Set By: IS

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2004
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question
- Write your **Student Number** and **Teacher's Initials** on the front cover of each writing booklet

NAME:____________________ TEACHER:____________________

Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^{1\cdot 5} \frac{2}{\sqrt{9-x^2}}\,dx$. **2**

(b) Find $\displaystyle\int \frac{1}{\sqrt{x^2-4x+5}}\,dx$, with the aid of the Table of Standard Integrals. **2**

(c) Find $\displaystyle\int \sin^2 x \cos^3 x\,dx$. **3**

(d) Using the substitution $x = 3\sec\theta$, evaluate $\displaystyle\int_3^6 \frac{1}{x^2\sqrt{x^2-9}}\,dx$. **4**

(e) (i) Find constants A, B, C such that $\dfrac{x^2+2}{x^2-x-2} \equiv A + \dfrac{Bx+C}{x^2-x-2}$. **1**

(ii) Hence find $\displaystyle\int \frac{x^2+2}{x^2-x-2}\,dx$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Simplify $\dfrac{1-i^3}{1-i}$. **2**

(b) Let $z = \dfrac{8-i}{2+i}$.

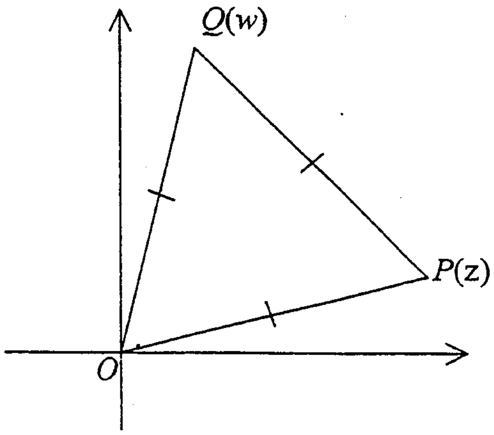
(i) Express z in the form $a+bi$ where a and b are real numbers. **2**

(ii) Hence, or otherwise, find $|z|$ and $\arg z$ (to 3 significant figures in the domain $-\pi < \theta \leq \pi$). **3**

(c) Sketch the region in the complex number plane where the inequalities $|z+1-2i| \leq 2$ and $\operatorname{Re}(z) \leq 0$ hold simultaneously. **2**

(d) Factorise $x^4 + 7x^2 - 18$ into the product of linear factors over the complex field. **2**

(e)



In the Argand diagram, OPQ is an equilateral triangle. P represents the complex number z and Q represents the complex number w.

(i) Explain why $w = z\,\text{cis}\dfrac{\pi}{3}$. **2**

(ii) Show that $w^3 + z^3 = 0$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = 2(x-1)(x-3)$.

Draw separate sketches of the following functions (at least one-third of a page), showing clearly the important features, including any intercepts on the axes, turning points, asymptotes, etc.

(i) $y = f(x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = 2 - f(x)$ **2**

(iv) $y = \sqrt{f(x)}$ **2**

(v) $y = \log_e f(x)$ **2**

(b) Let $I_n = \displaystyle\int_0^1 x^n e^{-x} dx.$

(i) Evaluate I_0. **1**

(ii) Prove that $I_n = n I_{n-1} - \dfrac{1}{e}$ for $n \geq 1$. **3**

(iii) Hence evaluate $\displaystyle\int_0^1 x^3 e^{-x}\, dx$. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\sqrt{9-12i}$. **3**

(b) $(2+i)$ is a zero of the polynomial $P(z)=z^3-z^2+az+b$, where a and b are real numbers. **4**

Find the other two zeros, and the values of a and b.

(c) α, β, γ are the roots of the equation $x^3-6x^2+12x-35=0$.

(i) Form a cubic equation whose roots are $\alpha-2, \beta-2, \gamma-2$. **2**

(ii) Hence, or otherwise, solve the equation $x^3-6x^2+12x-35=0$ over the complex field. **2**

(d) The roots of the equation $z^2+5z-2i=0$ are α and β. Without solving this equation, form the cubic equation whose roots are α, β and $(\alpha+\beta)$. **4**

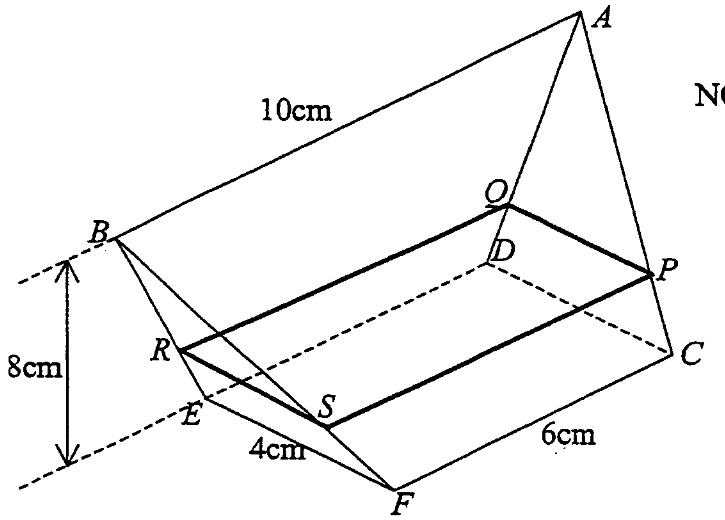
Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{16}=1$.

(i) Find its eccentricity. **1**

(ii) State the equations of the asymptotes. **1**

(b)



NOT TO SCALE

The diagram shows a wedge with the edge AB parallel to the horizontal rectangular base $CDEF$, and the plane $ABED$ is vertical. AB is 8 cm vertically above DE. $PQRS$ is a rectangular cross-section h cm above the base.

(i) Show that the area of the cross-section $PQRS$ is $\left(6+\dfrac{h}{2}\right)\left(4-\dfrac{h}{2}\right)$ cm². **2**

(ii) Hence find the volume of the wedge. **2**

(c) Consider the function $y=\dfrac{x^2-3x}{x+1}$.

(i) Find the equations of the two asymptotes. **2**

(ii) Find the coordinates of the stationary points and determine their nature. **5**

(iii) Sketch the graph of the function. **1**

(iv) For what values of k does the equation $\dfrac{x^2-3x}{x+1}=k$ have two real roots? **1**

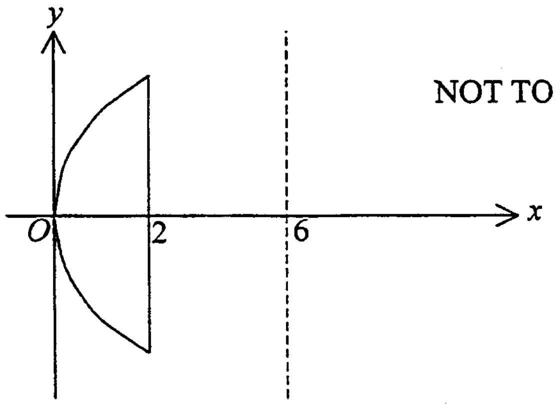
Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $f(x) = x\sqrt{4-x^2}$ is an odd function. 1

(ii) Hence, without finding any primitives, evaluate 2

$\int_{-2}^{2} \left(x\sqrt{4-x^2} - \sqrt{4-x^2}\right) dx$, giving reasons.

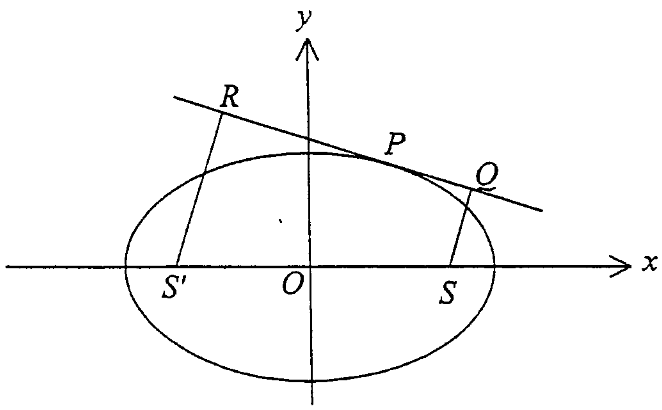
(b)



The region bounded by the parabola $y^2 = 4x$ and the line $x = 2$ is rotated about the line $x = 6$. 5

Using the method of cylindrical shells, find the volume of the solid formed.

(c)



(i) Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ is $(b\cos\theta)x + (a\sin\theta)y - ab = 0$. 3

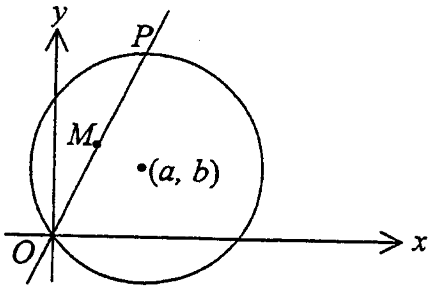
(ii) Q and R are the feet of the perpendiculars to the tangent from the foci S and S' respectively. 4

Prove that $SQ \times S'R = b^2$.

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the general solution of the inequality $\cos\theta \geq \frac{1}{2}$. 2

(b)



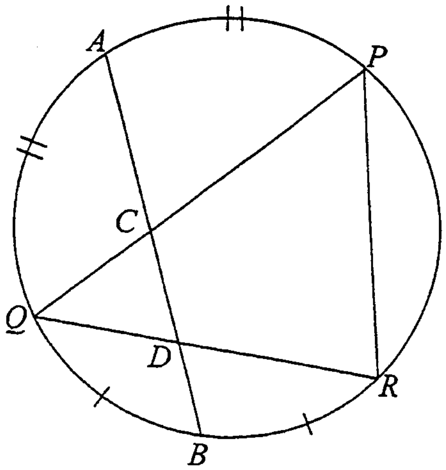
The diagram shows the graph of the circle $(x-a)^2+(y-b)^2=a^2+b^2$, which passes through the origin O. The line $y=mx$ cuts the circle at O and P.

(i) Show that the x coordinate of P is $\dfrac{2(a+bm)}{1+m^2}$. 2

(ii) Hence write down the coordinates of M, the midpoint of OP. 2

(iii) Hence show that the locus of M, as the gradient of OP varies, is a circle, and state its centre and radius. 4

(c)



A circle is drawn through the vertices of the triangle PQR. A is the midpoint of the arc PQ and B is the midpoint of the arc QR. The chord AB intersects PQ at C and QR at D.

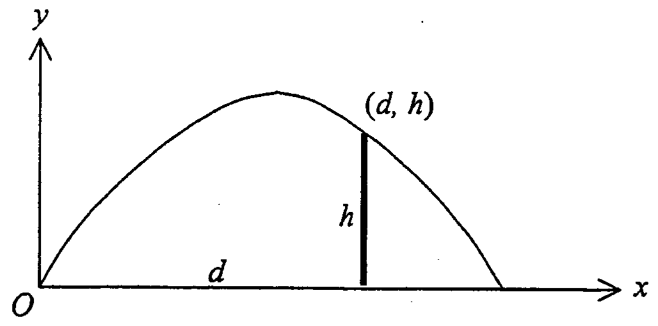
Copy or trace the diagram into your Writing Booklet.

(i) Explain why $\angle QPB = \angle BPR$. 1

(ii) Prove that $QC = QD$. 4

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A stone is projected from a point on the ground, and it just clears a fence d metres away. The height of the fence is h metres. The angle of projection to the horizontal is θ and the speed of projection is V m/s.
The displacement equations, measured from the point of projection, are:

$$x = V\cos\theta\, t \quad \text{and} \quad y = V\sin\theta\, t - \frac{1}{2}gt^2.$$

(i) Show that $V^2 = \dfrac{gd^{2}\sec^2\theta}{2(d\tan\theta - h)}$. **2**

(ii) Show that the maximum height reached is $\dfrac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ **3**

(iii) Show that the stone will just clear the fence at its highest point if $\tan\theta = \dfrac{2h}{d}$. **3**

(b) (i) Prove by mathematical induction that $\left(\sqrt{3}-1\right)^n = p_n + q_n\sqrt{3}$, where n is a positive integer and p_n and q_n are unique integers. **5**

(ii) Hence show that $p_n{}^2 - 3q_n{}^2 = (-2)^n$. **2**

End of paper

Suggested Solution (s) | **Comments**

QUESTION 1.

(a) $\displaystyle\int_0^{1.5} \frac{2}{\sqrt{9-x^2}} = 2\left[\sin^{-1}\frac{x}{3}\right]_0^{1.5}$ ✓

$= 2\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right]$

$= 2\left[\frac{\pi}{6} - 0\right]$

$= \frac{\pi}{3}$ ✓ ②

(b) $\displaystyle\int \frac{1}{\sqrt{x^2-4x+5}}\,dx = \int \frac{1}{\sqrt{(x-2)^2+1}}\,dx$ ✓

$= \log_e\left(x-2+\sqrt{(x-2)^2+1}\right) + c$

or $\log_e\left(x-2+\sqrt{x^2-4x+5}\right) + c$ ✓ ②

Comments: Ignore '+c' in marking.

(c) $\displaystyle\int \sin^2 x\cos^3 x\,dx$ $u = \sin x$, $du = \cos x\,dx$

$= \int \sin^2 x(1-\sin^2 x)\cos x\,dx$

$= \int u^2(1-u^2)\,du$ ✓

$= \int (u^2-u^4)\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c$ ✓

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$ ✓ ③

Comments: or equivalent without substitution

Ignore '+c'

(d) $\displaystyle\int_3^6 \frac{1}{x^2\sqrt{x^2-9}}\,dx$ $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$

$x=3,\ \theta=0$; $x=6,\ \theta=\frac{\pi}{3}$ ✓

$= \displaystyle\int_0^{\frac{\pi}{3}} \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$ ✓

$= \displaystyle\int_0^{\frac{\pi}{3}} \frac{3\sec\theta\tan\theta}{9\sec^2\theta\times 3\tan\theta}\,d\theta$

$= \frac{1}{9}\displaystyle\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$ ✓

$= \frac{1}{9}\left[\sin\theta\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{18}$ ✓ ④

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Suggested Solution (s) | **Comments**

QUESTION 1 (cont.)

(e) (i)

$$\begin{array}{r} 1 \\ x^2-x-2\overline{)\,x^2+0x+2} \\ \underline{x^2-x-2} \\ x+4 \end{array}$$

$\therefore \dfrac{x^2+2}{x^2-x-2} = 1 + \dfrac{x+4}{x^2-x-2}$

$A=1,\ B=1,\ C=4$ ①

Comments: OR

$\dfrac{x^2+2}{x^2-x-2} \equiv A + \dfrac{Bx+C}{x^2-x-2}$

$x^2+2 \equiv A(x^2-x-2)+Bx+C$

- equate coefficients

(ii) $\dfrac{x+4}{(x-2)(x+1)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+1}$

$x+4 \equiv A(x+1) + B(x-2)$

Sub. $x=-1$: $3 = -3B$

$\therefore B = -1$

Sub. $x=2$: $6 = 3A$

$A = 2$.

$\therefore \displaystyle\int \frac{x^2+2}{x^2-x-2} = \int\left(1+\frac{2}{x-2}-\frac{1}{x+1}\right)dx$

$= x + 2\ln(x-2) - \ln(x+1) + C$ ③

Comments: 1 mark - correct A, B.

2 marks for 3 correct primitives

Suggested Solution (s)	Comments

QUESTION 2

(a) $\dfrac{1-i^3}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}$ ✓ OR $\dfrac{(1-i)(1+i+i^2)}{1-i}$ ✓

$= \dfrac{1+2i+i^2}{1-i^2}$ $\qquad = 1+i-1$

$= \dfrac{2i}{2}$ $\qquad = i$ ✓

$= i$ ✓

(2)

(b) $z = \dfrac{8-i}{2+i}$

(i) $z = \dfrac{8-i}{2+i} \times \dfrac{2-i}{2-i}$

$= \dfrac{16-10i+i^2}{4-i^2}$ ✓

$= \dfrac{15-10i}{5}$

$= 3-2i$ ✓

(2)

(ii) $|z| = \sqrt{3^2+(-2)^2}$

$= \sqrt{13}$ ✓

$\tan\theta = -\dfrac{2}{3}$ ✓

$\theta = -0.588$ ✓

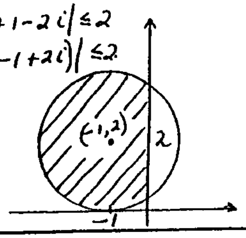
(3)

(c) $|z+1-2i| \le 2$

$|z-(-1+2i)| \le 2$

$Re(z) \le 0$

$x \le 0$



(2)

Comments: 1 - circle correct; 1- shading correct.

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Suggested Solution (s)	Comments

QUESTION 2 (cont)

(d) x^4+7x^2-18

$= (x^2+9)(x^2-2)$

$= (x^2-9i^2)(x^2-2)$

$= (x-3i)(x+3i)(x-\sqrt{2})(x+\sqrt{2})$

(2)

Comments: 1- for factors of x^2+9; 1- for factors of x^2-2.

(e)

(i) Since $OP = OQ$ and $\angle POQ = \frac{\pi}{3}$, multiplying z by $cis\,\frac{\pi}{3}$ doesn't change the modulus, as $|cis\,\frac{\pi}{3}| = 1$, but adds $\frac{\pi}{3}$ to the argument.

(2)

Comments: 1- modulus reason; 1- argument reason

(ii) $w = z\,cis\,\frac{\pi}{3}$

$w^3 = \left(z\,cis\,\frac{\pi}{3}\right)^3$

$= z^3\,cis\,\pi$ ✓

$= z^3(\cos\pi + i\sin\pi)$

$= z^3(-1+0i)$

$= -z^3$

$\therefore w^3 + z^3 = 0$. ✓

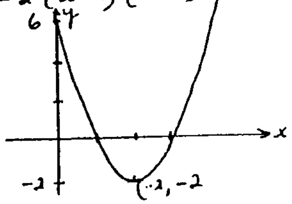
(2)

Suggested Solution (s)	Comments

QUESTION 3.

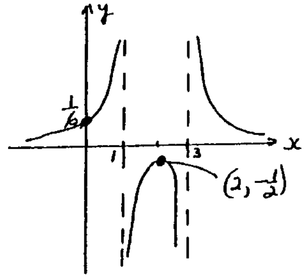
a) $f(x) = 2(x-1)(x-3)$

(i) $y = f(x)$



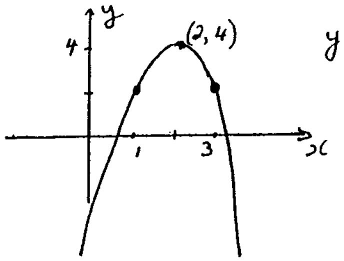
①

(ii) $y = \frac{1}{f(x)}$



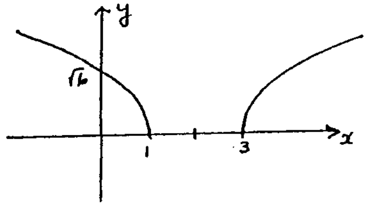
②

(iii) $y = 2 - f(x)$.



②

(iv) $y = \sqrt{f(x)}$

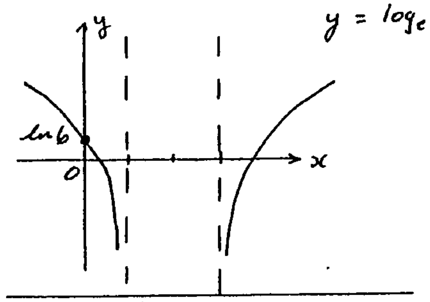


②

Suggested Solution (s)	Comments

QUESTION 3 (cont.)

(a)(v) $y = \log_e f(x)$.



②

(b) $I_n = \int_0^1 x^n e^{-x}\,dx$

(i) $I_0 = \int_0^1 e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^1$

$= (-e^{-1}) - (-e^0)$

$= 1 - \frac{1}{e}$ ①

(ii) $I_n = \int_0^1 x^n \frac{d}{dx}(-e^{-x})\,dx$

$= \left[-x^n e^{-x}\right]_0^1 - \int_0^1 (-e^{-x}) n x^{n-1}\,dx$ ✓

$= \left[-e^{-1} - 0\right]$ ✓ $+ n\int_0^1 x^{n-1} e^{-x}\,dx$ ✓

$= n I_{n-1} - \frac{1}{e}$ ③

(iii) $\int_0^1 x^3 e^{-x}\,dx = I_3$

$= 3I_2 - \frac{1}{e}$

$= 3\left[2I_1 - \frac{1}{e}\right] - \frac{1}{e}$ ✓

$= 6I_1 - \frac{4}{e}$

$= 6\left[1 \times I_0 - \frac{1}{e}\right] - \frac{4}{e}$

$= 6I_0 - \frac{10}{e}$

$= 6\left(1 - \frac{1}{e}\right) - \frac{10}{e}$

$= 6 - \frac{16}{e}$ ✓ ②

Suggested Solution (s)	Comments

QUESTION 4.

(a) Let $\sqrt{9-12i} = x + yi \quad (x, y \text{ real})$

$9 - 12i = x^2 - y^2 + 2xyi$

$x^2 - y^2 = 9 \quad , \quad 2xy = -12$ ✓

$x^2 - \frac{36}{x^2} = 9 \qquad y = -\frac{6}{x}$

$x^4 - 9x^2 - 36 = 0$

$(x^2 - 12)(x^2 + 3) = 0$

$x = \pm 2\sqrt{3} \quad (x \text{ real})$ ✓

If $x = 2\sqrt{3}$, $y = -\frac{6}{2\sqrt{3}} = -\sqrt{3}$

If $x = -2\sqrt{3}$, $y = -\frac{6}{-2\sqrt{3}} = \sqrt{3}$

$\therefore \sqrt{9-12i} = \pm(2\sqrt{3} - \sqrt{3}i)$ ✓ (3)

b) $P(z) = z^3 - z^2 + az + b$.

$(2+i)$ is a root $\therefore (2-i)$ is a root ✓

By sum of roots: $(2+i) + (2-i) + \alpha = 1$

$4 + \alpha = 1$ ✓

$\alpha = -3$

Roots are $2+i$, $2-i$, -3.

Factors are $(z+3)(z^2 - 4z + 5)$

$= z^3 - z^2 - 7z + 15$

$\therefore a = -7$ ✓, $b = 15$ ✓ (4)

Comments: Other methods could be: Subst. $z = 2+i$ & equate real & imag. parts, etc.

(c) $x^3 - 6x^2 + 12x - 35 = 0$.

i) Roots $\alpha - 2, \beta - 2, \gamma - 2$.

Let $y = x - 2$ i.e. $x = y + 2$

$(y+2)^3 - 6(y+2)^2 + 12(y+2) - 35 = 0$ ✓

$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 12y + 24 - 35 = 0$

$y^3 - 27 = 0$ ✓ (2)

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Suggested Solution (s)	Comments

QUESTION 4 (cont.)

(c)(ii) $(y-3)(y^2 + 3y + 9) = 0$

$y = 3 \text{ or } \frac{-3 \pm \sqrt{9-36}}{2}$ ✓

$y = 3 \text{ or } y = \frac{-3 + 3\sqrt{3}i}{2} \text{ or } y = \frac{-3 - 3\sqrt{3}i}{2}$

$x = 5, \ \frac{1}{2} + \frac{3\sqrt{3}}{2}i, \ \frac{1}{2} - \frac{3\sqrt{3}}{2}i$ ✓ (2)

Comments: OR 1-for $x = 5$ by any method (e.g. factor theorem) 1-other values correct → or equivalent → or equivalent.

(d) $z^2 + 5z - 2i = 0$.

Roots $\alpha, \beta, \alpha + \beta$.

$\Sigma\alpha$: $\alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta)$

$= 2 \times (-5)$

$= -10$ ✓

$\Sigma\alpha\beta$: $\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta)$

$= 3\alpha\beta + \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 + \alpha\beta$

$= (-5)^2 + (-2i)$

$= 25 - 2i$ ✓

$\alpha\beta\gamma$: $\alpha\beta(\alpha + \beta) = (-2i)(-5)$

$= 10i$ ✓

$\therefore$ Cubic equation with roots $\alpha, \beta, \alpha + \beta$:

$z^3 + 10z^2 + (25 - 2i)z - 10i = 0$ ✓ (4)

Comments: OR $\alpha + \beta = -5$. Equation is $(z+5)(z^2 + 5z - 2i) = 0$

$z^3 + 5z^2 - 2iz + 5z^2 + 25z - 10i = 0$

i.e. $z^3 + 10z^2 + (25 - 2i)z - 10i = 0$

Suggested Solution (s)	Comments

QUESTION 5.

a) $\frac{x^2}{4} - \frac{y^2}{16} = 1$

$a^2 = 4$
$b^2 = 16$
$a = 2$
$b = 4.$

(i) $b^2 = a^2(e^2 - 1)$

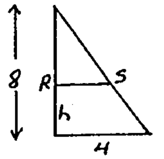
$16 = 4(e^2 - 1)$

$e = \sqrt{5}$ (1)

(ii) Asymptotes: $y = \pm \frac{b}{a} x$

$y = \pm 2x.$ (1)

b)(i) End view: $\frac{RS}{4} = \frac{8-h}{8}$



$RS = \frac{4}{8}(8-h)$

$= 4 - \frac{1}{2}h$ ✓

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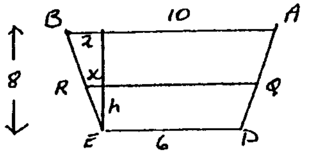


$\frac{x}{2} = \frac{h}{8}$

$x = \frac{h}{4}$

$RQ = 6 + 2 \times \frac{h}{4}$

$= 6 + \frac{h}{2}$ ✓

$\therefore$ Area $PQRS = (6 + \frac{h}{2})(4 - \frac{1}{2}h)$ (2)

(ii) $V = \int_0^8 (6 + \frac{h}{2})(4 - \frac{1}{2}h)\, dh$

$= \int_0^8 (24 - h - \frac{h^2}{4})\, dh$ ✓

$= [24h - \frac{1}{2}h^2 - \frac{1}{12}h^3]_0^8$

$= (24 \times 8 - \frac{1}{2} \times 64 - \frac{1}{12} \times 8^3)$

Volume $= 117\frac{1}{3}\ cm^3$ ✓ (2)

Suggested Solution (s)	Comments

QUESTION 5 (cont.)

(c) $y = \frac{x^2 - 3x}{x+1}$

(i)
$$\begin{array}{r} x - 4 \\ x+1 \overline{) x^2 - 3x + 0} \\ \underline{x^2 + x} \\ -4x + 0 \\ \underline{-4x - 4} \\ 4 \end{array}$$

$y = x - 4 + \frac{4}{x+1}$

Asymptotes : $x = -1$, ✓ $y = x - 4$ ✓ (2)

(ii) $\frac{dy}{dx} = \frac{(x+1)(2x-3) - (x^2-3x)(1)}{(x+1)^2}$ ✓

$= \frac{x^2 + 2x - 3}{(x+1)^2}$

$= \frac{(x+3)(x-1)}{(x+1)^2}$

stat. pts. at $x = -3$, $x = 1$. ✓

Stat. pts. at $(-3, -9)$, $(1, -1)$ ✓

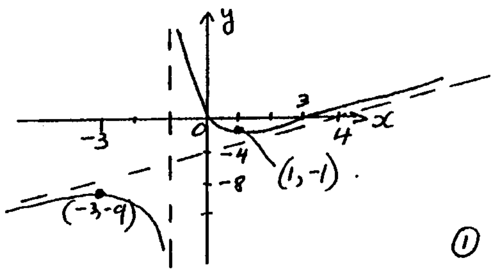
$f'(-3-\epsilon) = \frac{(-)(-)}{(+)} > 0$; $f'(-3+\epsilon) = \frac{(+)(-)}{(+)} < 0$

$\therefore$ Rel. maximum at $(-3, -9)$ ✓

$f'(1-\epsilon) = \frac{(+)(-)}{(+)} < 0$; $f'(1+\epsilon) = \frac{(+)(+)}{(+)} > 0$

$\therefore$ Rel. minimum at $(1, -1)$ ✓ (5)

(iii).



(1)

(iv). Two real roots for $k > -1$, $k < -9$. (1)

Suggested Solution (s)	Comments

QUESTION 6

a)(i) $f(x) = x\sqrt{4-x^2}$

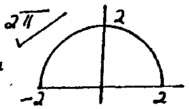
$f(-x) = (-x)\sqrt{4-(-x)^2} = -x\sqrt{4-x^2} = -f(x)$

$\therefore f(x)$ is odd function (1)

(ii) $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$ since $f(x)$ is odd

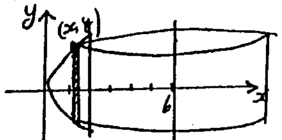
$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\times\pi\times 2^2 = 2\pi$

since it represents area of a semi-circle



$\therefore \int_{-2}^{2}\left(x\sqrt{4-x^2} - \sqrt{4-x^2}\right)dx = 0 - 2\pi = -2\pi$ (2)

b).



$y^2 = 4x$

$y = \pm 2\sqrt{x}$

Radius $= 6 - x$.

$\delta V \doteqdot 2\pi(6-x)2y\,\delta x$

$\doteqdot 4\pi(6-x)\,y\,\delta x$

$V = 4\pi\int_0^2 (6-x)\,y\,dx$

$= 4\pi\int_0^2 (6-x)\,2\sqrt{x}\,dx$

$= 4\pi\int_0^2 \left(12x^{\frac{1}{2}} - 2x^{3/2}\right)dx$

$= 4\pi\left[12\times\frac{2}{3}x^{3/2} - 2\times\frac{2}{5}x^{5/2}\right]_0^2$

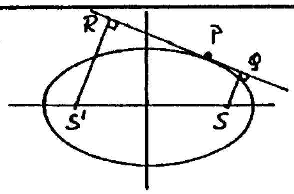
$= 4\pi\left[8\times 2\sqrt{2} - \frac{4}{5}\times 2^2\sqrt{2}\right]$

$= 4\pi\left[\frac{80\sqrt{2} - 16\sqrt{2}}{5}\right]$

Volume $= \frac{256\sqrt{2}\,\pi}{5}$ unit^3

Suggested Solution (s)	Comments

QUESTION 6 (cont.)



(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

At $P(a\cos\theta, b\sin\theta)$, $\frac{dy}{dx} = -\frac{b^2a\cos\theta}{a^2b\sin\theta}$

$= -\frac{b\cos\theta}{a\sin\theta}$

Tangent: $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$(a\sin\theta)y - ab\sin^2\theta = -(b\cos\theta)x + ab\cos^2\theta$

$(b\cos\theta)x + (a\sin\theta)y = ab(\cos^2\theta + \sin^2\theta)$

$(b\cos\theta)x + (a\sin\theta)y - ab = 0$. (3)

(ii) $S(ae, 0)$ $S'(-ae, 0)$

$SQ \times S'R = \left|\frac{(b\cos\theta)(ae) - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right| \times \left|\frac{(b\cos\theta)(-ae) - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$

$= \left|\frac{ab(e\cos\theta - 1)(-ab)(e\cos\theta + 1)}{b^2\cos^2\theta + a^2\sin^2\theta}\right|$

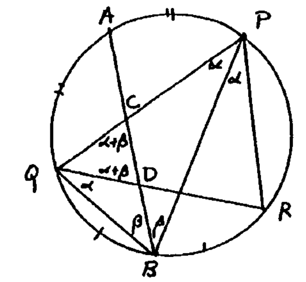
$= \left|\frac{a^2b^2(e^2\cos^2\theta - 1)}{a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta}\right|$

$= \left|\frac{b^2(e^2\cos^2\theta - 1)}{\cos^2\theta - e^2\cos^2\theta + \sin^2\theta}\right|$

$= \left|\frac{b^2(e^2\cos^2\theta - 1)}{1 - e^2\cos^2\theta}\right|$

$= b^2$ (4)

Suggested Solution (s)	Comments
QUESTION 7.	
(a) $\cos\theta \geq \frac{1}{2}$ $2n\pi - \frac{\pi}{3} \leq \theta \leq 2n\pi + \frac{\pi}{3}$ (2)	1 mark for part of the solution eg $\frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$
(b) $y = mx$; $(x-a)^2+(y-b)^2 = a^2+b^2$ (i) i.e. $x^2+y^2-2ax-2by=0$ Sub. $y=mx$: $x^2+m^2x^2-2ax-2bmx=0$ $x^2(1+m^2)-x(2a+2bm)=0$ ✓ $x=0$ or $x=\frac{2a+2bm}{1+m^2}$ ✓ x-coord of P is $\frac{2(a+bm)}{1+m^2}$ (2)	
(ii) y-coord. of P is $\frac{2m(a+bm)}{1+m^2}$ ✓ $M: \left(\frac{a+bm}{1+m^2}, \frac{m(a+bm)}{1+m^2}\right)$ ✓ (2)	
(iii) $x = \frac{a+bm}{1+m^2}$ $y = mx$ $\therefore m = \frac{y}{x}$ $x(1+m^2) = a+bm$ $x\left(1+\frac{y^2}{x^2}\right) = a+b\left(\frac{y}{x}\right)$ ✓ $x^2+y^2 = ax+by$ ✓ $\left(x^2-ax+\frac{a^2}{4}\right)+\left(y^2-by+\frac{b^2}{4}\right) = \frac{a^2}{4}+\frac{b^2}{4}$ $\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2 = \frac{a^2+b^2}{4}$ ✓ Locus is a circle, centre $\left(\frac{a}{2},\frac{b}{2}\right)$ radius $\frac{\sqrt{a^2+b^2}}{2}$ ✓ (4)	or equivalent.

Suggested Solution (s)	Comments
QUESTION 7 (cont).	
(c). 	
(i) $\angle QPB = \angle BPR$ because angles at the circumference standing on equal arcs are equal. (1)	Must mention "circumference"
(ii) Let $\angle QPB = \angle BPR = \alpha$ $\angle BQR = \angle BPR = \alpha$ (angles at circumference on same arc) ✓ $\angle ABQ = \angle ABP = \beta$ (angles at circumference on equal arcs) ✓ $\angle QDC = \angle QBD + \angle DQB$ (ext. angle ΔQDB) ✓ $= \beta + \alpha$ $\angle QCD = \angle CPB + \angle CBP$ (ext. angle ΔBCP) ✓ $= \alpha + \beta$ $\therefore \angle QDC = \angle QCD$ $\therefore QC = QD$. (4)	

Suggested Solution (s)	Comments
QUESTION 8	
a) $x = V\cos\theta\, t$, $y = V\sin\theta t - \frac{1}{2}gt^2$	
(i) Subst. $x = d$, $y = h$ and eliminate t.	
$d = V\cos\theta\, t$, $h = V\sin\theta t - \frac{1}{2}gt^2$	
$t = \frac{d}{V\cos\theta}$, $h = V\sin\theta \cdot \frac{d}{V\cos\theta} - \frac{g}{2} \times \frac{d^2}{V^2\cos^2\theta}$	
$h = d\tan\theta - \frac{gd^2}{2V^2\cos^2\theta}$ ✓	
$\frac{gd^2}{2V^2\cos^2\theta} = d\tan\theta - h$.	
$V^2 = \frac{gd^2}{2\cos^2\theta(d\tan\theta - h)}$	
$= \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$ ✓ (2)	
(ii) $\dot{y} = V\sin\theta - gt$	
$= 0$ when $t = \frac{V\sin\theta}{g}$ ✓	
Max. ht. $= V\sin\theta \times \frac{V\sin\theta}{g} - \frac{g}{2} \times \frac{V^2\sin^2\theta}{g^2}$	
$= \frac{V^2\sin^2\theta}{2g}$ ✓	
$= \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)} \times \frac{\sin^2\theta}{2g}$ (3)	
$= \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ ✓ since $\sec^2\theta\sin^2\theta = \frac{1}{\cos^2\theta}\sin^2\theta = \tan^2\theta$.	
(iii) $h = \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ ✓	OR Subst. $\tan\theta = \frac{2h}{d}$ into (iii) and show it equals 'h'.
$d^2\tan^2\theta = 4h(d\tan\theta - h)$	
$d^2\tan^2\theta - 4dh\tan\theta + 4h^2 = 0$	
$(d\tan\theta - 2h)^2 = 0$ ✓	Need description conclusion for third mark.
$d\tan\theta - 2h = 0$ (3)	
$\therefore \tan\theta = \frac{2h}{d}$ ✓	

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Suggested Solution (s)	Comments
QUESTION 8 (cont.)	
(b)(i) Prove $(\sqrt{3}-1)^n = p_n + q_n\sqrt{3}$; p_n, q_n integers	
When $n=1$, LHS $= (\sqrt{3}-1)^1 = -1 + \sqrt{3}$.	
$p_1 = -1$, $q_1 = 1$ are unique $\therefore$ true for $n=1$ ✓	
Assume it is true for $n=k$,	
i.e. assume $(\sqrt{3}-1)^k = p_k + q_k\sqrt{3}$ (p_k, q_k unique integers) ✓	
when $n = k+1$,	
LHS $= (\sqrt{3}-1)^{k+1}$	
$= (\sqrt{3}-1)(\sqrt{3}-1)^k$	
$= (\sqrt{3}-1)(p_k + q_k\sqrt{3})$ by assumption ✓	
$= p_k\sqrt{3} + 3q_k - p_k - q_k\sqrt{3}$	
$= (3q_k - p_k) + (p_k - q_k)\sqrt{3}$	
$= p_{k+1} + q_{k+1}\sqrt{3}$ ✓	Note: Mark for conclusion not awarded if second last mark not awarded
where $p_{k+1} = 3q_k - p_k$, $q_{k+1} = p_k - q_k$	
$\therefore$ If it is true for $n=k$, it is true for $n=k+1$	
Since it is true for $n=1$, it is true for $n=2,3,\ldots$ ✓ (5)	
(ii) $p_n^2 - 3q_n^2 = (3q_{n-1} - p_{n-1})^2 - 3(p_{n-1} - q_{n-1})^2$	OR Prove by induction (but only 2 marks).
$= 9q_{n-1}^2 - 6q_{n-1}p_{n-1} + p_{n-1}^2 - 3p_{n-1}^2 + 6p_{n-1}q_{n-1} - 3q_{n-1}^2$	
$= 6q_{n-1}^2 - 2p_{n-1}^2$	
$= -2(p_{n-1}^2 - 3q_{n-1}^2)$ ✓	
$= (-2)(-2)(p_{n-2}^2 - 3q_{n-2}^2)$	
$= (-2)(-2)\ldots\ldots(-2)(p_1^2 - 3q_1^2)$	
$= (-2)(-2)\ldots(-2)(1-3)$	
$= (-2)^n$ ✓ (2)	