Set By: EH

Teachers:

EH
RD
AJ

KNOX GRAMMAR SCHOOL
MATHEMATICS FACULTY

2005
TRIAL HSC EXAMINATION

Mathematics Extension 2 (Year 12)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question
- Write your **Board of Studies Student Number** and **Class Teacher's Initials** on the front cover of each of your writing booklets

Board of Studies Student Number: ________________

Class Teacher's Initials: ________________

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find $\int xe^{-2x} \; dx$. **2**

(b) Find $\int \sin^3 x \; dx$. **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{d\theta}{4+2\sin 2\theta}$, using the substitution, $t = \tan\theta$. **4**

(d) (i) Find real numbers A, B and C such that: **3**

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+4} \equiv \frac{1}{(x-2)(x^2+4)}$$

(ii) Evaluate $\int_{-2}^{0} \frac{8}{(x-2)(x^2+4)} dx$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z_1 = 2+3i$ and $z_2 = 4-5i$, find $z_1 \times \overline{z}_2$, in the form $x+yi$. **2**

(b) A unit circle has its centre at the origin O. The point z_1 moves on this circle and $z_2 = \dfrac{\sqrt{2}-3i}{z_1}$.

(i) Calculate $|\, z_1 z_2 \,|$. **1**

(ii) Hence find the Cartesian equation of the locus of z_2. **2**

(c) Sketch the locus of z in the Argand plane such that:

(i) $|\, z+2 \,| = |\, z-3i \,|$ **1**

(ii) $\arg(z-i) = \dfrac{3\pi}{4}$ **2**

(d) (i) Determine the Cartesian equation of the locus of z, such that: **2**

$$\text{Re}\left(z - \frac{1}{z}\right) = 0$$

(ii) Hence or otherwise, sketch the locus of z, in the Argand plane, if $\text{Re}\left(z - \dfrac{1}{z}\right) = 0$. **2**

Question 2 continues on the next page

Question 2 continued:

(e)

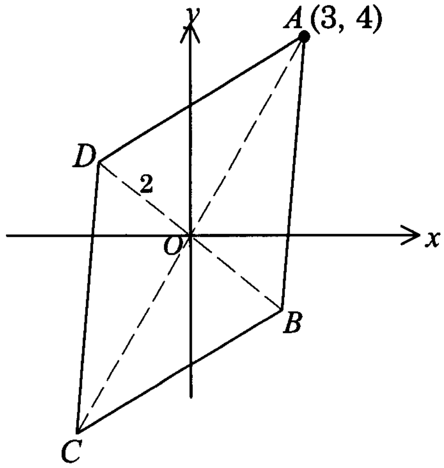


Diagram is not to scale

In the diagram, *ABCD* is a rhombus whose diagonals meet at *O*, the origin. *A* represents the complex number $3 + 4i$ and $OD = 2$ units.

Find the complex number represented by:

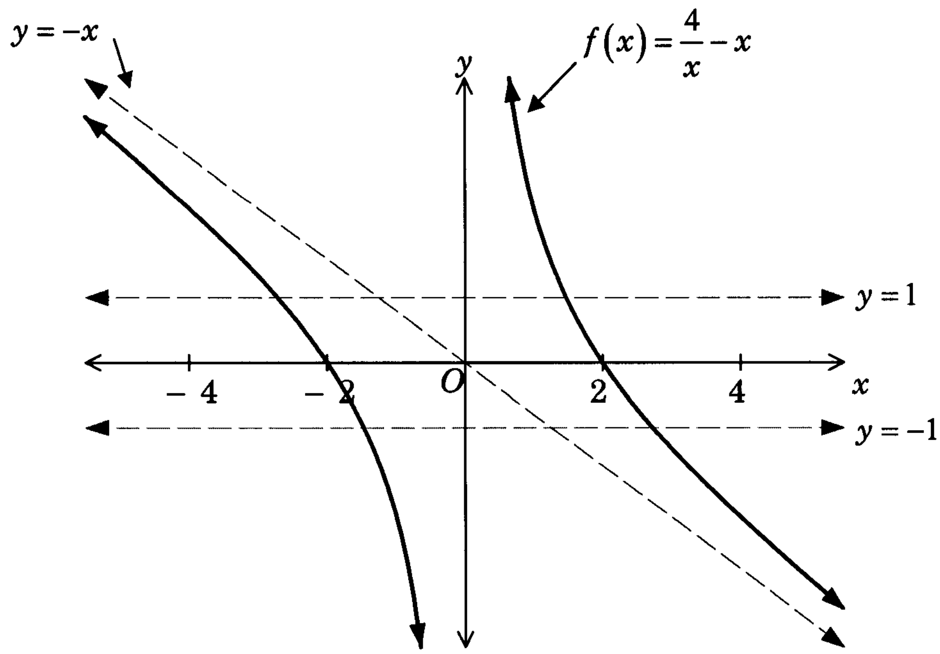
(i) *C* **1**

(ii) *D* **2**

Question 3 commences on the next page

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



For this part of Question 3, your answers are to be superimposed on the appropriate sketches on the separate answer sheet and then handed in with your writing booklet for this question.

The diagram above shows the graph of $y=f(x)$, where $f(x)=\frac{4}{x}-x$.

Sketch on separate number planes the graphs of:

(i) $y=\sqrt{f(x)}$ **2**

(ii) $y^2=f(x)$ **2**

(iii) $y=\frac{1}{f(x)}$ **2**

(iv) $y=f(|x|)$ **2**

(v) $y=e^{f(x)}$ **2**

(b) Evaluate $\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2-4}}{x}\,dx$, using the substitution $x=\frac{2}{5}\sec\theta$. **5**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The equation $x^3 - 3x + 2 = 0$ has roots α, β and γ.

(i) Form the cubic polynomial equation with roots: **1**

$$\frac{1}{\alpha}, \frac{1}{\beta} \text{ and } \frac{1}{\gamma}$$

(ii) Form the cubic polynomial equation with roots: **3**

$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma} \text{ and } \frac{\gamma}{\alpha\beta}$$

(b) (i) Prove that if a polynomial $P(x)$ has a root α of multiplicity r then $P'(x)$ has a root α of multiplicity $(r-1)$. **2**

(**Hint**: Start with $P(x) = (x-\alpha)^r Q(x)$)

(ii) Given $x = 1$ is a double root of the equation $x^4 - 5x^3 + 16x^2 - 21x + 9 = 0$, and using the result of b(i), or otherwise, find the other roots. **3**

(c) Given $z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$ and $w = \sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

(i) Express $\frac{w}{z}$ in modulus argument form. **1**

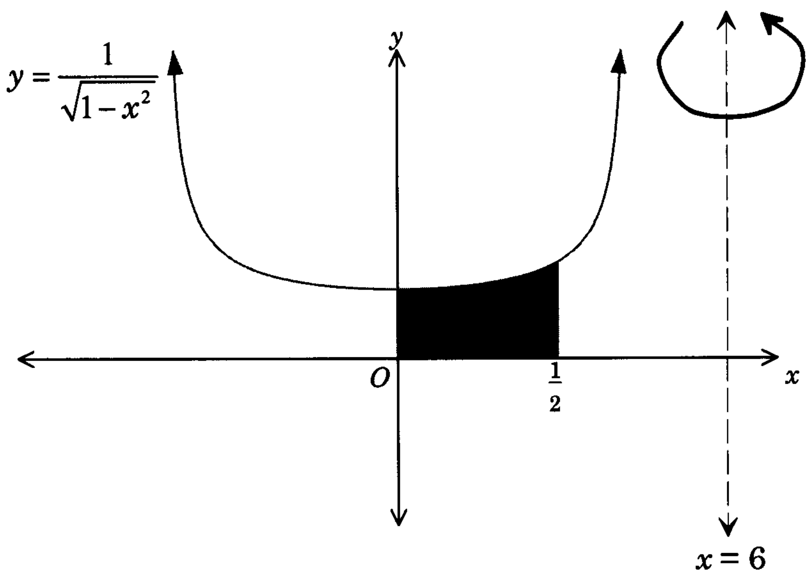
(ii) Use Mathematical Induction to prove for positive integers n that: **3**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(iii) Hence, using the result in c(ii), find the value of $\left(\frac{w}{z}\right)^{12}$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The shaded region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$, the coordinate axes and the line $x = \frac{1}{2}$ is rotated through one complete revolution about the line $x = 6$. **4**

Use the method of cylindrical shells to find the volume of the solid of revolution formed in cubic units.

(b) Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \; dx$.

(i) Show that if n is a positive integer greater than one, then: **2**

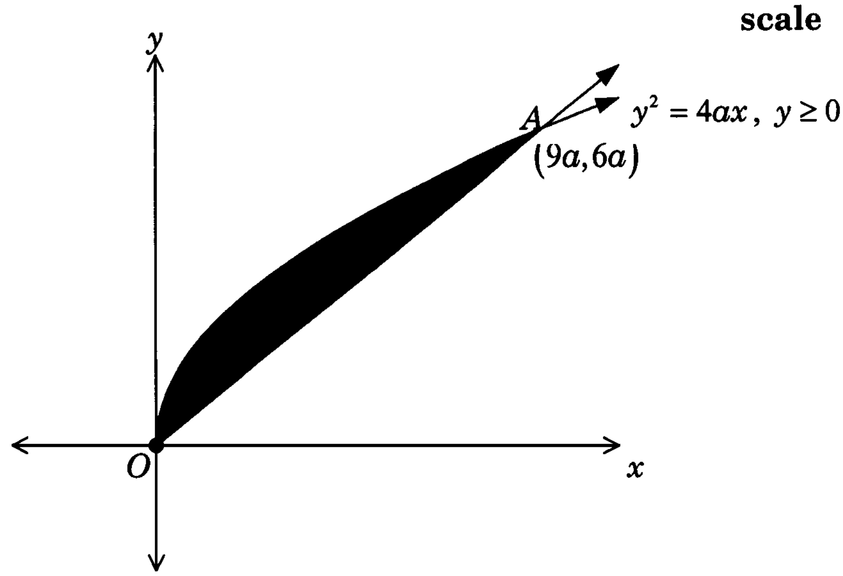
$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

(ii) Evaluate I_4. **2**

Question 5 continues on the next page

Question 5 continued:

(c) **Diagram is not to scale**



The base of a certain solid is the area bounded by the parabola $y^2 = 4ax$ (for $y \geq 0, a \geq 0$) and the chord joining $(0, 0)$ and $A(9a, 6a)$. Cross-sections of this solid, determined by planes taken perpendicular to the x-axis, are semicircles with the diameter completely in the base of the solid. **4**

By using the method of slicing, find the total volume of the solid formed.

(d) (i) Sketch on the same number plane the graphs of $y = |x| - 2$ and $y = 4 + 3x - x^2$. **1**

(ii) Hence or otherwise solve $\dfrac{|x| - 2}{4 + 3x - x^2} > 0$. **2**

Question 6 commences on the next page

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Solve for x if $\cos x = \sin\left(\frac{x}{2}\right)$. **3**

(b)

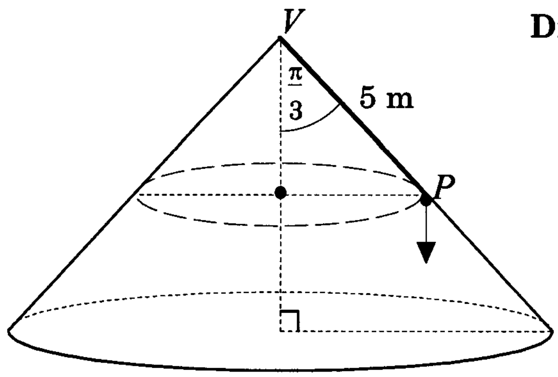


Diagram is not to scale

A circular cone of semi-vertical angle $\frac{\pi}{3}$ is fixed with its vertex upwards as shown. A particle P of mass m kg is attached to the vertex at V by a light inextensible string of length 5 metres. The particle P rotates with uniform angular velocity ω rad/sec in a horizontal circle whose centre is vertically below V, on the outside surface of the cone and in contact with it. Let T be the tension in the string and N the normal reaction force at P.

(i) Draw a diagram showing all the forces acting on the particle. **1**

(ii) Find the tension T in the string and the normal force N on P in Newtons. Leave your answers in terms of m, g and ω. **3**

(iii) Show that for the particle to remain in uniform circular motion on the surface of the cone, then $\omega^2 < \frac{2g}{5}$, where g is the acceleration due to gravity. **2**

Question 6 continues on the next page

Question 6 continued:

(c)

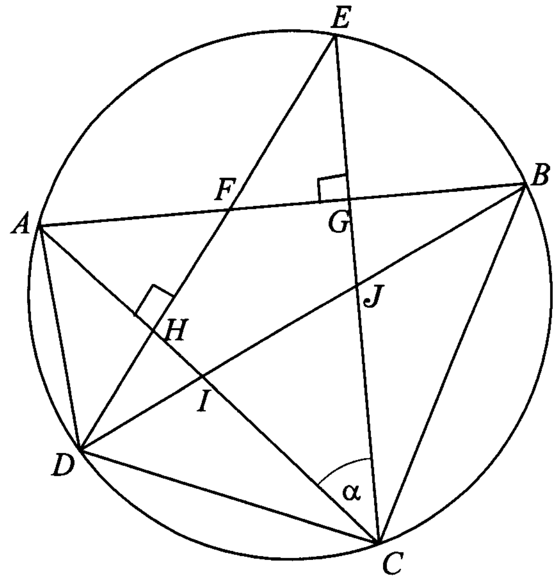


Diagram is not to scale

For this part of Question 6, your answers are to be placed on the answer sheet supplied and then handed in with your writing booklet for this question.

In the diagram, EC and ED are perpendicular to BA and AC at G and H respectively. The chords AC and BD meet at I. Let $\angle ECA = \alpha$.

(i) Prove that ΔBCD is isosceles. **2**

(ii) Prove that $\Delta CID \,|||\, \Delta CDA$. **2**

(iii) Given that $\Delta CIB \,|||\, \Delta CBA$ and $AB + AD = 2BC$. **2**

Prove that $CI = \dfrac{BD}{2}$.

Question 7 commences on the next page

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A projectile of unit mass is moving through air and experiences a resistance force R proportional to the square of its speed v. That is, $R = kv^2$, where k is a positive constant. *In this question, regard the direction of motion as positive.*

(i) Suppose the projectile is fired vertically upwards from the ground with an initial speed of u metres per second. Prove that the maximum height H reached by the projectile, where g is the acceleration due to gravity, is given by: **3**

$$H = \frac{1}{2k}\log_e\left(1 + \frac{ku^2}{g}\right)$$

(ii) Prove that the time T taken to reach this maximum height is given by: **2**

$$T = \frac{1}{\sqrt{gk}}\tan^{-1}\left(\frac{u\sqrt{k}}{\sqrt{g}}\right)$$

(b) (i) Differentiate with respect to x the function $h(x) = \frac{\log_{10} x}{x}$. **2**

(ii) Given that the only stationary point of $h(x)$ is a maximum, deduce, without calculating any numerical values, $e^\pi > \pi^e$. **3**

Question 7 continues on the next page

Question 7 continued:

(c) (i) A vehicle of mass m (in kg) is moving with speed v (in m/s) around a curve of radius r (in metres) banked at angle α with the horizontal. The normal reaction between the road and the vehicle is N, the friction (taken to be up the slope) is F_r, and the acceleration due to gravity is g (in m/s²). **2**

Draw a diagram that represents the forces on the vehicle.

By resolving forces parallel and perpendicular to the road, show that:

$$F_r = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$$

(ii) A train is travelling around a curve of radius 3000 metres at a speed of 180 km/h. The width of the rails is 1.5 metres. **3**

Taking the acceleration due to gravity to be 9.8 m/s², find how much higher than the inner rail must the outer rail be, in order for lateral thrust (F_r) on the rails to be avoided? Give your answer to the nearest centimetre.

Question 8 commences on the next page

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find all the roots of $z^5 - 1 = 0$ and then show that these roots can be represented as $1, \omega, \omega^2, \omega^3$ and ω^4 where $0 < \arg \omega < \frac{\pi}{2}$. **4**

(b) Prove that $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$. **2**

(c) Show that $(1-\omega)(1-\omega^4) = 2 - 2\cos\frac{2\pi}{5}$. **2**

(d) Hence or otherwise, show that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$. **3**

(e) Suppose P_0, P_1, P_2, P_3, P_4 are the corresponding points of $1, \omega, \omega^2, \omega^3$ and ω^4 in the Argand plane.

(i) Show that $\left|\overrightarrow{P_0P_1}\right| = 2\sin\frac{\pi}{5}$. **2**

(ii) Hence, or otherwise deduce that $\left|\overrightarrow{P_0P_1}\right| \times \left|\overrightarrow{P_0P_2}\right| \times \left|\overrightarrow{P_0P_3}\right| \times \left|\overrightarrow{P_0P_4}\right| = 5$. **2**

End of Paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \,, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

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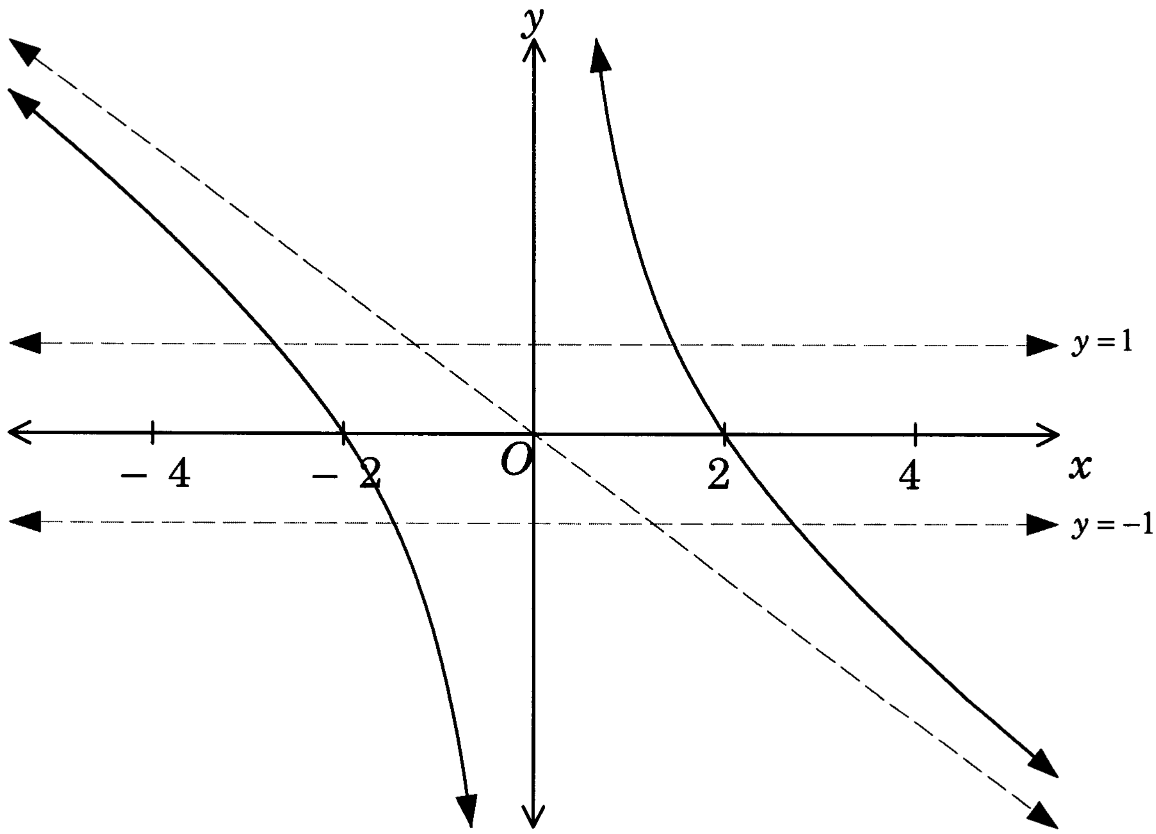
Knox Grammar School
Year 12
Mathematics Extension 2
TRIAL HSC - 2005
ANSWER SHEET FOR QUESTION 3(a)
(10 Marks)

Hand in with your writing booklet for Question 3

(i) $y=\sqrt{f(x)}$ **2**



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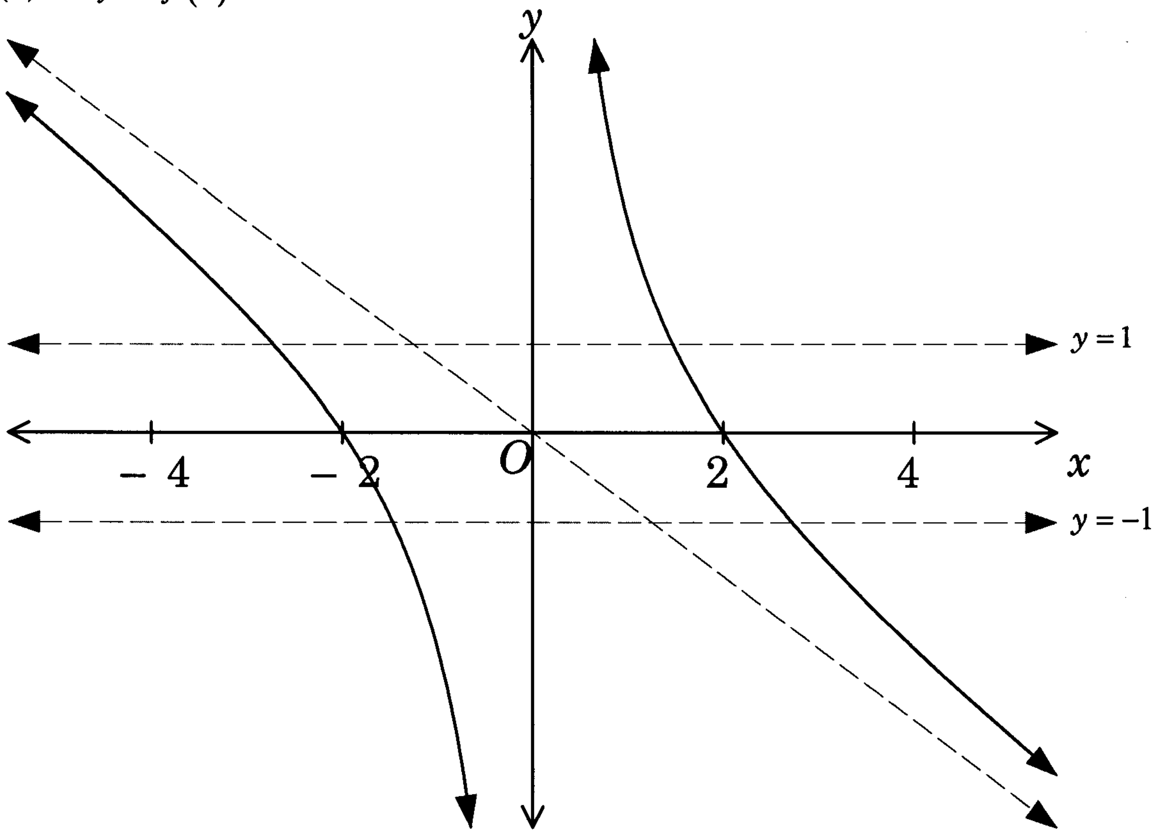
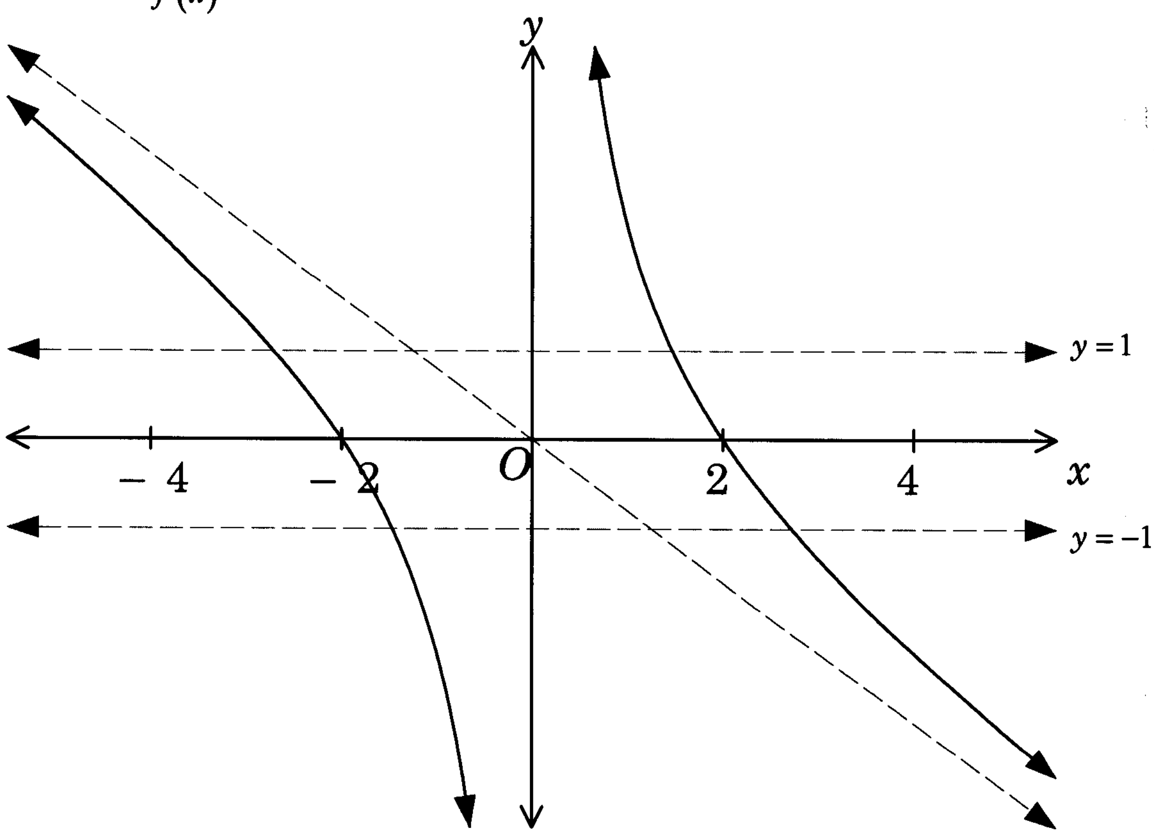
ANSWER SHEET FOR QUESTION 3(a)

(ii) $y^2 = f(x)$ **2**



(iii) $y = \dfrac{1}{f(x)}$ **2**



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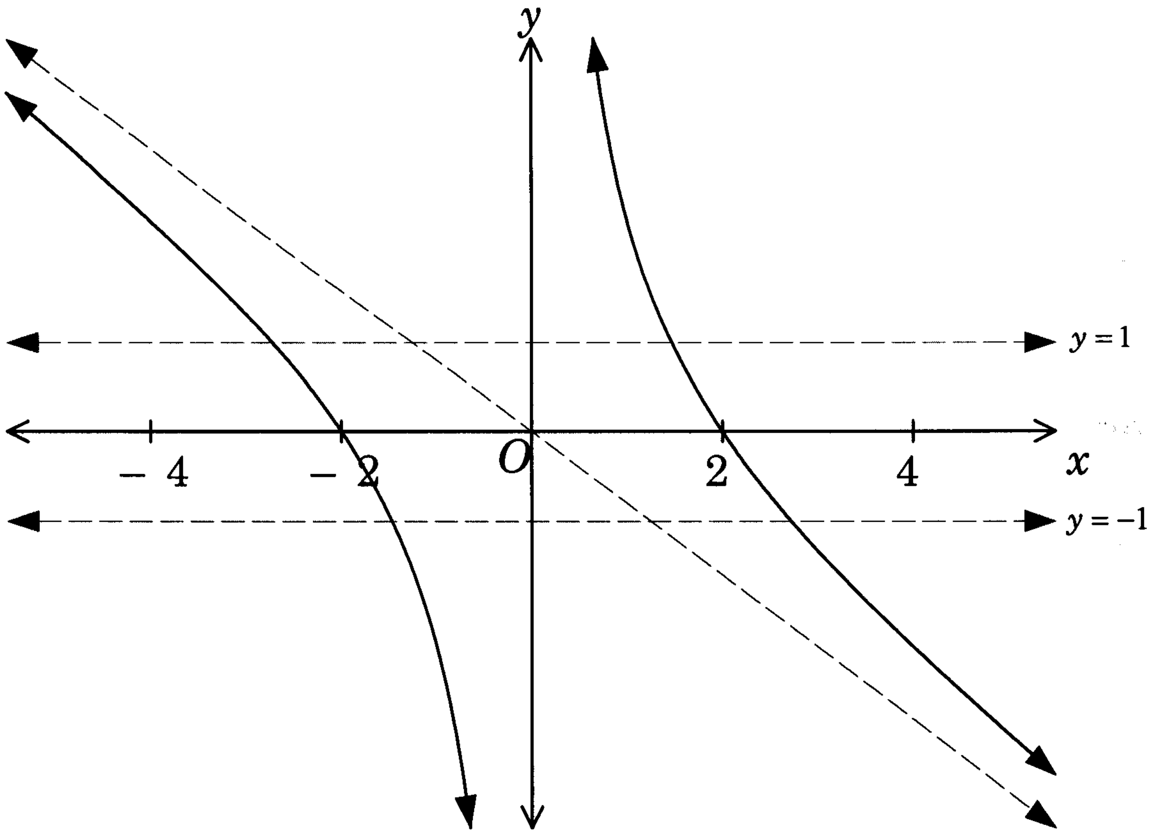
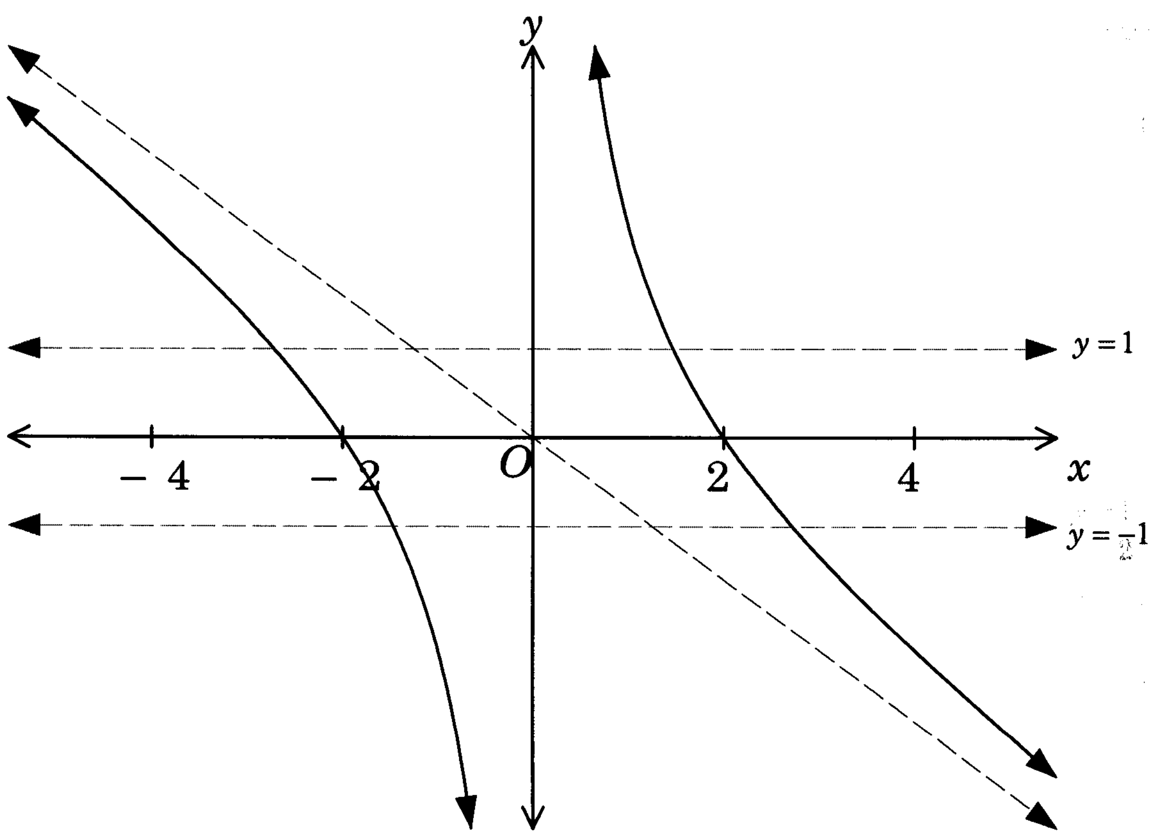
ANSWER SHEET FOR QUESTION 3(a)

(iv) $y = f(|x|)$ **2**



(v) $y = e^{f(x)}$ **2**



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ANSWER SHEET FOR QUESTION 6(c)
(6 Marks)

Hand in with your writing booklet for Question 6

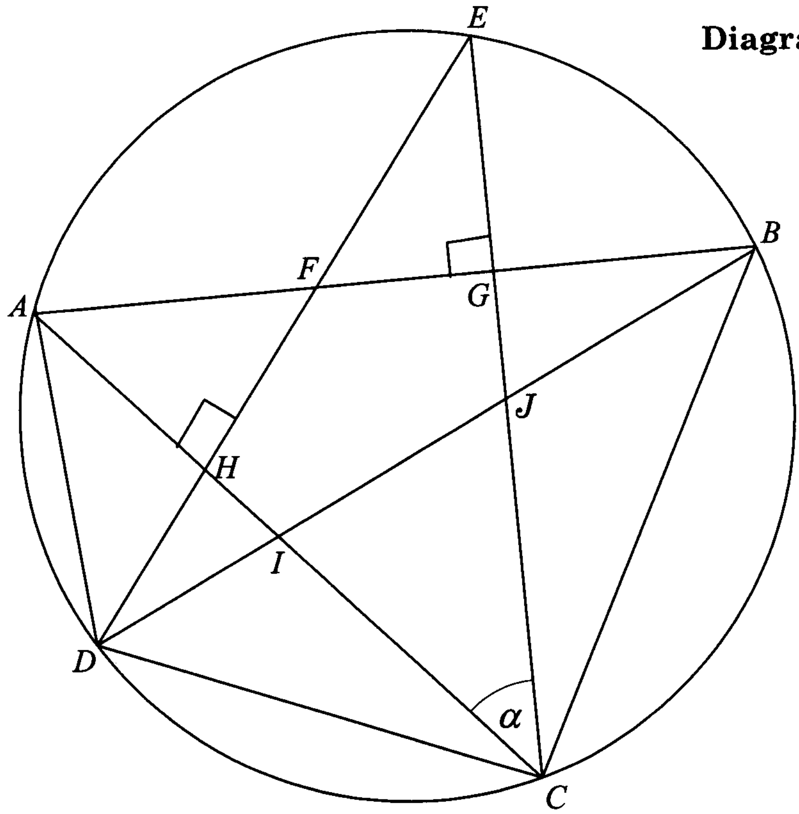

Diagram is not to scale

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(i) Prove that ΔBCD is isosceles. **2**

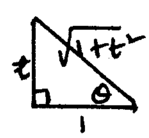
(ii) Prove that $\Delta CID \,|||\, \Delta CDA$. **2**

(iii) Given that $\Delta CIB \,|||\, \Delta CBA$ and $AB + AD = 2BC$. **2**

Prove that $CI = \dfrac{BD}{2}$.

MARKERS: *Q1: RD *Q3: EH Q5: AJ *Q7: RD
*Q2: RD Q4: AJ *Q6: EH *Q8: EH.

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments
QUESTION 1 : (15 MARKS)	
(a) $\int x e^{-2x}\,dx$	
using Integration by Parts: let $u = x$, $dv = e^{-2x}$; $du = 1$, $v = -\frac{1}{2}e^{-2x}$	✓ or implied
$\therefore \int x e^{-2x}\,dx = uv - \int v\,du$	
$= -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$ $= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$	✓
(b) $\int \sin^3 x\,dx$	
$= \int \sin^2 x \sin x\,dx$ $= \int \sin x - \sin x\cos^2 x\,dx$	✓ method
$= -\cos x + \frac{\cos^3 x}{3} + c$	✓
(c) let $t = \tan\theta$ 	
$\therefore \frac{dt}{d\theta} = \sec^2\theta$	
ie $\frac{d\theta}{dt} = \frac{1}{\tan^2\theta + 1}$ $\frac{d\theta}{dt} = \frac{1}{t^2+1}$	✓
when $\theta = 0$, $t = 0$; $\theta = \frac{\pi}{4}$, $t = 1$	✓
$\therefore \int_0^{\frac{\pi}{4}} \frac{d\theta}{4 + 2\sin 2\theta}$	
$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{4 + 4\sin\theta\cos\theta}$	
$= \int_0^1 \frac{\frac{1}{1+t^2}}{4 + 4\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)}\,dt$	✓
$= \frac{1}{4}\int_0^1 \frac{1}{t^2+t+1}\,dt$	
$= \frac{1}{4}\int_0^1 \frac{1}{(t+\frac{1}{2})^2 + \frac{3}{4}}\,dt$	
$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\Big]_0^1$	
$= \frac{1}{2\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$	
$= \frac{\pi\sqrt{3}}{36}$ $\left(\text{or } \frac{\pi}{12\sqrt{3}}\right)$.	✓
(d) (i) We want: $A(x^2+4) + (Bx+C)(x-2) \equiv 1$	
when $x = 2$: $8A = 1$ $\therefore A = \frac{1}{8}$	
when $x = 0$: $4A - 2C = 1$ $\therefore 2C = -\frac{1}{2}$, $C = -\frac{1}{4}$	
when $x = 1$: $5A + (B+C)(-1) = 1$ $\therefore \frac{5}{8} - B + \frac{1}{4} = 1$	
$\therefore B = -\frac{1}{8}$	
$\therefore A = \frac{1}{8}$, $B = -\frac{1}{8}$, $C = -\frac{1}{4}$.	✓✓✓

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q1 ctd:

(d)(ii) $\displaystyle\int_{-2}^{0} \frac{8}{(x-2)(x^2+4)}\,dx$

$= 8\displaystyle\int_{-2}^{0} \frac{\frac{1}{8}}{x-2} - \frac{\left(\frac{1}{8}x+\frac{1}{4}\right)}{x^2+4}\,dx$ ✓

$= \displaystyle\int_{-2}^{0} \frac{1}{x-2} - \frac{x+2}{x^2+4}\,dx$

$= \displaystyle\int_{-2}^{0} \frac{1}{x-2} - \frac{x}{x^2+4} - \frac{2}{x^2+4}\,dx$ ✓

$= \left[\ln|x-2| - \frac{1}{2}\ln(x^2+4) - \tan^{-1}\frac{x}{2}\right]_{-2}^{0}$

$= \log_e\left|\frac{x-2}{\sqrt{x^2+4}}\right| - \tan^{-1}\frac{x}{2}\Big]_{-2}^{0}$ ✓

$= \ln|(-1)| - 0 - \ln\left|\frac{-4}{2\sqrt{2}}\right| - \frac{\pi}{4}$

$= -\ln\sqrt{2} - \frac{\pi}{4}$

$= -\frac{1}{2}\ln 2 - \frac{\pi}{4}$. ✓

QUESTION 2: (15 MARKS)

(a) $z_1 = 2+3i$

$z_2 = 4-5i$

$\bar{z}_2 = 4+5i$

$\therefore z_1 \times \bar{z}_2 = (2+3i)(4+5i)$

$= 8+10i+12i-15$ ✓

$= -7+22i$ ✓

(b)(i) $|z_1| = 1$ (given)

$z_2 = \dfrac{\sqrt{2}-3i}{z_1}$ $(z_1 \neq 0)$

$\therefore z_1 z_2 = \sqrt{2}-3i$

ie $|z_1 z_2| = |\sqrt{2}-3i|$

$|z_1||z_2| = \sqrt{2+9} = \sqrt{11}$ ✓

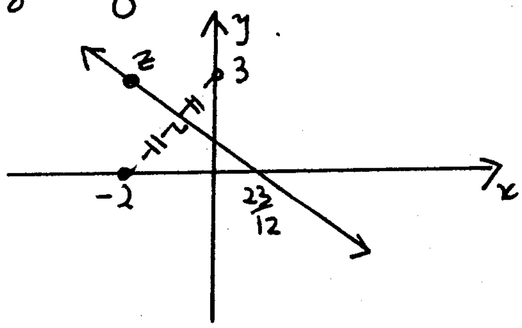
(ii) $\therefore |z_2| = \sqrt{11}$, NB: $|z_1| = 1$

$\therefore$ The locus of z_2 is $x^2+y^2 = 11$. ✓✓

(c)(i) $|z+2| = |z-3i|$

ie $|z-(-2)| = |z-3i|$

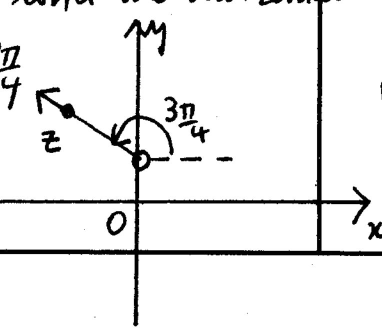
$\therefore z$ lies on the $\perp$ bisector joining $(-2,0)$ and $(0,3)$



✓

(ii) This is the half-ray excluding $(0,1)$ making an angle of $\frac{3\pi}{4}$ with the horizontal.

$\arg(z-i) = \frac{3\pi}{4}$



✓✓

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q2 ctd:

(d)(i) There are several ways to do this problems:

method 1: Let $z = r \operatorname{cis}\theta$

$\therefore \frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\theta)$

$\therefore z - \frac{1}{z} = r\operatorname{cis}\theta - \frac{1}{r}\operatorname{cis}(-\theta)$

$= r\cos\theta - \frac{1}{r}\cos\theta + i r\sin\theta + \frac{1}{r}\sin\theta\, i$

we want $\operatorname{Re}(z - \frac{1}{z}) = 0$

ie $r\cos\theta - \frac{1}{r}\cos\theta = 0$

ie $\cos\theta\,(r - \frac{1}{r}) = 0$

$\therefore$ either $\cos\theta = 0$, $r \neq 0$

ie $\theta = \pm\frac{\pi}{2}$

& $r - \frac{1}{r} = 0$

ie $r^2 = 1$

$\therefore r = \pm 1$

Since $r = |z|$

The the locus of z is a unit circle, centred at $(0,0)$ and on the y-axis, excluding $(0,0)$

ie $x^2+y^2=1$ & $x=0$ $(x,y)\neq(0,0)$

method 2: Let $z = x+iy$

$\therefore \operatorname{Re}\left(x+iy - \frac{1}{x+iy}\right) = 0$

ie $\operatorname{Re}\left[x+iy - \frac{x-iy}{x^2+y^2}\right] = 0$

ie $\operatorname{Re}\left[\frac{x(x^2+y^2) - x + iy(x^2+y^2)+y}{x^2+y^2}\right] = 0$

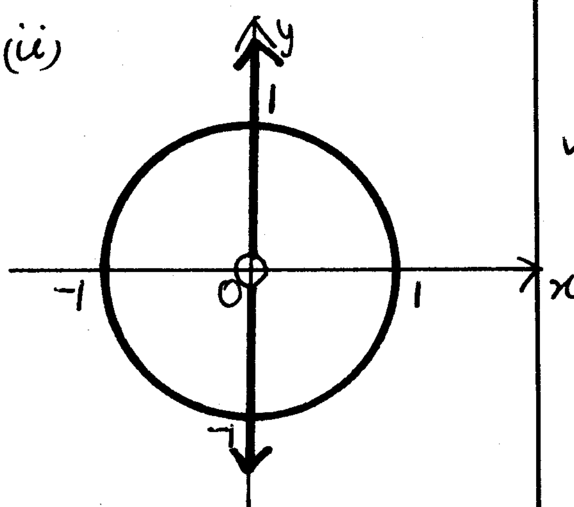
ie $x(x^2+y^2) - x = 0$

$x(x^2+y^2-1) = 0$

$\therefore x = 0$ or $x^2+y^2=1$ ✓✓

except when $x+iy=0$.

(ii)



✓✓

(e) (i) $\vec{OC} = c = (-3,-4)$

$\therefore \vec{OC} = -3-4i$ ✓

(ii) Since the diagonals of a Rhombus bisect each other at right-angles, then we first seek a constant k such that:

$\vec{OD} = k\,i\,\vec{OA}$

ie $\vec{OD} = k(3i-4)$

Since $|\vec{OD}| = 2$

we want $|k(3i-4)| = 2$

ie $9k^2 + 16k^2 = 4$

ie $k = 2/5$ $(k>0)$ - can you see why? $\Rightarrow \vec{OB}$.

$\therefore \vec{OD} = -\frac{8}{5} + \frac{6}{5}i$ ✓✓

Board of Studies Student Number: ________________

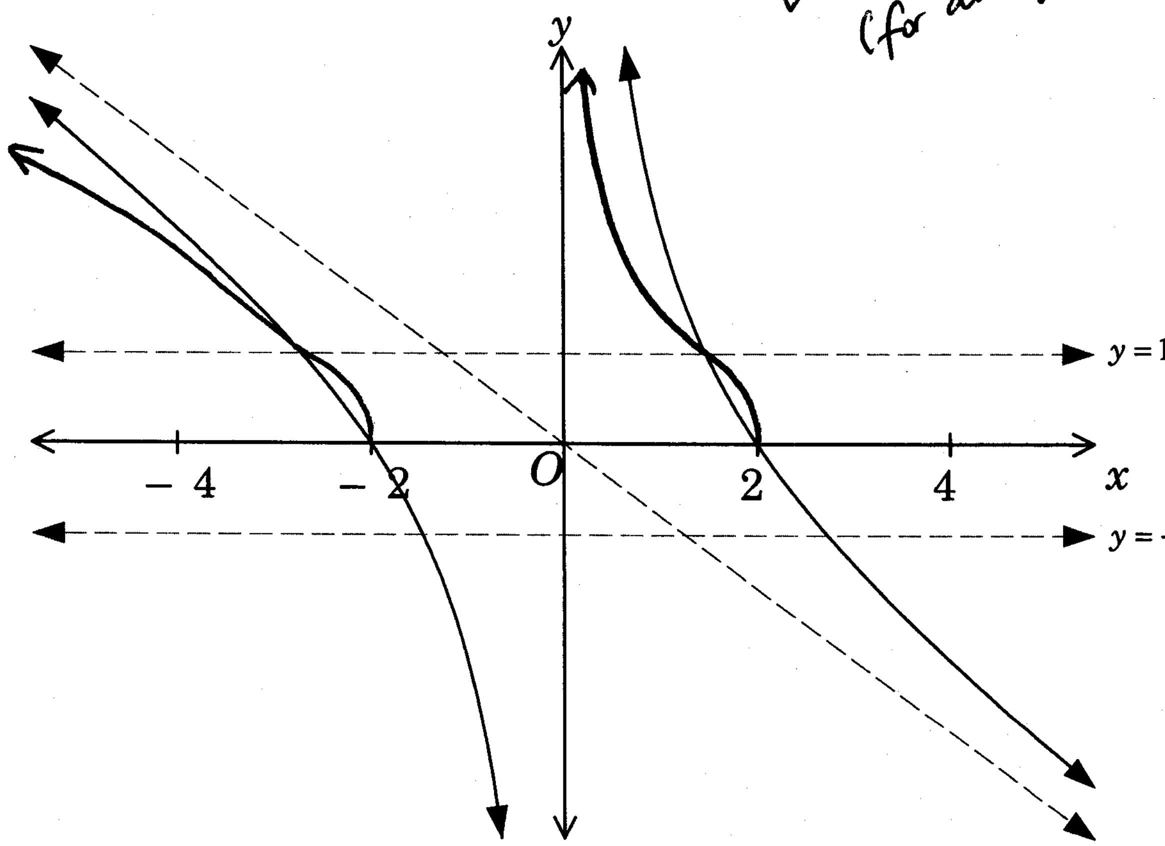
Class Teacher's Initials: ________________

Knox Grammar School
Year 12
Mathematics Extension 2
TRIAL HSC - 2005
ANSWER SHEET FOR QUESTION 3(a)
(10 Marks)

Hand in with your writing booklet for Question 3

✓ correct shape.
✓ correct position.
(for all graphs).

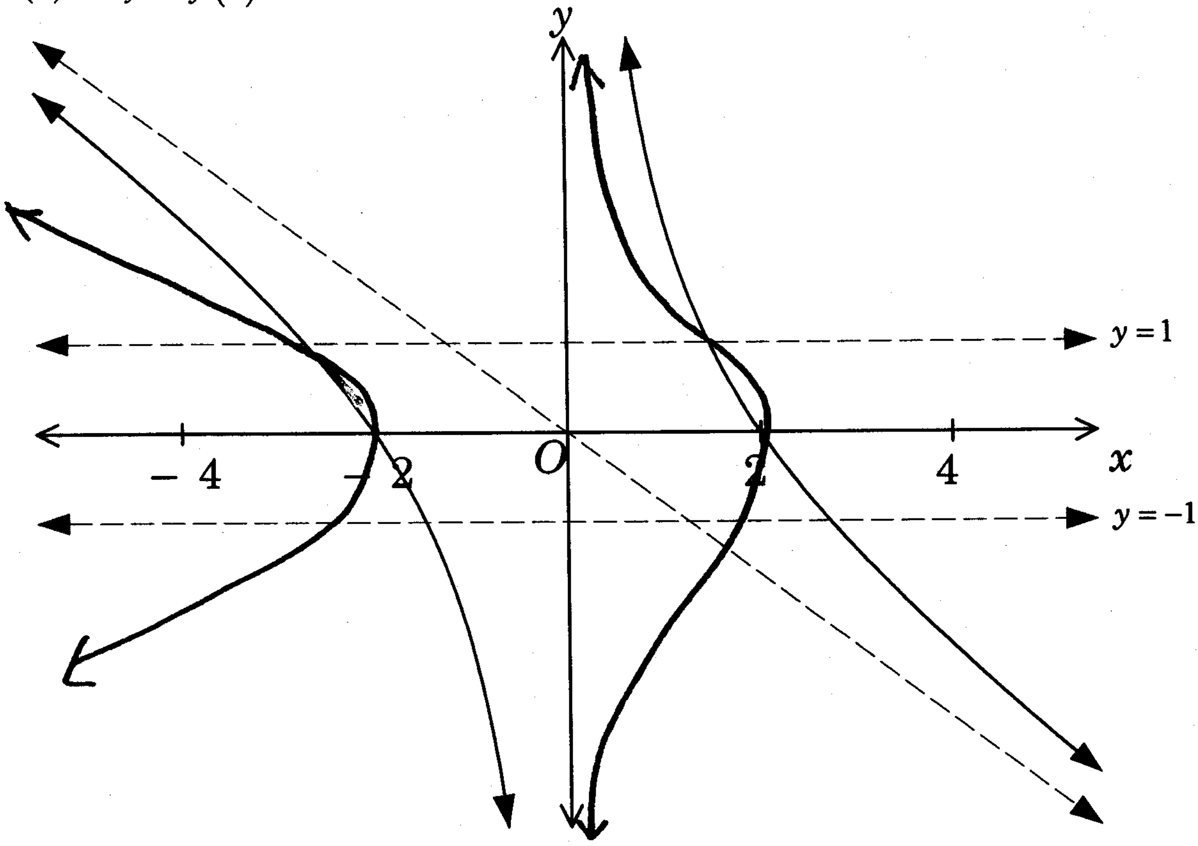
(i) $y = \sqrt{f(x)}$ **2**



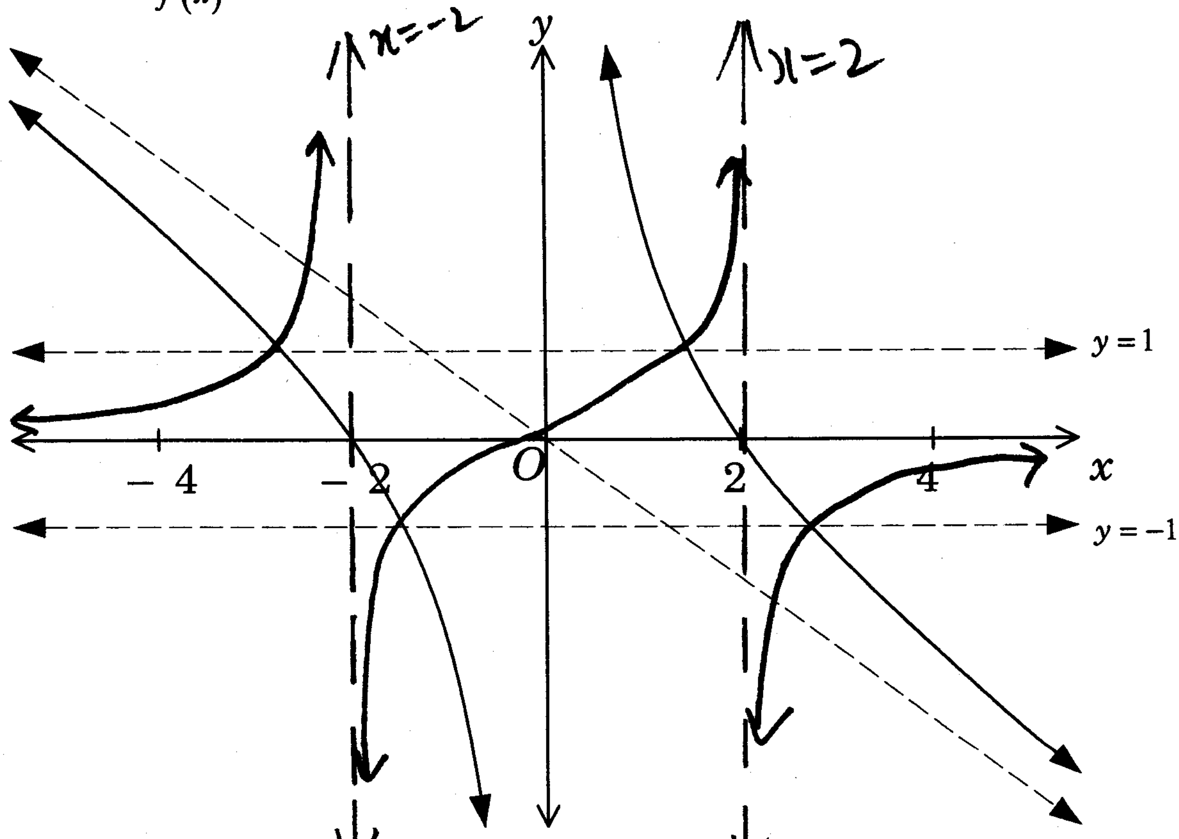
Please turn over …

ANSWER SHEET FOR QUESTION 3(a)

(ii) $y^2 = f(x)$ **2**



(iii) $y = \dfrac{1}{f(x)}$ **2**



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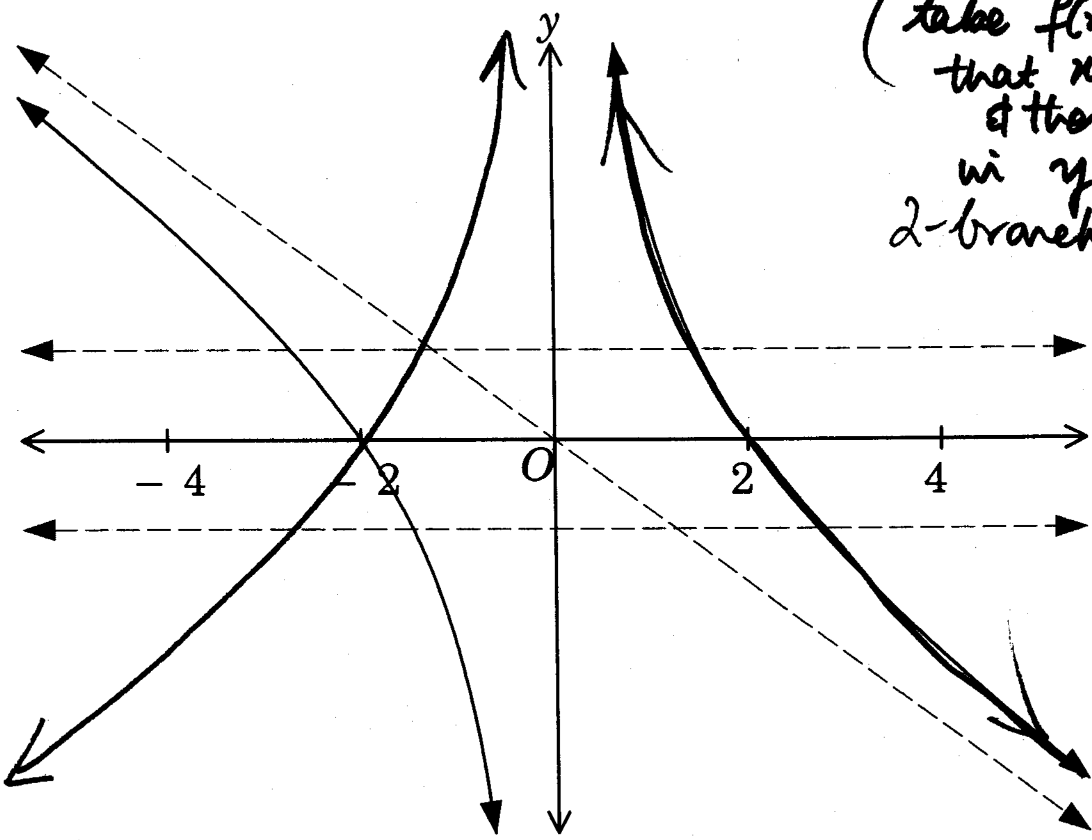
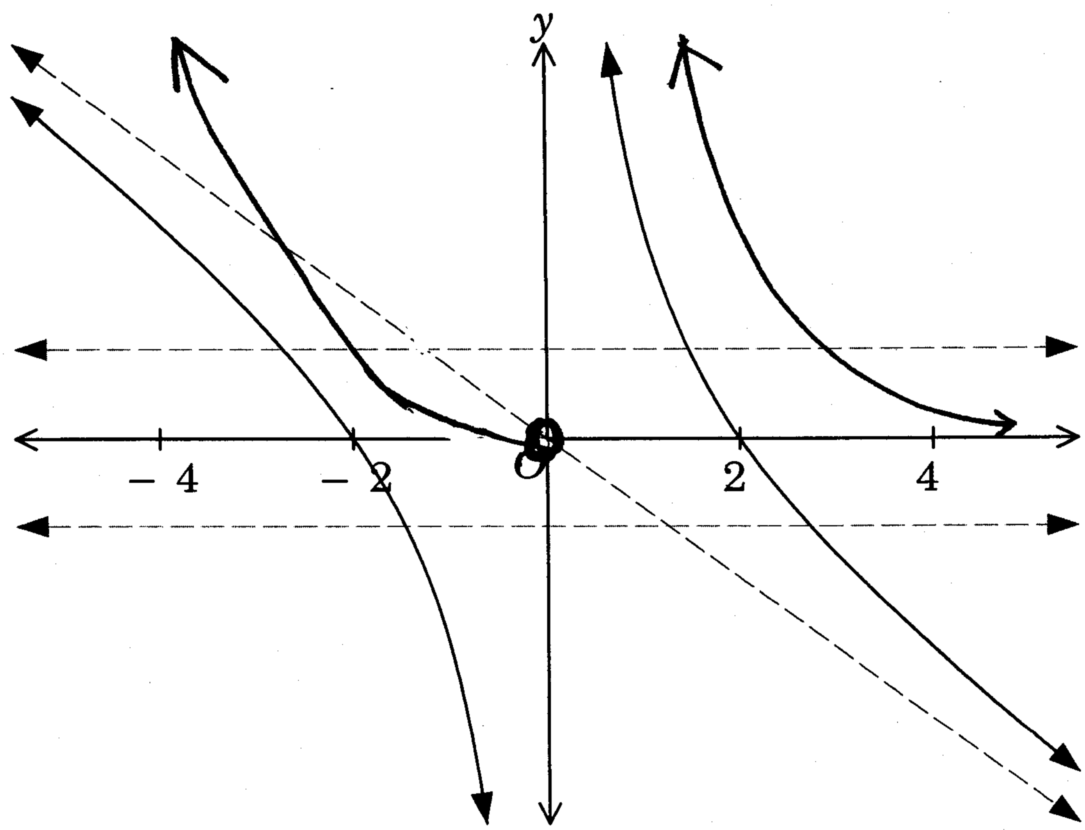
ANSWER SHEET FOR QUESTION 3(a)

(iv) $y = f(|x|)$ **2**



(take f(x) such that x ≥ 0 & then reflect in y-axis). 2-branches.

(v) $y = e^{f(x)}$ **2**



Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments
Q3(b)	
Let $x = \frac{2}{5}\sec\theta$ $\therefore \frac{dx}{d\theta} = \frac{2}{5}\sec\theta\tan\theta$	✓
when $x = -\frac{4}{5}$, $-2 = \sec\theta$ taking the smallest positive 'argument' $\theta = \frac{2\pi}{3}$. when $x = -\frac{2}{5}$, $-1 = \sec\theta$ as above $\theta = \pi$.	✓
$\therefore \int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2-4}}{x}\frac{dx}{d\theta}\,d\theta$	
$= \int_{\frac{2\pi}{3}}^{\pi} \frac{\sqrt{25(\frac{4}{25}\sec^2\theta - 4)}}{\frac{2}{5}\sec\theta}\frac{2}{5}\sec\theta\tan\theta\,d\theta$	✓
$= \int_{\frac{2\pi}{3}}^{\pi} \frac{2\sqrt{\tan^2\theta}}{\frac{2}{5}\sec\theta}\cdot\frac{2}{5}\tan\theta\sec\theta\,d\theta$	
$= \int_{\frac{2\pi}{3}}^{\pi} 2\lvert\tan\theta\rvert\tan\theta\,d\theta$.	
since $\tan\theta < 0$ for $\theta \in [\frac{2\pi}{3}, \pi]$ then take $\lvert\tan\theta\rvert = -\tan\theta$ in this region.	
$\therefore$ we want: $\int_{\frac{2\pi}{3}}^{\pi} -2\tan^2\theta\,d\theta$	✓ must have correct "sign"

(−1 if (−) sign ignored).

Suggested Solution (s)	Comments
$= -2\int_{\frac{2\pi}{3}}^{\pi}(\sec^2\theta - 1)\,d\theta$ $= -2\left[\tan\theta - \theta\right]_{\frac{2\pi}{3}}^{\pi}$ $= -2\left(\tan\pi - \pi - \tan\frac{2\pi}{3} + \frac{2\pi}{3}\right)$ $= -2\left(-\pi + \sqrt{3} + \frac{2\pi}{3}\right)$ $= -2\left[\sqrt{3} - \frac{\pi}{3}\right]$ $= \frac{2\pi}{3} - 2\sqrt{3}$ ✓	[NB: $\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3}$]
QUESTION 4: (15 MARKS)	
(a) $x^3 - 3x + 2 = 0$ $\alpha, \beta, \gamma \Rightarrow$ roots.	
(i) let $y = \frac{1}{x}$ $\therefore x = \frac{1}{y}$ $\therefore (\frac{1}{y})^3 - 3(\frac{1}{y}) + 2 = 0$ ie $1 - 3y^2 + 2y^3 = 0$ is the required equation.	✓
(ii) Observe that: $\frac{\alpha}{\beta\gamma} = \frac{\alpha^2}{\alpha\beta\gamma} = \frac{\alpha^2}{-2}$ $\frac{\beta}{\alpha\gamma} = \frac{\beta^2}{\alpha\beta\gamma} = \frac{\beta^2}{-2}$ $\frac{\gamma}{\alpha\beta} = \frac{\gamma^2}{\alpha\beta\gamma} = \frac{\gamma^2}{-2}$ $\therefore$ let $y = \frac{x^2}{-2}$ ie $x = \sqrt{-2y}$ ✓	

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q4(a) ctd:

$\therefore (\sqrt{-2y})^3 - 3\sqrt{-2y} + 2 = 0$

ie $\sqrt{-2y}\left[(\sqrt{-2y})^2 - 3\right] = -2$

ie $\sqrt{-2y}\,(-2y-3) = -2$

ie $-2y\,(2y+3)^2 = 4$ ✓

ie $-2y(4y^2+12y+9) = 4$

ie $-8y^3 - 24y^2 - 18y = 4$

ie $8y^3 + 24y^2 + 18y + 4 = 0$ ✓

is the required equation.

(b)(i)

Let $P(x) = (x-\alpha)^r Q(x)$

$\therefore P'(x) = Q(x)\left[r(x-\alpha)^{r-1}\right] + (x-\alpha)^r Q'(x)$

ie $P'(x) = (x-\alpha)^{r-1}\left[rQ(x) + (x-\alpha)Q'(x)\right]$ ✓

$\therefore P'(x)$ has a root $x=\alpha$ of multiplicity $r-1$ since $P(\alpha) = P'(\alpha) = 0$ by substitution ✓

(ii) Let $P(x) = x^4 - 5x^3 + 16x^2 - 21x + 9$

$P'(x) = 4x^3 - 15x^2 + 32x - 21$

we know $P(1) = P'(1) = 0$

$P'(1) = 4 - 15 + 32 - 21$
$= 0$
$= P(1)$

To find the other roots, there are several approaches:

NB:
$P(x) = (x-1)^2\,Q(x)$

by inspection $Q(x) = x^2 - 3x + 9$ ✓

or Divide $(x-1)^2$ into $P(x)$ and use long division to find $Q(x)$

or NB: $1, 1, \alpha, \bar{\alpha}$ are the roots since in $Q(x)$, $\Delta < 0$

$x^2 - 3x + 9 = 0$

$\therefore x = \dfrac{3 \pm \sqrt{-27}}{2}$ ✓

$= \dfrac{3 \pm 3\sqrt{3}i}{2}$ ✓

$\therefore$ The roots are:

$1, 1, \dfrac{3+3\sqrt{3}i}{2}, \dfrac{3-3\sqrt{3}i}{2}$.

(c)(i) $z = \text{cis}\,\dfrac{2\pi}{3}$

$\omega = \sqrt{3}\,\text{cis}\,\dfrac{\pi}{3}$

$\dfrac{\omega}{z} = \sqrt{3}\,\text{cis}\left(\dfrac{\pi}{3} - \dfrac{2\pi}{3}\right)$ ✓

ie $\dfrac{\omega}{z} = \sqrt{3}\,\text{cis}\left(-\dfrac{\pi}{3}\right)$

c(ii) $\left(\dfrac{\omega}{z}\right)^{12} = \left[\sqrt{3}\,\text{cis}\left(-\dfrac{\pi}{3}\right)\right]^{12}$

$= 3^6\,\text{cis}(-4\pi)$ ✓

$= 729 \times 1$

$= 729$ ✓

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q4(c) ctd

(ii) Let $S(n)$ represent the statement:

$(\text{cis}\,\theta)^n = \text{cis}(n\theta)$

Step 1: Prove $S(1)$ is true:

$LHS = (\text{cis}\,\theta)^1$
$= \cos\theta + i\sin\theta$

$RHS = \text{cis}(1\times\theta)$
$= \text{cis}\,\theta$
$= \cos\theta + i\sin\theta$

$\therefore$ The Statement is true for $n=1$. ✓

Step 2: Assume $S(k)$ is true for $1\le k\le n$, $(k,n)\in\mathbb{Z}^+$

i.e. Assume: $(\text{cis}\,\theta)^k = \text{cis}(k\theta)$

RTP: $(\text{cis}\,\theta)^{k+1} = \text{cis}[(k+1)\theta]$ —— ①

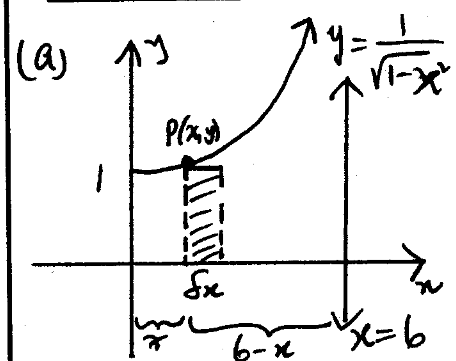
From ①

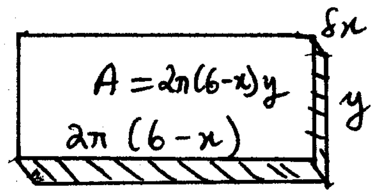
$LHS = (\text{cis}\,\theta)^{k+1}$
$= (\text{cis}\,\theta)^k \cdot (\text{cis}\,\theta)^1$
$= (\text{cis}\,k\theta)\,\text{cis}\,\theta$ from assumption
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ ✓
$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + \cos k\theta\cdot i\sin\theta + i^2\sin k\theta\sin\theta$
$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$ ✓
$= \cos(k\theta+\theta) + i\sin(k\theta+\theta)$ ✓
$= \cos\theta(k+1) + i\sin\theta(k+1)$

Step 3: The statement is true for $n=1$. Whenever it is true for $n=k$, it is also true for all positive integer values n.

QUESTION 5: (15 MARKS)

(a)





Slicing parallel to the axis of rotation:

The area of a typical slice $A = 2\pi(6-x)y$

Volume of a typical shell

$\delta y = 2\pi(6-x)y\,\delta x$

$\therefore$ Total Volume

$V = \lim_{\delta x\to 0}\sum_{x=0}^{x=\frac{1}{2}} 2\pi(6-x)y\,\delta x$ ✓

$= 2\pi\int_0^{\frac{1}{2}} \frac{6-x}{\sqrt{1-x^2}}\,dx$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q5(a) ctd:

$V = 2\pi\int_0^{\frac{1}{2}}\left(\frac{6}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right)dx$ ✓

$= 2\pi\left[6\sin^{-1}x + \sqrt{1-x^2}\right]_0^{1/2}$ ✓

$= 2\pi\left[6\sin^{-1}\frac{1}{2} + \sqrt{1-\frac{1}{4}} - \sqrt{1}\right]$

$= 2\pi\left[\pi + \frac{\sqrt{3}}{2} - 1\right]$ ✓

$V = \pi\left[2\pi + \sqrt{3} - 2\right]$ cubic units.

(b) $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\, dx$

(i) let $u = x^n$ $dv = \cos x$

$du = nx^{n-1}$ $v = \sin x$

$\therefore I_n = uv - \int v\, du$

$= x^n \sin x\Big]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \sin x \cdot n \cdot x^{n-1} dx$

$= \left[\left(\frac{\pi}{2}\right)^n \sin\frac{\pi}{2} - 0\right] - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\, dx$ ✓

↓ Apply integration by parts again

$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n\left[x^{n-1}(-\cos x)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\pi/2} x^{n-2}\cos x\, dx\right]$ ✓

i.e. $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\, dx$

$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\, I_{n-2}$

b(ii) $I_4 = \int_0^{\frac{\pi}{2}} x^4 \cos x\, dx$

$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\, I_2$

$= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2(1)\, I_0\right]$

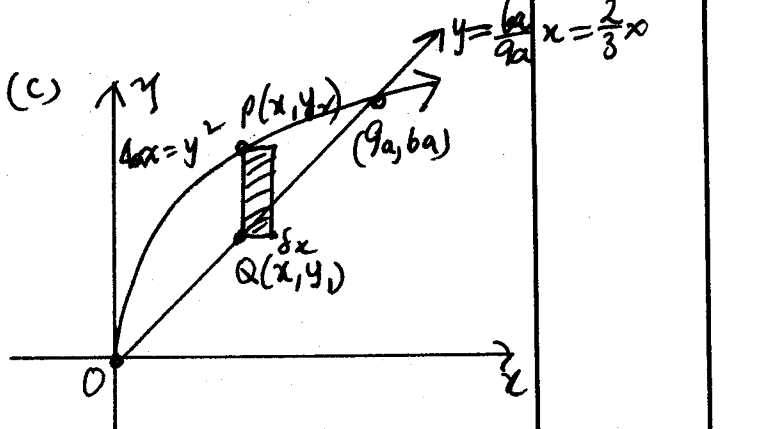
$= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\int_0^{\frac{\pi}{2}}\cos x\, dx\right]$ ✓

$= \frac{\pi^4}{16} - 12\left[\frac{\pi^2}{4} - \left[2\sin x\right]_0^{\frac{\pi}{2}}\right]$

$= \frac{\pi^4}{16} - 12\left[\frac{\pi^2}{4} - 2\sin\frac{\pi}{2}\right]$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$. ✓

(c)



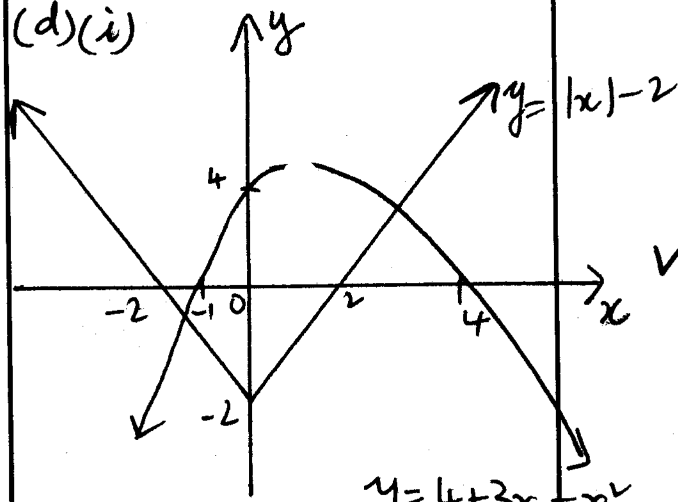
Length of $PQ = |y_2 - y_1|$

$\therefore$ Radius of semi-circle, in base is $\frac{|y_2 - y_1|}{2}$

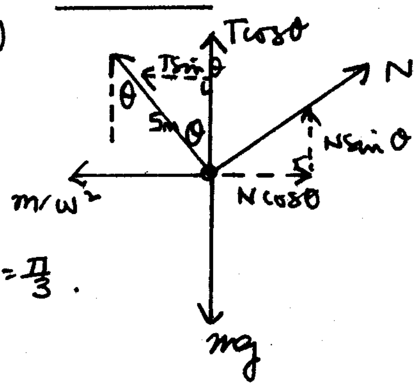
$\therefore$ Area of typical slice is $A = \frac{1}{2}\pi\left|\frac{y_2 - y_1}{2}\right|^2$

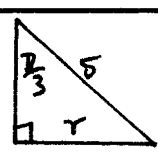
i.e. $A = \frac{\pi}{8}(y_2 - y_1)^2$ ✓

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
but y_1 & $y_2 > 0$ (as given) $\therefore$ let $y_2 = \sqrt{4ax}$, $y_1 = \frac{2}{3}x$ } see diagram $\therefore A \approx \frac{\pi}{8}\left[\sqrt{4ax} - \frac{2}{3}x\right]^2$ $= \frac{\pi}{8}\left(2\sqrt{ax} - \frac{2}{3}x\right)^2$ $\therefore$ Volume, δV of a typical slice is: $\delta V = \frac{\pi}{8}\left(2\sqrt{ax} - \frac{2}{3}x\right)^2 \delta x$	✓	(d)(i)  $y = 4 + 3x - x^2$ $y = (4-x)(x+1)$	✓
$\therefore$ Total Volume of solid: $V = \lim_{\delta x \to 0} \sum_{x=0}^{x=9a} \frac{\pi}{8}\left(2\sqrt{ax} - \frac{2}{3}x\right)^2 \delta x$ $V = \frac{\pi}{8}\int_0^{9a}\left(2\sqrt{ax} - \frac{2}{3}x\right)^2 dx$ $= \frac{\pi}{8}\left[2ax^2 - \frac{16}{15}x^2\sqrt{ax} + \frac{4x^3}{27}\right]_0^{9a}$	✓	(ii) Solving $\frac{\lvert x\rvert - 2}{4 + 3x - x^2} > 0$ We want x such that $\lvert x\rvert - 2$ & $4 + 3x - x^2$ are the same "sign": When both are (+): $2 < x < 4$ When both are (−): $-2 < x < -1$	✓ ✓
$= \frac{\pi}{8}\left[2a(81a^2) - \frac{16}{15} \times 81a^2\sqrt{9a^2} + \frac{4(9a)^3}{27}\right]$ $V = \frac{27\pi a^3}{20}$ cubic units.	✓	$\therefore$ Total solution is: $\{x : -2 < x < -1$ and $2 < x < 4\}$.	

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments
QUESTION 6: (15 MARKS)	
(a) $\cos x = \sin\frac{x}{2}$	
Various methods are available –	
method 1:	
$\cos x = \sin\left(\frac{x}{2}\right)$	
$\cos x = \cos\left(\frac{\pi}{2} - \frac{x}{2}\right)$	✓ method
$\therefore x = 2n\pi \pm \left(\frac{\pi}{2} - \frac{x}{2}\right)$	
$\therefore x = 2n\pi + \left(\frac{\pi}{2} - \frac{x}{2}\right)$ or $x = 2n\pi - \left(\frac{\pi}{2} - \frac{x}{2}\right)$	
i.e. $\frac{3x}{2} = 2n\pi + \frac{\pi}{2}$ or $\frac{x}{2} = 2n\pi - \frac{\pi}{2}$	
i.e. $x = \frac{\pi}{3}(4n+1)$ or $x = \pi(4n-1)$	✓✓
where n is an integer.	
method 2:	alternate
$\cos x = \sin\frac{x}{2}$	
i.e. $\cos 2\left(\frac{x}{2}\right) = \sin\frac{x}{2}$	
i.e. $1 - 2\sin^2\frac{x}{2} = \sin\frac{x}{2}$	✓
i.e. $2\sin^2\frac{x}{2} + \sin\frac{x}{2} - 1 = 0$	
$\left(2\sin\frac{x}{2} - 1\right)\left(\sin\frac{x}{2} + 1\right) = 0$	
$\therefore \sin\frac{x}{2} = \frac{1}{2}$ or $\sin\frac{x}{2} = -1$	✓✓
$\therefore \frac{x}{2} = n\pi + (-1)^n \sin^{-1}\frac{1}{2}$	
i.e. $x = 2n\pi + (-1)^n \frac{\pi}{3}$	
or $\frac{x}{2} = n\pi + (-1)^n \sin^{-1}(-1)$	
i.e. $\frac{x}{2} = n\pi - (-1)^n\left(\frac{\pi}{2}\right)$	
$\therefore x = 2n\pi - \pi(-1)^n$	
$x = \pi(2n - (-1)^n)$	
(b) At P:	
(i) 	✓ must have ↑ → components.
$\theta = \frac{\pi}{3}$.	
(ii) Resolving forces at P:	
Vertically: $m\ddot{y} = \Sigma F_y$	
$T\cos\theta + N\sin\theta - mg = 0$	
$T \times \cos\frac{\pi}{3} + N\sin\frac{\pi}{3} - mg = 0$	
i.e. $\frac{T}{2} + \frac{N\sqrt{3}}{2} - mg = 0$ (1)	
i.e. $T + N\sqrt{3} = 2mg$ (1)	✓
Radially:	
$T\sin\theta - N\cos\theta = mr\omega^2$	
$T \times \sin\frac{\pi}{3} - N\cos\frac{\pi}{3} = mr\omega^2$	✓
$\frac{T\sqrt{3}}{2} - \frac{N}{2} = mr\omega^2$	



$\frac{r}{5} = \sin\frac{\pi}{3}$

$\therefore r = \frac{5\sqrt{3}}{2}$ ✓

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q6 ctd:

$$\frac{T\sqrt{3}}{2} - \frac{N}{2} = m\left(\frac{5\sqrt{3}}{2}\right)w^2$$

$\therefore T\sqrt{3} - N = 5\sqrt{3}mw^2$ (2)

ie* $N = T\sqrt{3} - 5\sqrt{3}mw^2$ (3)

sub (3) into (1)

$T + (T\sqrt{3} - 5\sqrt{3}mw^2)\sqrt{3} = 2mg$

ie $T + 3T - 15mw^2 = 2mg$

ie $4T = 2mg + 15mw^2$

ie* $\boxed{T = \frac{m}{4}(2g + 15w^2)}$ (4)

we want $T > 0$, $N > 0$

since $w > 0$, $m > 0$, $g > 0$

$T > 0$.

$\therefore$ we need $N > 0$.

This will occur if:

$N = \left[\frac{m}{4}(2g + 15w^2)\right]\sqrt{3} - 5\sqrt{3}mw^2 > 0$

ie when:

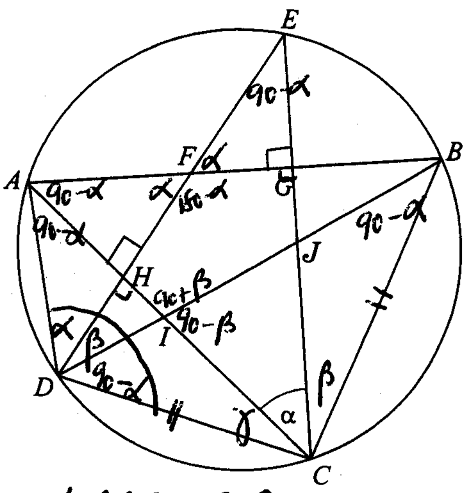
$\frac{m\sqrt{3}}{4}(2g + 15w^2 - 20w^2) > 0$

ie $\boxed{N = \frac{m\sqrt{3}}{4}(2g - 5w^2)} > 0$

ie when $2g - 5w^2 > 0$

ie $w^2 < \frac{2g}{5}$ as expected.

(c)



(i) $\angle AGC = 90°$

$\therefore$ in ΔAGC, $\angle CAG = 90 - \alpha$ (complementary angle)

Similarly $\angle CHE = 90°$

$\therefore \angle DEC = 90 - \alpha$

$\therefore \angle BDC = 90 - \alpha$ ($\angle$'s in the same segment)

$\therefore \angle DBC = 90 - \alpha$ (as above)

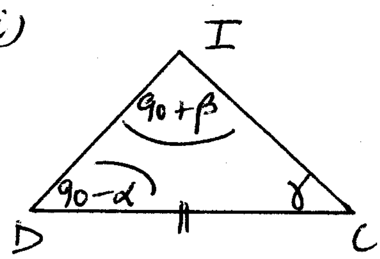
$\therefore \Delta BCD$ is isosceles

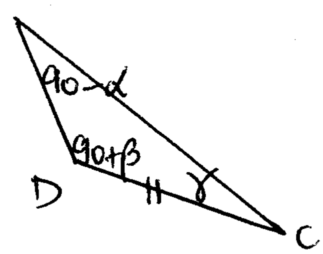
$\therefore DC = BC$

NB: These are "equal chords"

$\therefore \angle DAC = \angle CDB = 90 - \alpha$.

(ii)





Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments

In ΔCID & ΔCDA,

1. $\angle IDC = \angle DAC = 90 - \alpha$
$= \angle BDC$

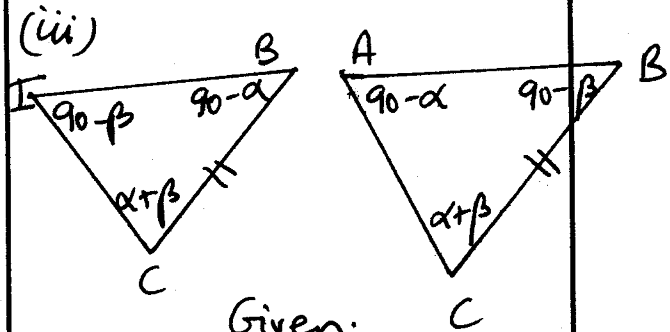
(equal chords subtend equal angles) ✓

2. $\angle DIC = \angle ADC = 90 + \beta$

$(\alpha + \beta + 90 - \alpha)$
$= 90 + \beta$ ✓

$\therefore \Delta CID \parallel\!| \Delta CDA$ (A.A.A)

(iii)



Given:

$\Delta CIB \parallel\!| \Delta CBA$ ①

$AB + AD = 2BC$ ②

Proven: $\Delta CIB \parallel\!| \Delta CDA$ ③

RTP: $CI = \frac{BD}{2}$.

from (ii) $\frac{CI}{CD} = \frac{CD}{CA} = \frac{ID}{DA}$ ④

from (iii) $\frac{CI}{CB} = \frac{CB}{CA} = \frac{IB}{BA}$ ⑤ ✓

Corresponding sides in $\parallel\!|$ Δ's are in proportion.

Observe: $BD = ID + IB$ (check)

from ④ $ID = \frac{IC \times AD}{CD}$ ⑥

from ⑤ $IB = \frac{CI \times AB}{CB}$ ⑦

$\therefore$ ⑥ + ⑦

$ID + IB = \frac{IC \times AD}{CD} + \frac{CI \times AB}{CB}$

i.e. $BD = IC\left[\frac{AD}{CD} + \frac{AB}{CB}\right]$

but ΔCDB is isosceles

$\therefore CD = CB$ ✓

$\therefore BD = IC\left[\frac{AD}{CB} + \frac{AB}{CB}\right]$

$BD = \frac{IC}{BC}(AD + AB)$

but $AD + AB = 2BC$

$\therefore BD = \frac{IC}{BC} \times 2BC$

$BD = 2IC$

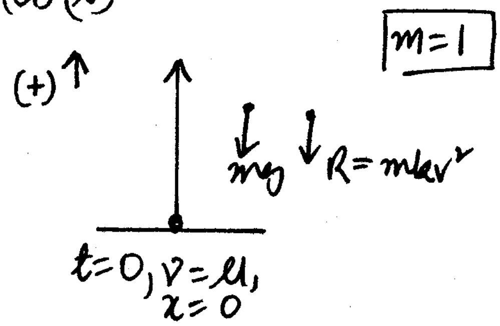
$\therefore IC = \frac{BD}{2}$ as required.

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

QUESTION 7: (15 MARKS)

(a) (i)



$m\ddot{x} = \sum F_x$

i.e. $\ddot{x} = -mg - mkv^2$

$\ddot{x} = -g - kv^2$ ✓

let $v\dfrac{dv}{dx} = -(g+kv^2)$

$\therefore \dfrac{dx}{dv} = -\left(\dfrac{v}{g+kv^2}\right)$ ✓

$\therefore x = -\dfrac{1}{2k}\log_e(g+kv^2)+c_2$

when $x=0,\ v=u$

$\therefore c_2 = \dfrac{1}{2k}\log_e(g+ku^2)$

$\therefore x = \dfrac{1}{2k}\log_e(g+ku^2) - \dfrac{1}{2k}\log_e(g+kv^2)$

when $x=H,\ v=0$

$\therefore H = \dfrac{1}{2k}\log_e\left(\dfrac{g+ku^2}{g}\right)$ ✓

i.e. $H = \dfrac{1}{2k}\log_e\left(1+\dfrac{ku^2}{g}\right)$. (as expected)

NB: we would have obtained the same result/expression if we had evaluated:

$$\int_0^H dx = \int_u^0 -\left(\frac{v}{g+kv^2}\right)dv$$

(Can you see why?)

(ii) Let $\dfrac{dv}{dt} = -g - kv^2$ ✓

$\therefore \dfrac{dv}{dt} = -(g+kv^2)$

i.e. $\dfrac{dt}{dv} = -\left(\dfrac{1}{kv^2+g}\right)$

$\dfrac{dt}{dv} = -\left(\dfrac{1}{k(v^2+g/k)}\right)$

i.e. $t = -\dfrac{1}{k}\left[\dfrac{1}{\sqrt{\frac{g}{k}}}\tan^{-1}\left[\dfrac{v}{\sqrt{\frac{g}{k}}}\right]\right]+c_2$

when $t=0;\ v=u$ $\therefore$

$c_2 = \dfrac{1}{k}\left(\dfrac{1}{\sqrt{\frac{g}{k}}}\right)\tan^{-1}\left[u\sqrt{\dfrac{k}{g}}\right]$

$\therefore$

when $t=T,\ v=0$

$\therefore T = \dfrac{1}{k}\sqrt{\dfrac{k}{g}}\left[\tan^{-1} u\sqrt{\dfrac{k}{g}}\right]$

$T = \sqrt{\dfrac{k}{k^2 g}}\tan^{-1}\left(\dfrac{u\sqrt{k}}{\sqrt{g}}\right)$

i.e. $T = \dfrac{1}{\sqrt{kg}}\tan^{-1}\left(\dfrac{u\sqrt{k}}{\sqrt{g}}\right)$ seconds. ✓

OR: we could have evaluated:

$$\int_0^T dt = -\frac{1}{k}\int_u^0 \frac{1}{v^2+g/k}\,dv$$

Suggested Solution (s)	Comments

Q7(b)

(i) Let $h(x) = \frac{\log_{10} x}{x}$

Changing base:

$h(x) = \frac{\log_e x}{\log_e 10} \times \frac{1}{x}$

ie $h(x) = \frac{\ln x}{x \ln 10}$ ✓

$\therefore h'(x) = \frac{1}{\ln 10}\left[\frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}\right]$ ✓ Quotient Rule.

ie $h'(x) = \frac{1}{\ln 10}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)$

(ii) $h(x)$ has SP's when $h'(x) = 0$

ie when

$\frac{1}{x^2} - \frac{\ln x}{x^2} = 0$

ie $1 - \ln x = 0$

ie when $x = e$. ✓

We know this is the only SP's and the value of x that makes $h(x)$ maximum ✓

$\therefore$ for each x in its domain:

$h(x) < h(e)$

ie $h(\pi) < h(e)$ by definition

$\therefore \frac{1}{\pi}\left(\frac{\ln \pi}{\ln 10}\right) < \frac{1}{e}\left(\frac{\ln e}{\ln 10}\right)$

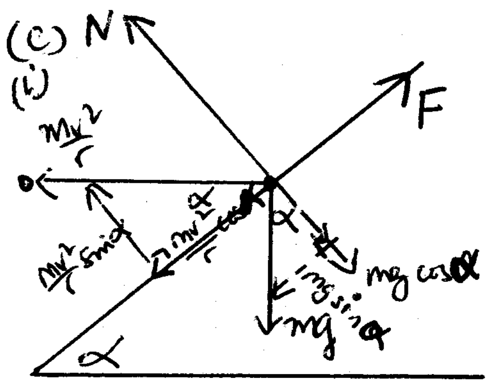
ie $\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$ ✓

$\therefore \pi \ln e > e \ln \pi$

ie $\ln e^{\pi} > \ln \pi^{e}$

$\therefore e^{\pi} > \pi^{e}$ as required.

(c) (i)



(ii) Resolving forces parallel to the slope (and downward)

$mg\sin\alpha - F = \frac{mv^2}{r}\cos\alpha$ ✓✓

ie $F = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q7 (c) (iii)

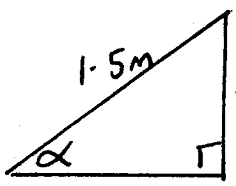
$r = 3000\text{m}$

$v = 180\text{km/h}$

$= \frac{180 \times 1000}{3600}$ m/s

$= 50$ m/s.

Also, $F = 0$ (no lateral thrust).



$h \leftarrow$ required height.

$\sin\alpha = \frac{h}{1 \cdot 5}$ *

but for small α, $\tan\alpha \approx \sin\alpha$.

from (ii)

$mg\sin\alpha = \frac{mv^2}{r}\cos\alpha$

ie $\tan\alpha = \frac{v^2}{rg}$

$\therefore \tan\alpha = \frac{50^2}{3000 \times 9 \cdot 8}$

$\alpha = 4 \cdot 9° \ldots$

from * $h \doteqdot 1 \cdot 5 \sin\alpha$

ie $h \approx 1 \cdot 5 \sin \tan^{-1}\left(\frac{50^2}{3000 \times 9 \cdot 8}\right)$

$= 0 \cdot 128 \ldots$ m

$h \doteqdot 13$ cms.

QUESTION 8 : (15 MARKS)

(a) $z^5 - 1 = 0$

let $z^5 = 1$

ie $z = \text{cis}\left(\frac{2k\pi}{5}\right)$

$\therefore$ The roots are:

$k=0, \; z_1 = \text{cis}\, 0 = 1$

$k=1, \; z_2 = \text{cis}\left(\frac{2\pi}{5}\right)$

$k=2, \; z_3 = \text{cis}\left(\frac{4\pi}{5}\right)$

$k=3, \; z_4 = \text{cis}\left(\frac{6\pi}{5}\right)$

$= \text{cis}\left(\frac{6\pi}{5} - 2\pi\right)$

$= \text{cis}\left(-\frac{4\pi}{5}\right)$

$k=4, \; z_5 = \text{cis}\left(\frac{8\pi}{5}\right)$

$= \text{cis}\left(-\frac{2\pi}{5}\right)$

If $0 < \arg w < \pi/2$

then select $w = \text{cis}\frac{2\pi}{5}$.

$\therefore w^2 = \text{cis}\frac{4\pi}{5} = z_3$

$w^3 = \text{cis}\frac{6\pi}{5} = z_4$

$w^4 = \text{cis}\frac{8\pi}{5} = z_5$

$\therefore 1, w, w^2, w^3, w^4$ are the 5th roots of $z^5 - 1 = 0$.

Year 12- 2005 Trial HSC Mathematics EXTENSION 2 Assessment Task 4
Suggested Solutions and Marking Scheme

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

(b) $z^5 - 1$

$= (z-1)(z-w)(z-w^2)(z-w^3)(z-w^4)$, where $(z-w)(z-w^2)(z-w^3)(z-w^4) = z^4+z^3+z^2+z+1$

ie $(z-w)(z-w^2)(z-w^3)(z-w^4) = z^4+z^3+z^2+z+1$ ✓

let $z=1$.

$\therefore (1-w)(1-w^2)(1-w^3)(1-w^4) = 1+1+1+1+1 = 5$.

(c) $(1-w)(1-w^4)$

$= 1 - w^4 - w + w^5$ but $w^5 = 1$

$= -w^4 - w + 2$ ✓✓

$= -\text{cis}\frac{8\pi}{5} - \text{cis}\frac{2\pi}{5} + 2$

$= -\text{cis}\left(-\frac{2\pi}{5}\right) - \text{cis}\left(\frac{2\pi}{5}\right) + 2$

$= -\left[\cos\frac{-2\pi}{5} + i\sin\left(\frac{-2\pi}{5}\right)\right] - \left[\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right] + 2$

$= -2\cos\frac{2\pi}{5} + 2$

$\therefore (1-w)(1-w^4) = 2 - 2\cos\frac{2\pi}{5}$

(d) $(1-w)(1-w^4) = 2 - 2\cos\frac{2\pi}{5}$ ①

also: Consider:

$(1-w^2)(1-w^3) = 1 - w^2 - w^3 + w^5$

$= 2 - \text{cis}\frac{4\pi}{5} - \text{cis}\left(-\frac{4\pi}{5}\right)$ ✓

ie $(1-w^2)(1-w^3) = 2 - 2\cos\frac{4\pi}{5}$ ② ✓

① × ②

$(1-w)(1-w^2)(1-w^3)(1-w^4) = \left[2 - 2\cos\frac{2\pi}{5}\right]\left[2 - 2\cos\frac{4\pi}{5}\right]$

from (b) RHS $= 5$ ✓

$\therefore \left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$

ie $\left[2\sin^2\frac{\pi}{5}\right]\left[2\sin^2\frac{2\pi}{5}\right] = \frac{5}{4}$

[using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$]

ie $\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{16}$

since $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5} > 0$

then

$\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$.

(e) $|\overrightarrow{P_0P_1}| = |w - 1|$

$= \left|\text{cis}\frac{2\pi}{5} - 1\right|$ ✓

$= \sqrt{\cos^2\frac{2\pi}{5} - 2\cos\frac{2\pi}{5} + 1 + \sin^2\frac{2\pi}{5}}$

$= \sqrt{2 - 2\cos\frac{2\pi}{5}}$

$= \sqrt{2}\sqrt{1 - \cos\frac{2\pi}{5}}$ ✓

$= \sqrt{2}\sqrt{2\sin^2\frac{\pi}{5}} = 2\sin\frac{\pi}{5}$.

Q8 (e) ctd.

(ii)

$$|\overrightarrow{P_0P_1}| \times |\overrightarrow{P_0P_2}| \times |\overrightarrow{P_0P_3}| \times |\overrightarrow{P_0P_4}|$$

$$= |w-1|\,|w^2-1|\,|w^3-1|\,|w^4-1| \checkmark$$

$$= |1-w|\,|1-w^2|\,|1-w^3|\,|1-w^4|$$

$$= |(1-w)(1-w^2)(1-w^3)(1-w^4)| \checkmark$$

$$= |5|$$

$$= 5 \quad \text{from (b).}$$

End of Paper.

Several questions throughout this paper can be done in alternative ways.