = \angle BDE = \beta$ (alternate segment theorem (as above))

$\angle FAD = \angle DEF = \beta$ (angles standing on the same arc DF subtend equal angles at the circumference)

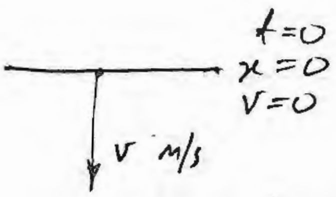
$\therefore$ $\angle BAD = \angle FAD$

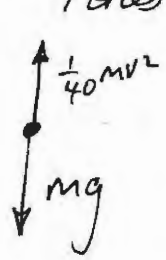
$\therefore$ AD bisects $\angle BAC$ as required.

QUESTION 15

Forces diagram

(a)





down = positive

$g = 10 m/s^2$ downwards

(i) Resultant Force $= m\ddot{x} = mg - \frac{1}{40}mv^2$

$$\ddot{x} = g - \frac{1}{40}v^2$$
$$= \frac{40g}{40} - \frac{v^2}{40}$$
$$= \frac{40g - v^2}{40}$$
$$= \frac{400 - v^2}{40}$$
$$= \frac{1}{40}\left(400 - v^2\right) \quad \text{as required.}$$

(ii) $\frac{dV}{dt} = \frac{1}{40}\left(400 - V^2\right)$

$$\frac{dt}{dv} = \frac{40}{400 - v^2}$$

$$t = 40\int \frac{1}{400 - v^2}\,dv$$

$$= 40\int \frac{1}{(20+v)(20-v)}\,dv$$

$$= \frac{40}{40}\int \frac{1}{20+v} + \frac{1}{20-v}\,dv$$

$$= \left[\ln(20+v) - \ln(20-v)\right] + c$$

$$= \ln\left(\frac{20+v}{20-v}\right) + c$$

$$\frac{1}{(20+v)(20-v)} = \frac{a}{20+v} + \frac{b}{20-v}$$

$$1 = a(20-v) + b(20+v)$$

$b - a = 0$, $a = b$ $\qquad$ $20a + 20b = 1$, $a + b = \frac{1}{20}$

$a = b = \frac{1}{40}.$

when $t=0$ $v=0$ $\therefore c=0$

$$\therefore \quad t = \ln\left(\frac{20+v}{20-v}\right)$$

QUESTION 15 (continued)

(a) (iii)

$$t = \ln\left(\frac{20+v}{20-v}\right)$$

$$e^t = \frac{20+v}{20-v}$$

$$(20-v)e^t = 20+v$$

$$20e^t = 20 + v(1+e^t)$$

$$20(e^t-1) = v(1+e^t)$$

$$v = 20\left(\frac{e^t-1}{1+e^t}\right)$$

$$= 20\left(\frac{1+e^t-2}{1+e^t}\right)$$

$$= 20\left(1-\frac{2}{1+e^t}\right) \quad \text{as required.}$$

(b)

$$P(x) = x^4 - 2x^3 - x^2 + 6x - 6$$

Since $P(1-i)=0$ $x-(1-i)$ is a factor of $P(x)$ and so is $x-(1+i)$

$$\big(x-(1-i)\big)\big(x-(1+i)\big) = (x-1+i)(x-1-i)$$

$$= (x-1)^2 - i^2$$

$$= x^2-2x+1+1$$

$$= x^2-2x+2$$

$$P(x) = (x^4-2x^3-x^2+6x-6) = (x^2-2x+2)(x^2+0x-3)$$

$$= (x^2-2x+2)(x^2-3)$$

$$= \big(x-(1-i)\big)\big(x-(1+i)\big)(x-\sqrt{3})(x+\sqrt{3})$$

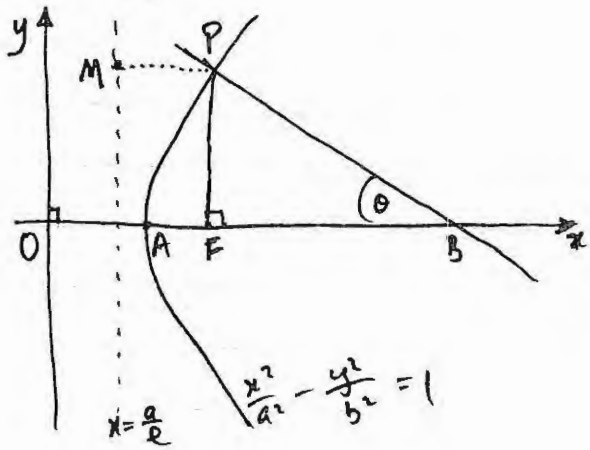
$\therefore$ the four solutions to $P(x)=0$ are:

$$x = 1-i,\ 1+i,\ \sqrt{3},\ -\sqrt{3}$$

i.e. $x = \pm\sqrt{3},\ 1\pm i$

QUESTION 15 (continued)

(c) (i)



$A(a, 0) \quad F(ae, 0)$

$$\begin{aligned} PF &= e\,PM \\ &= e\left(ae - \frac{a}{e}\right) \\ &= ae^2 - a \\ &= a(e^2 - 1) \end{aligned}$$

$\therefore P\left(ae,\ a(e^2-1)\right)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$b^2 = a^2(e^2-1)$$
$$\frac{b^2}{a^2} = e^2 - 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{a^2} \div \frac{2y}{b^2} \\ &= \frac{b^2 x}{a^2 y} \\ &= (e^2-1)\,\frac{x}{y} \end{aligned}$$

at P, $\frac{dy}{dx} = (e^2-1)\,\frac{ae}{a(e^2-1)} = e$

$\therefore$ Normal at P has gradient $= -\frac{1}{e}$

$$\begin{aligned} \therefore \tan(180 - \theta) &= -\frac{1}{e} \\ -\tan\theta &= -\frac{1}{e} \\ \tan\theta &= \frac{1}{e} \end{aligned}$$ as required.

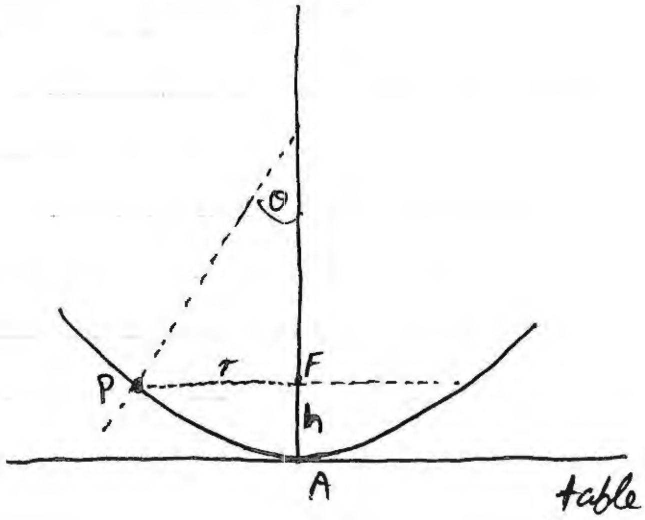
(ii) $AF = h = ae - a$

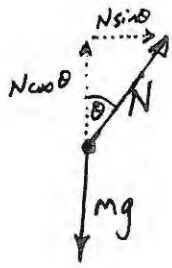
$h = a(e-1)$

$$\begin{aligned} PF &= a(e^2-1) \\ &= a(e-1)(e+1) \\ &= h(e+1) \end{aligned}$$ as required.

QUESTION 15 (continued)

(e) (iii)





Resolving forces <u>Vertically</u> $N\cos\theta = mg$ —①

<u>Radially</u> $N\sin\theta = m\omega^2 r$ —②

② ÷ ① $\tan\theta = \dfrac{m\omega^2 r}{mg}$

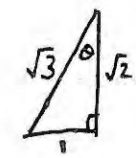
$\dfrac{1}{e} = \dfrac{\omega^2}{g} h(e+1)$ since $r = PF = h(e+1)$

$\dfrac{g}{e} = \omega^2 h(e+1)$

$\therefore \omega^2 = \dfrac{g}{he(e+1)}$ as required.

(iv) for a rectangular hyperbola $e = \sqrt{2}$

$N\cos\theta = mg$ $\tan\theta = \frac{1}{e} = \frac{1}{\sqrt{2}}$



$N \dfrac{\sqrt{2}}{\sqrt{3}} = mg$

$N\sqrt{\dfrac{2}{3}} = mg$

$\therefore N = mg\sqrt{\dfrac{3}{2}}$

QUESTION 16

(a) (i) The hyperbola $\left(ct, \frac{c}{t}\right)$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

when $$\frac{(ct)^2}{a^2} + \frac{\left(\frac{c}{t}\right)^2}{b^2} = 1$$

$$b^2c^2t^2 + \frac{a^2c^2}{t^2} = a^2b^2$$

$$b^2c^2t^4 + a^2c^2 = a^2b^2t^2$$

$$(bc)^2t^4 - (ab)^2t^2 + (ac)^2 = 0$$

The number of solutions is 0, 2, or 4 = # of possible intersection points.

For the hyperbola to "TOUCH" the ellipse at P where $t=p$,

p must be a double root, and to "TOUCH" at Q

where $t=-p$, $-p$ must be a double root.

$\therefore$ the roots of the equation are $p, p, -p, -p$.

(ii) Σ roots in pairs $= p^2 - p^2 - p^2 - p^2 - p^2 + p^2 = \frac{-(ab)^2}{(bc)^2}$

$$-2p^2 = -\frac{a^2}{c^2}$$

$$p^2 = \frac{a^2}{2c^2}$$

$$p = \frac{a}{\sqrt{2}c}$$

product of roots $= p^4 = \frac{(ca)^2}{(bc)^2}$

$$p^4 = \frac{a^2}{b^2}$$

$$p^2 = \frac{a}{b} \qquad \text{but } p^2 = \frac{a^2}{2c^2} \quad \therefore \frac{a}{b} = \frac{a^2}{2c^2}$$

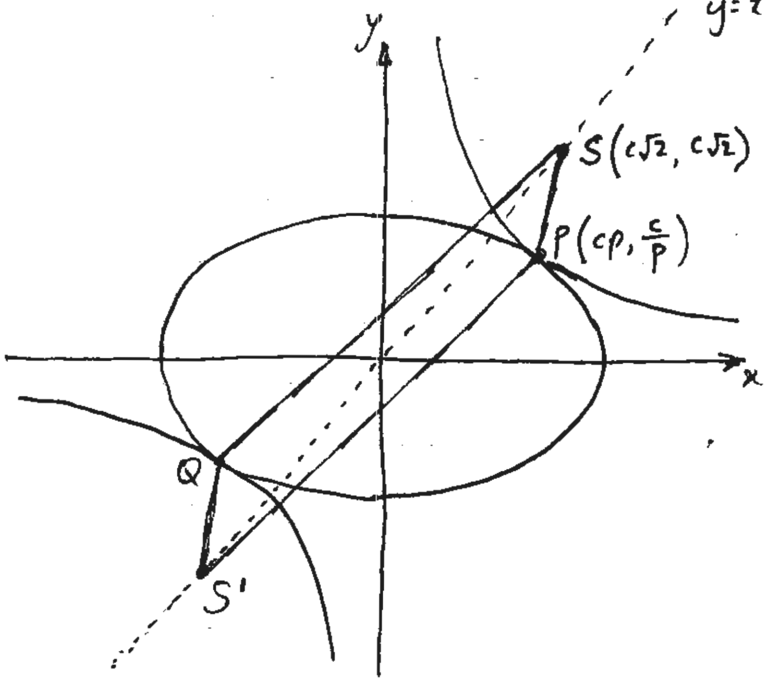
$$2ac^2 = a^2b$$

$$\therefore ab = 2c^2$$

$$\therefore p = \frac{a}{\sqrt{2}c} \quad \text{and} \quad ab = 2c^2$$

QUESTION 16 continued

(a) (iii)



$y = x$

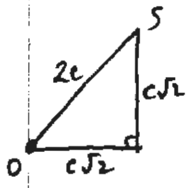
$x - y + 0 = 0$

general form.

Area of $PSQS' = 2 \times$ Area PSS'

$= 2 \times \frac{1}{2} \times SS' \times$ ($\perp$ distance from P to $y = x$.)

$$= 2 \times \tfrac{1}{2}(2c + 2c) \times \left| \frac{1 \times cp - 1 \times \frac{c}{p} + 0}{\sqrt{1^2 + (-1)^2}} \right|$$



$$= 1 \times 4c \times \frac{cp - \frac{c}{p}}{\sqrt{2}}$$

$$= \frac{4c}{\sqrt{2}} \times \left(cp - \frac{c}{p}\right)$$

$p = \frac{a}{c\sqrt{2}}$

$cp = \frac{a}{\sqrt{2}}$

$$= \frac{4c}{\sqrt{2}} \times \left(\frac{a}{\sqrt{2}} - c \times \frac{c\sqrt{2}}{a}\right)$$

$$= \frac{4c}{\sqrt{2}} \times \left(\frac{a}{\sqrt{2}} - \frac{c^2\sqrt{2}}{a}\right)$$

$$= 2ac - \frac{8c^3}{a}$$

$ab = 2c^2$

$b = \frac{2c^2}{a}$

$$= 2c\left(a - \frac{2c^2}{a}\right)$$

$$= 2c(a - b)$$

QUESTION 16 (continued)

(b) (i) RTP $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$

i.e. RTP $\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b} \geq 0$ since $P \geq Q \Leftrightarrow P - Q \geq 0$

$$\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b} = \frac{b+a}{ab} - \frac{4}{a+b}$$
$$= \frac{(a+b)^2 - 4ab}{ab(a+b)}$$
$$= \frac{a^2+b^2+2ab-4ab}{ab(a+b)}$$
$$= \frac{a^2+b^2-2ab}{ab(a+b)}$$
$$= \frac{(a-b)^2}{ab(a+b)}$$
$$\geq 0$$

since $(a-b)^2 \geq 0$ and $ab(a+b) \geq 0$ when $a, b > 0$.

(ii) $\frac{x+y}{2} \geq \sqrt{xy}$

$$\left.\begin{array}{l} x = \frac{1}{a^2} \\ y = \frac{1}{b^2} \end{array}\right\} \Rightarrow \frac{\frac{1}{a^2} + \frac{1}{b^2}}{2} \geq \sqrt{\frac{1}{a^2}\frac{1}{b^2}}$$

$$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$$

$$\geq \frac{8}{(a+b)^2}$$

Also, $\frac{a+b}{2} \geq \sqrt{ab}$

$\therefore \frac{(a+b)^2}{4} \geq ab$

$\therefore (a+b)^2 \geq 4ab$

$\therefore \frac{1}{4ab} \geq \frac{1}{(a+b)^2}$

$\therefore \frac{2}{ab} \geq \frac{8}{(a+b)^2}$

QUESTION 16 (continued)

(c) $f(x) = \dfrac{x^n}{e^x} \qquad n > 1$

(i) $$f'(x) = \frac{e^x \times nx^{n-1} - x^n \times e^x}{(e^x)^2}$$

$$= \frac{e^x x^{n-1}(n-x)}{(e^x)^2}$$

$$= \frac{x^{n-1}(n-x)}{e^x}$$

(ii) stationary points when $f'(x) = 0$

$$\frac{x^{n-1}(n-x)}{e^x} = 0$$

$$x^{n-1}(n-x) = 0$$

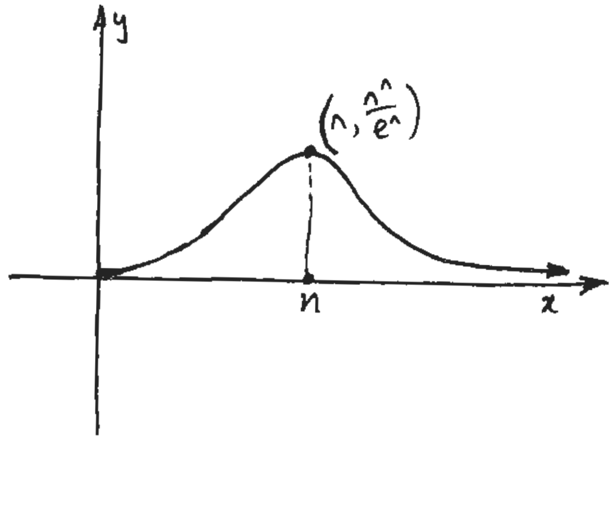
$$x = 0 \quad \text{OR} \quad x = n$$

x	—	0	$\frac{n}{2}$	n	$2n$
$f'(x)$		0	$+$	0	$-$
gradient ⇒		—	/	—	\

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0^+$$

when $x = 0 \quad f(x) = 0$

when $x = n \quad f(n) = \dfrac{n^n}{e^n}$



QUESTION 16 (continued)

(c) (iii) when $x > n$ the curve is decreasing

$\therefore$ the y-coordinate of the graph for $x > n$ is always less than the maximum value $\frac{n^n}{e^n}$

$$\therefore \quad \frac{x^n}{e^x} < \frac{n^n}{e^n} \quad \text{for } x > n.$$

(iv) $$\frac{x^n}{e^x} < \frac{n^n}{e^n} \quad \text{for } x > n$$

use $x = n+1$, $x > n$

$$\frac{(n+1)^n}{e^{n+1}} < \frac{n^n}{e^n}$$

$$\frac{(n+1)^n}{n^n} < \frac{e^{n+1}}{e^n}$$

noting $n^n > 0$ and $e^{n+1} > 0$.

$$\left(\frac{n+1}{n}\right)^n < \frac{e^n e}{e^n}$$

$$\left(\frac{n+1}{n}\right)^n < e$$

$$\therefore \left(1 + \frac{1}{n}\right)^n < e$$