E.H

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2001
TRIAL HSC EXAMINATION

Mathematics Extension 2

- **General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on page 10
- All necessary working should be shown in every question

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a SEPARATE writing booklet for each question

NAME:____________________ **TEACHER:**____________________

Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find:

(i) $\int \frac{x}{\sqrt{9-4x^2}}\,dx$ **2**

(ii) $\int \frac{x^2}{x+1}\,dx$ **2**

(iii) $\int_0^{\ln 2} xe^x\,dx$ **3**

(b) (i) Find real numbers A, B and C such that $\frac{2}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$. **3**

(ii) Hence, find $\int_0^1 \frac{2}{(t+1)(t^2+1)}\,dt$. **3**

(iii) By using the substitution $t = \tan\left(\frac{x}{2}\right)$ evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x - \cos x}\,dx$. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Suppose $z = 2+2i$ and $w = -1+\sqrt{3}i$.

(i) Express z and w in modulus – argument form. **2**

(ii) Find $\left|\frac{z}{w}\right|^4$. **1**

(iii) Find the principal argument of $\left(\frac{z}{w}\right)^4$. **2**

(b) Sketch separately the following loci in an Argand plane and state the cartesian equations in each case given that:

(i) $|z-3i| = |z-4|$ **2**

(ii) $\text{Re}\left(\frac{z-2}{2}\right) = 0$ **2**

(iii) $\arg(z+2) = -\frac{\pi}{6}$ **2**

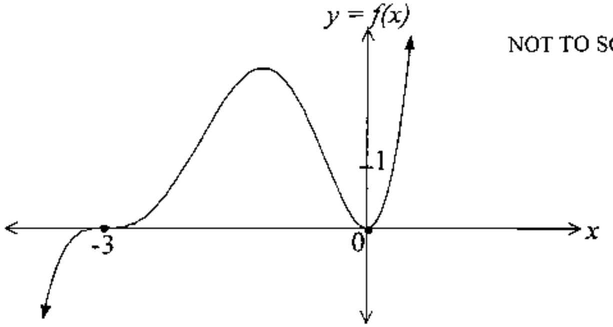
(c) (i) Show that if $z = x+iy$ then $|z|^2 = z\bar{z}$. **1**

(ii) Using the result of (c)(i), or otherwise, prove that for any two complex numbers z and w that:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$ **2**

(iii) Interpret this result geometrically. A vector diagram may be useful. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**



NOT TO SCALE

(a) Consider the graph of $y = f(x)$ as shown above.

On the answer sheet provided on pages 11 & 12, use the graph of $y = f(x)$ to clearly sketch ***separately*** the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f'(x)$. **1**

(b) Suggest a possible polynomial equation for the graph of $y = f(x)$ shown in ***part (a) of Question 3.*** **1**

(c) (i) Show that $x = 1$ is a zero of $x^3 + 3x^2 - 4$. **1**

(ii) Sketch the curve with the equation $y = x^3 + 3x^2 - 4$, giving the coordinates of any maximum or minimum points and the intercepts made on each axis. **3**

(iii) Use your results in (c)(ii) above to sketch the curves: **4**

(α) $y = \left| x^3 + 3x^2 - 4 \right|$

(β) $y = \ln\left| x^3 + 3x^2 - 4 \right|$

(iv) Hence, or otherwise, determine the value of m, where m is a constant such that the equation $2\ln|x+2| + \ln|x-1| = m$. **1**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Draw a sketch graph of the hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ and shade clearly the region bounded by the lines $x = \pm a$ and the upper and lower branches of this hyperbola. **1**

(ii) Show $\dfrac{d}{d\theta}\ln(\sec\theta + \tan\theta) = \sec\theta$. **1**

(iii) Explain why the area, A, of the shaded region drawn in (a)(i) above can be by: **2**

$$A = \int_0^a \frac{4b}{a}\sqrt{a^2 + x^2}\, dx.$$

(iv) By using the substitution $x = a\tan\theta$ in (a)(iii), show that $A = 4ab\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$. **2**

(v) Show that the integral stated in (a)(iv) simplifies to $2ab\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$. **3**

(***Hint***: Write $\sec^3\theta$ in the form $\sec\theta.\sec^2\theta$ and then use integration by parts)

(vi) Use the ***method of cylindrical shells*** to show that the volume (in cubic units) of the solid generated by revolving this area about the y-axis is given by: **3**

$$V = \frac{4\pi ba^2}{3}\left(2\sqrt{2} - 1\right).$$

(b) A solid has a base, which is the *standard ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with major axis of length $2a$ units and minor axis of length $2b$ units ($a > b$). In the vertical plane, the cross–sections of the solid are always isosceles triangles with perpendicular height h and whose base is parallel to the major axis. **3**

Use the ***method of slicing*** to find the volume of the solid.

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The point T with coordinates $(at^2, 2at)$, $t \neq 0$, $a > 0$, lies on the parabola with equation $y^2 = 4ax$. The tangent to the parabola at T meets the axis of the parabola at R. The normal at T meets the axis of the parabola at Q and the parabola again at P. The coordinates of P are $(ap^2, 2ap)$.

(i) Represent this information on a clear and well-labelled diagram. **1**

(ii) Derive the equations of the tangent and normal to the parabola at T. **2**

(iii) Show that the length of RQ is $2a(1+t^2)$ units. **1**

(iv) Show that the values of t for which R will lie on the directrix of the parabola satisfy $t^2 = 1$. **2**

(v) Show that if $t \neq p$, then $p = -\left(t + \frac{2}{t}\right)$. **2**

(vi) Find TP, in terms of a and simplify your expression as far a possible. **1**

(vii) Hence, or otherwise, prove that the area of ΔTPR is $16a^2$ square units. **1**
(You may assume R lies on the directrix)

(b) The equation of a rectangular hyperbola in cartesian form is given by $xy = c^2$ where $c > 0$.

(i) Verify that the point $P\left(cp, \frac{c}{p}\right)$ lies on $xy = c^2$, where p is a non-zero real number. **1**

(ii) Q has coordinates $\left(cq, \frac{c}{q}\right)$ where q is a non-zero real number. **2**
Show that the equation of the chord PQ is given by $x + pqy = c(p + q)$.

(iii) Find the equation of the locus of the midpoint of the chord PQ if it is known that the chord must always pass through the point $(0, 2)$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle of mass m units is projected vertically upward from the ground with initial speed u. The air resistance at any instance is proportional to the velocity v at that instant. For this question you may assume $R = kmv$ where k is a constant.

(i) With the aid of a suitable diagram show that $\frac{dv}{dt} = -(g + kv)$? **1**

(ii) Show at any time t, that $t = \frac{1}{k}\ln\left|\frac{g+ku}{g+kv}\right|$ seconds. **3**

(iii) Prove that the particle reaches it highest point in time T seconds when: **1**

$$T = \frac{1}{k}\ln\left(\frac{ku}{g} + 1\right)$$

(iv) The highest point reached by the particle is at H metres above the ground.

(α) Prove that $x = \frac{1}{k^2}(g + ku)(1 - e^{-kt}) - \frac{gt}{k}$. **3**

(β) Prove that $H = \frac{1}{k}(u - gT)$. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$ where n is a positive integer such that $n \geq 2$.

(i) By replacing $\sin^n\theta$ with $\sin^{n-1}\theta.\sin\theta$, and using integration by parts or otherwise, show that $I_n = \frac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, or otherwise, evaluate I_{10}. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let α, β, and γ be the non – zero roots of the equation $x^3 + rx + s = 0$.

(i) Find in terms of r, the simplified value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

(ii) Find in terms of r and s, the simplified value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. **2**

(iii) Find in terms of r and s, the cubic equations (in general form) whose roots are

(A) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$, and $\dfrac{1}{\gamma}$; **2**

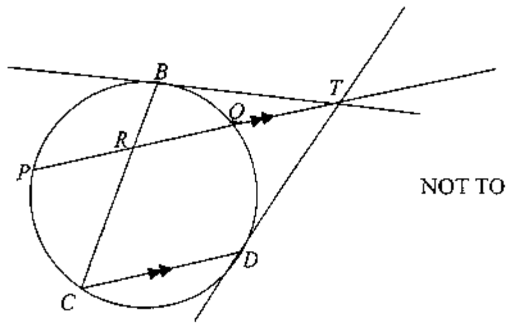
(B) $\alpha + \beta - \gamma, \beta + \gamma - \alpha$ and $\gamma + \alpha - \beta$ **3**

(b) Suppose $x^3 + rx + s = 0$ (with r and s being non–zero and real) has a double root. **2**

Show that $x = -\dfrac{3s}{2r}$.

(c) Find all the roots of $p(x) = x^4 - 8x^3 + 39x^2 - 122x + 170$ given that $3 - i$ is one of the roots. **4**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**



NOT TO SCALE

(a) In the diagram, PQ and CD are parallel chords of a circle. The tangent at D meets PQ produced externally at T. B is the point of contact of the other tangent from the circle. BC meets PQ internally at R.

Copy or trace this diagram into your writing booklet

(i) Explain why $\angle BDT = \angle BRT$? **2**

(ii) Show that B, T, D and R are concylic points. **2**

(iii) Prove that $\angle BRT = \angle DRT$. **2**

(iv) Show that ΔRCD is isosceles. **2**

(v) Show that BC bisects PQ. **2**

(b) (i) Show that $\cos x = \sin\left(x + \dfrac{\pi}{2}\right)$. **1**

(ii) Given that $y = 3\sin x + 4\cos x$, prove by the Principle of Mathematical Induction that $\dfrac{d^n y}{dx^n} = 5\sin\left(x + \alpha + \dfrac{n\pi}{2}\right)$ where $\dfrac{d^n y}{dx^n}$ means the nth derivative of y with respect to x and $n = 1, 2, 3, \ldots\ldots$ **4**

You are advised to first express $y = 3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$.

End of Paper

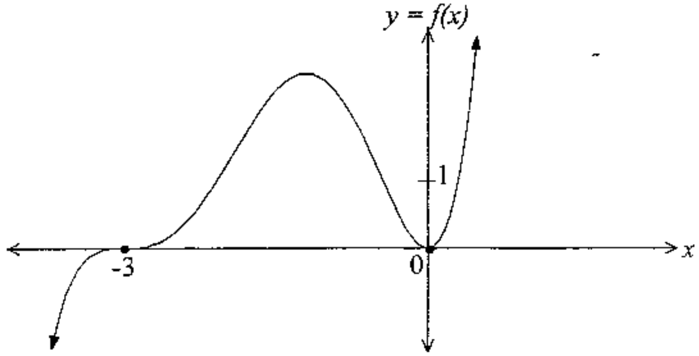
Year 12 Trial HSC Mathematics Extension 2 – 2001
Answer Sheet for Question 3 (a) only!

Detach and submit this page with your solutions to Question 3

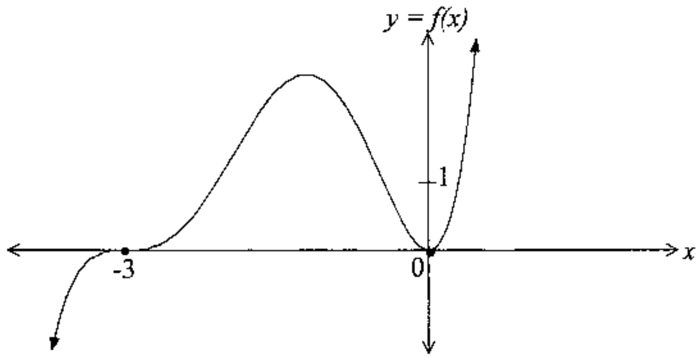
Student Name: ______________________

Question 3 (a) *In each case use the graph of y = f(x) to clearly sketch the following:*

(i) $y = \frac{1}{f(x)}$.

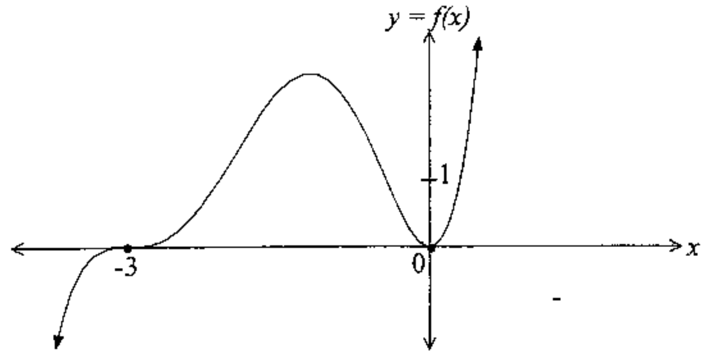


(ii) $y^2 = f(x)$.



Please turn over for part (a)(iii).

(iii) $y = f'(x)$



Question 3 (b):

Possible polynomial equation for y = f(x): ______________________

Suggested Solutions

Mathematics Extension 2 – Yr 12 TRIAL HSC KGS 2001

Question 1

(a) (i) $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$

by inspection,

$= -\frac{1}{4}\sqrt{9-4x^2} + c$

(or let $u = 9-4x^2$)

ie $\frac{du}{dx} = -8x$

separating variables,

$\therefore x\,dx = -\frac{1}{8}\,du$

$\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{8}\int u^{-1/2}\,du$

$= -\frac{1}{8}\left(\frac{u^{1/2}}{1/2}\right) + c$

$= -\frac{1}{4}\sqrt{u} + c$

$= -\frac{1}{4}\sqrt{9-4x^2} + c$

(ii) $\displaystyle\int \frac{x^2}{x+1}\,dx$

$= \displaystyle\int \frac{(x-1)(x+1)+1}{(x+1)}$

$= \displaystyle\int (x-1) + \frac{1}{x+1}\,dx$

$= \frac{x^2}{2} - x + \ln|x+1| + c$

(or a substitution will)

(iii) $\displaystyle\int_0^{\ln 2} x e^x\,dx$

let $u = x$, $dv = e^x$

$du = 1$, $v = e^x$

$\therefore$ using $\int u\,dv = uv - \int v\,du$

$= [xe^x]_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx$

$= [\ln 2(e^{\ln 2}) - 0] - [e^x]_0^{\ln 2}$

$= (\ln 2)(2) - e^{\ln 2} + e^0$

$= 2\ln 2 - 2 + 1$

$= 2\ln 2 - 1$

(b) (i) $\frac{2}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$

$\Rightarrow 2 \equiv A(t^2+1) + (t+1)(Bt+C)$

by substitution:

$t=-1$; $2 = 2A$ $\therefore A = 1$

$t=0$; $2 = A + C$, $2 = 1 + C$ $\therefore C = 1$

$t=1$; $2 = 2A + 2(B+C)$, $2 = 2 + 2(B+1)$ ie $B = -1$

(or use $t = \pm i$)

hence: $A = 1$, $B = -1$, $C = 1$

(ii) $\displaystyle\int_0^1 \frac{2}{(t+1)(t^2+1)}\,dt$

$= \displaystyle\int_0^1 \frac{1}{t+1} + \frac{1-t}{t^2+1}\,dt$

$= \displaystyle\int_0^1 \frac{1}{t+1} + \frac{1}{t^2+1} - \frac{t}{t^2+1}\,dt$

$= \left[\ln|t+1| + \tan^{-1}t - \frac{1}{2}\ln(t^2+1)\right]_0^1$

$= \left[\tan^{-1}t + \ln\left|\frac{t+1}{\sqrt{t^2+1}}\right|\right]_0^1$

$= \tan^{-1}1 + \ln\left|\frac{2}{\sqrt{2}}\right| - 0$

$= \frac{\pi}{4} + \ln\sqrt{2}$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$

(iii) $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(\tan^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}(t^2+1)$

Separating varia

$\frac{2\,dt}{t^2+1} = dx$

also when $x = 0$,
$x = \frac{\pi}{2}$ t

$\sin x = \frac{2t}{1+t^2}$, cos

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x - \cos x}\,dx$

$= \displaystyle\int_0^1 \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2} - \left(\frac{1-t^2}{1+t^2}\right)} \cdot \left(\frac{2}{t^2+1}\right.$

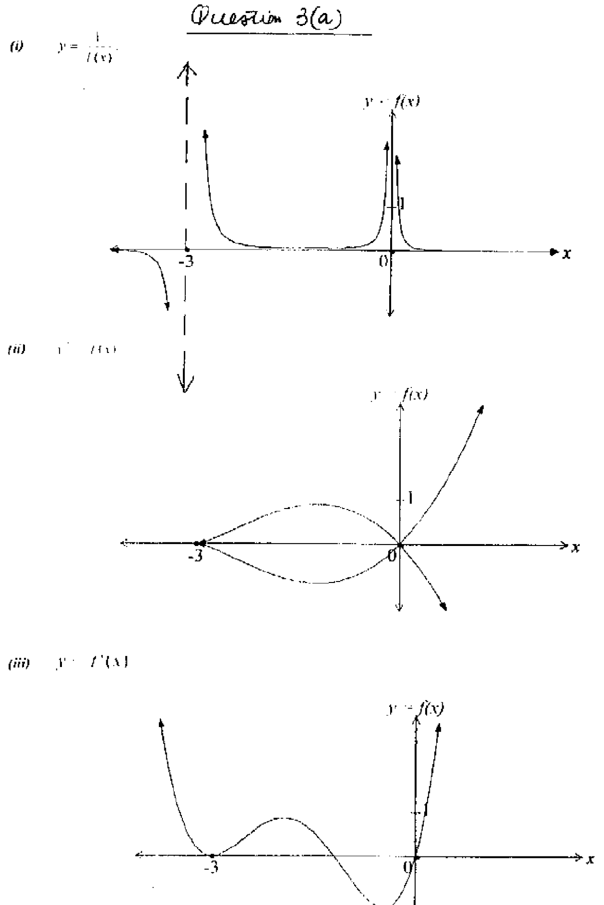
$= \displaystyle\int_0^1 \frac{4t}{(t^2+1)(2t^2+2t)}$

$= \displaystyle\int_0^1 \frac{2}{(t+1)(t^2+1)}$

from (b)(ii)

$= \frac{\pi}{4} + \frac{\ln 2}{2}$

Question 3(a)



(b) Possible equation: $y = ax^2(x+3)^3$ $(a \neq 0)$

Question 3

(c) (i) $1^3 + 3(1)^2 - 4$

$= 1 + 3 - 4$

$= 4 - 4$

$= 0$

$\therefore x = 1$ is a zero of $x^3 + 3x^2 - 4$.

(ii) other intercepts:

when $x = 0$, $y = -4 \Rightarrow (0,-4)$

also $x = 1$.

N.B. $y = (x-1)(x+2)^2$

$\therefore$ x-intercepts are $x = 1$, $x = -2$ (double root)

$\frac{dy}{dx} = 3x^2 + 6x$

TP's when $y' = 0$

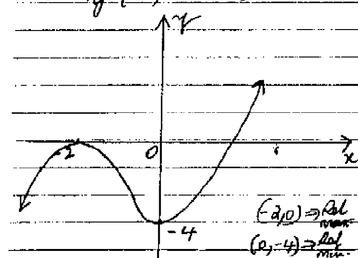
ie when $3x(x+2) = 0$

$x = 0$, $x = -2$

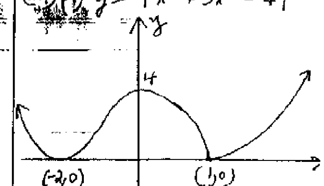
$y'' = 6x + 6$

$y''(0) = 6 > 0$ Rel min

$y''(-2) = -6 < 0$ Rel max



(iii) (α) $y = |x^3 + 3x^2 - 4|$



(β) $y = \ln|x^3 + 3x^2 - 4|$

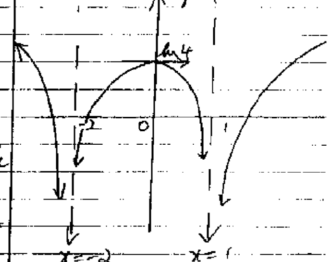
This is defined only when $y = |x^3+3x^2-4| > 0$

However, we use the graph we observe $y = \ln|x^3+3x^2-4|$ is undefined when $x = -2, 1$

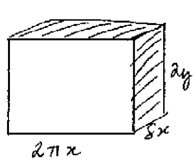
These are asymptotes.

Also, when $x = 0$, $y = \ln 4$ a max T. Point as well.

$\lim_{x \to \pm\infty} \ln|x^3+3x^2-4| = \infty$



(vi)



$V = 4\pi \int_0^a xy\,dx$

$= 4\pi \int_0^a x\left(\frac{b}{a}\sqrt{a^2+x^2}\right)dx$

$(y>0)$

$V = \frac{4\pi b}{a}\int_0^a x\sqrt{a^2+x^2}\,dx$

$= \frac{4\pi b}{a}\left[\frac{(a^2+x^2)^{3/2}}{3}\right]_0^a$

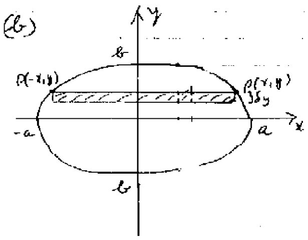
$= \frac{4\pi b}{3a}\left[(a^2+a^2)^{3/2} - (a^2)^{3/2}\right]$

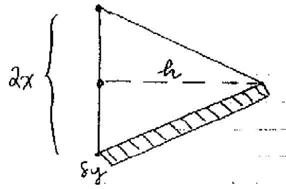
$= \frac{4\pi b}{3a}\left[(2a^2)^{3/2} - a^3\right]$

$= \frac{4\pi b}{3a}\left[2^{3/2}\cdot a^3 - a^3\right]$

$= \frac{4\pi b a^2}{3}\left[2\sqrt{2}-1\right]\ u^3.$

(b)





Area of a typical cross-sectional slice

$A(y) = \frac{1}{2}\cdot 2x\cdot h$

$A(y) = xh$

Volume of a typical slice

$\delta V = xh\,\delta y$

$\therefore$ Total Volume $= \lim_{\delta y\to 0}\sum_{y=-b}^{b} xh\,\delta y$

ie $V = 2h\int_0^b x\,dy$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$

$x^2 = \frac{a^2(b^2-y^2)}{b^2}$

$\therefore x = \frac{a}{b}\sqrt{b^2-y^2}$

$\therefore V = \frac{2ha}{b}\int_0^b \sqrt{b^2-y^2}\,dy$

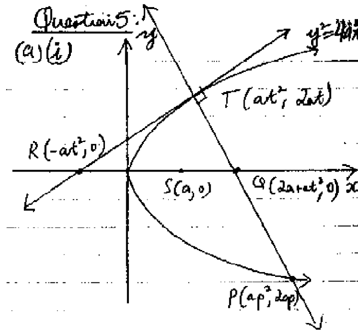
$= \frac{2ha}{b}\left[\frac{1}{4}\cdot\pi\cdot b^2\right]$

$= \frac{2ah\pi b^2}{4b}$

$V = \frac{\pi abh}{2}$ cubic units

Question 5:

(a)(i)



(ii) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\therefore \frac{dy}{dx} = \frac{2a}{y}$

when $y = 2at$, at T

then $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Also, the slope of the normal at T is $-t$

Equation of tangent:

$y - 2at = \frac{1}{t}(x - at^2)$

$x - ty = -at^2$ ①

Equation of normal:

$y - 2at = -t(x - at^2)$

$tx + y = 2at + at^3$ ②

(iii) To find the length of RQ we need the coordinates of R & Q.

Equation of tangent meets x-axis at R ie when y

$\therefore$ in ① $x = -at^2$

$\therefore R(-at^2, 0)$

Equation of normal meets x-axis at Q ie when

$\therefore$ in ② $tx = 2at + a$

$(t\neq 0)$ $\therefore x = 2a + at$

$\therefore Q(2a + at^2, 0)$

$\therefore RQ = |-at^2| + |2a + a$

$= at^2 + 2a + at$

$= 2at^2 + 2a$

$= 2a(t^2+1)$

(iv) R will lie on the directrix if when $x =$

$y = 0$ (by definition y^2

in ① $x - ty = -at^2$

$-a = -at^2$

$\therefore t^2 = 1$

$\Rightarrow t = \pm 1$

(v) Since $P(ap^2, 2ap)$ sat equation of normal at T

then in ②

$t(ap^2) + 2ap = 2at + a$

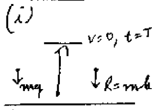
ie $p^2t + 2p = 2t + t^3$

ie $2(p-t) = t(t^2 - p^2$

$-2 = t(t+p)$ (t

Question 6.

(a) (i)



$m\ddot{x} = \sum F_x$

$m\ddot{x} = -mg - mkv$

$\therefore \ddot{x} = -g - kv$

ie $\frac{dv}{dt} = -(g+kv)$

(ii) $\frac{dt}{dv} = -\frac{1}{g+kv}$

$t = -\frac{1}{k}\ln|g+kv| + c_1$

when $t=0$, $v=u$

$\therefore 0 = -\frac{1}{k}\ln|g+ku| + c_1$

ie $c_1 = \frac{1}{k}\ln|g+ku|$

hence $t = \frac{1}{k}\ln|g+ku| - \frac{1}{k}\ln|g+kv|$

ie $t = \frac{1}{k}\ln\left|\frac{g+ku}{g+kv}\right|$.

(iii) when $t=T$, $v=0$ (at max height)

$\therefore T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$

$T = \frac{1}{k}\ln\left(1+\frac{ku}{g}\right)$ from (ii).

(iv) Considering $kt = \ln\left[\frac{g+ku}{g+kv}\right]$

and solving for v.

$e^{kt} = \frac{g+ku}{g+kv}$

ie $kv = (g+ku)e^{-kt} - g$

$v = \frac{1}{k}\left[(g+ku)e^{-kt} - g\right]$

let $v = \frac{dx}{dt}$

thence: $\frac{dx}{dt} = \frac{1}{k}\left[(g+ku)e^{-kt} - g\right]$

hence: $x = \frac{1}{k}\left[-\frac{1}{k}(g+ku)e^{-kt} - gt\right] +$

when $t=0$, $x=0$ $\therefore C_2 = \frac{1}{k^2}(g+ku)$

hence: $x = \frac{1}{k^2}(g+ku)\left[1 - e^{-kt}\right] - \frac{gt}{k}$

at $x = H$, $t = T$

$\therefore H = \frac{1}{k^2}(g+ku)\left[1 - e^{-kT}\right] - \frac{gT}{k}$

since $T = \frac{1}{k}\ln\left(1+\frac{ku}{g}\right)$

then $kT = \ln\left(1+\frac{ku}{g}\right)$

$e^{kT} = \ \ \ \ e\ \ \overline{(g+ku)}$

$H = \frac{1}{k^2}(g+ku)\left[1 - e^{-\ln\left(\frac{g+ku}{g}\right)}\right] - \frac{gT}{k}$

$= \frac{1}{k^2}(g+ku)\left[1 - \frac{g}{g+ku}\right] - \frac{gT}{k}$

$= \frac{g+ku}{k^2} - \frac{g}{k^2} - \frac{gT}{k}$

$H = \frac{g+ku-g}{k^2} - \frac{gkT}{}$

$H = \frac{ku - gkT}{k^2}$

$H = \frac{u - gT}{k}$

ie $H = \frac{1}{k}(u - gT)$ //.

(b) $I_n = \int_0^{\pi/2}\sin^n\theta\,d\theta$

$= \int_0^{\pi/2}\sin\theta\cdot\sin^{n-1}\theta\,d\theta$

let $u = \sin^{n-1}\theta$ $\quad dv = \sin\theta$

$du = (n-1)\cos\theta\sin^{n-2}\theta$ $\quad v = -\cos\theta$

$\therefore I_n = uv - \int v\,du$

$= \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2}\cos^2\theta\sin^{n-2}\theta\,d\theta$

$I_n = 0 + (n-1)\int_0^{\pi/2}(1-\sin^2\theta)\sin^{n-2}\theta\,d\theta$

$I_n = (n-1)\left[\int_0^{\pi/2}\sin^{n-2}\theta - \sin^n\theta\,d\theta\right]$

$I_n = (n-1)\left[I_{n-2} - I_n\right]$

$I_n = (n-1)I_{n-2} - (n-1)$

ie $nI_n = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(n:

(ii) $I_{10} = \frac{9}{10}I_8$

$= \frac{9}{10}\times\frac{7}{8}I_6$

$= \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}I_4$

$= \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}I_2$

$= \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}I_0$

$I_0 = \int_0^{\pi/2}1\,d\theta$

$= \pi/2$

$\therefore I_{10} = \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}$

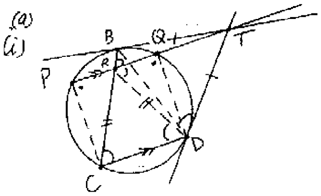
$= \frac{945\pi}{7680}$

$= \frac{63\pi}{512}$ //

Question 8:

(a)

(i)



(ii) Join B to D – see diagram.

$PQ \parallel CD$ ∴ $\angle BRT = \angle BCD$ (corresp. ∠'s in ∥ lines are equal)

but $\angle TDB = \angle DCB$ (angle between tangent TD and chord BD is equal to the ∠ in the alt. segment)

hence: $\angle BDT = \angle BRT$

(iii) Join DR – not necessary though

since $\angle BDT = \angle BRT$ then BRDT is a cyclic quadrilateral because the chord BT subtends equal angles at R & D – see diagram

(i.e. 2 ∠'s standing on the same chord are equal; in this case they are)

(iv) $BT = TD$ (two tangents drawn from the same ext. pt are =)

since BRDT is a cyclic quad. and since equal chords BT & TD subtend equal angles on the circumference then $\angle TRD = \angle BRT$ as well.

(v) $\angle DRT = \angle RDC$ (alt. ∠'s in parallel lines are equal)

$\angle BRT = \angle BCD$ from (ii)

$\angle BRT = \angle DRT$ (from iv)

∴ $\angle RDC = \angle BCD = \angle RCD$

∴ $\triangle RCD$ is isosceles. ($RC = RD$ ∴)

(vi) Join PC & QD

∴ PQDC is a cyclic quad now

$\angle RPC = 180 - \angle PCD$ (cointerior ∠'s in ∥ lines).

$\angle PQD = 180 - \angle PCD$ (opp. ∠'s of a cyclic quad add to 180)

∴ $\angle RPC = \angle PQD$

In $\triangle PRC$ & $\triangle RQD$

$RC = RD$ (see (v))

& $\angle RPC = \angle RQD$ (from above)

& $\angle QRD = \angle PRC$ (alt ∠'s in ∥ lines & isosceles △).

∴ $\triangle PRC \equiv \triangle QRD$ (AAS)

∴ $PR = RQ$ (corresp. sides in ≡ △'s are equal)

∴ BC bisects PQ as required

(b) (i) $\cos x = \cos(-x)$

$= \sin\left(\frac{\pi}{2} - (-x)\right)$

$= \sin\left(\frac{\pi}{2} + x\right)$

or: $\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos x$

$= \cos x$

(ii) $y = 3\sin x + 4\cos x$

$y = \sqrt{3^2+4^2}\,\sin(x+\alpha)$

$y = 5\sin(x+\alpha)$

By induction:

Prove the statement $\frac{d^n y}{dx^n} = 5\sin\left(x+\alpha+\frac{n\pi}{2}\right)$ is true for $n = 1$

$LHS = \frac{dy}{dx} = 5\cos(x+\alpha)$

$RHS = 5\sin\left(x+\alpha+\frac{\pi}{2}\right)$

$= 5\sin\left(\left(x+\frac{\pi}{2}\right)+\alpha\right)$

$= 5\sin\left((x+\alpha)+\frac{\pi}{2}\right)$

$= 5\cos(x+\alpha)$ from (i).

just replace x with $(x+\alpha)$

Assume the statement is true for $n = k$ [$1 \le k \le n$, $(k,n) \in \mathbb{Z}^+$]

RTP the statement is true for $n = k+1$.

Assume:

$\frac{d^k y}{dx^k} = 5\sin\left[(x+\alpha)+\frac{\pi}{2}k\right]$

RTP:

$\frac{d^{k+1} y}{dx^{k+1}} = 5\sin\left[(x+\alpha)+(k+1)\frac{\pi}{2}\right]$

From (2)

$LHS = \frac{d^{k+1} y}{dx^{k+1}}$

$= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\left[5\sin\left[(x+\alpha)+\frac{k\pi}{2}\right]\right]$ from

$= 5\cos\left[(x+\alpha)+\frac{k\pi}{2}\right]$

$= 5\sin\left[\frac{\pi}{2}+(x+\alpha)\right.$

[just replace $x \to x+$ in (i)]

$= 5\sin\left[\frac{\pi}{2}(k+1)+(x+\alpha)\right]$

The statement is true for $n = 1$. Whenever it is true for n it is also true for $n = k+1$ and for all positive integer values for $n \geq 1$.