VIRILE AGITUR

Student Number

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Subject Teachers
Mr I Bradford
Mr M Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 40

BLANK PAGE

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$, or otherwise, find $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\int \frac{4x^3 - 2x^2 + 1}{2x-1}\,dx$. **3**

(d) (i) Find constants a, b and c such that **2**

$$\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}.$$

(ii) Hence, find $\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$. **2**

(e) Show, using integration by parts, that **4**

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\;dx = \frac{\pi\sqrt{3}}{3} - \ln 2.$$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\sqrt{3}-i$ in modulus-argument form. **4**

(ii) Hence evaluate $\left(\sqrt{3}-i\right)^6$.

(b) (i) Simplify $(2i)^3$. **2**

(ii) Hence find all complex numbers z such that $z^3 = 8i$. **2**
Express your answers in the form $x+iy$.

(c) Sketch the region where the inequalities $|z-3+i| \le 5$ and $|z+1| \le |z-1|$ both hold. **3**

(d) Let $w = \dfrac{3+4i}{5}$ and $z = \dfrac{5+12i}{13}$, so that $|w| = |z| = 1$.

(i) Find wz and $w\overline{z}$ in the form $x+iy$. **2**

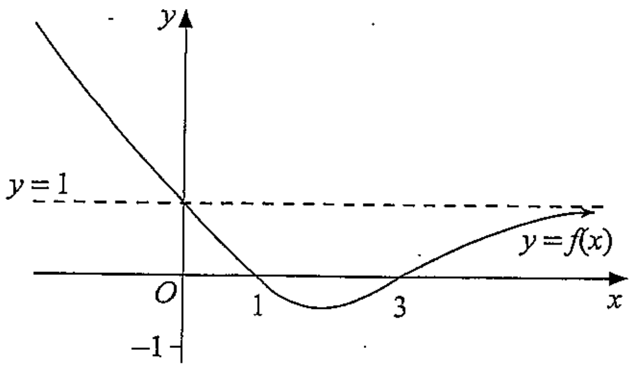
(ii) Hence find two distinct ways of writing 65^2 as the sum of a^2+b^2, where a and b are integers and $0 < a < b$. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch, without using calculus, the curve $y = \dfrac{4x^2}{x^2 - 9}$ showing all asymptotes. **3**

(b)



The diagram shows the graph of the $y = f(x)$. The graph has a horizontal asymptote at $y = 1$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \ln f(x)$. **2**

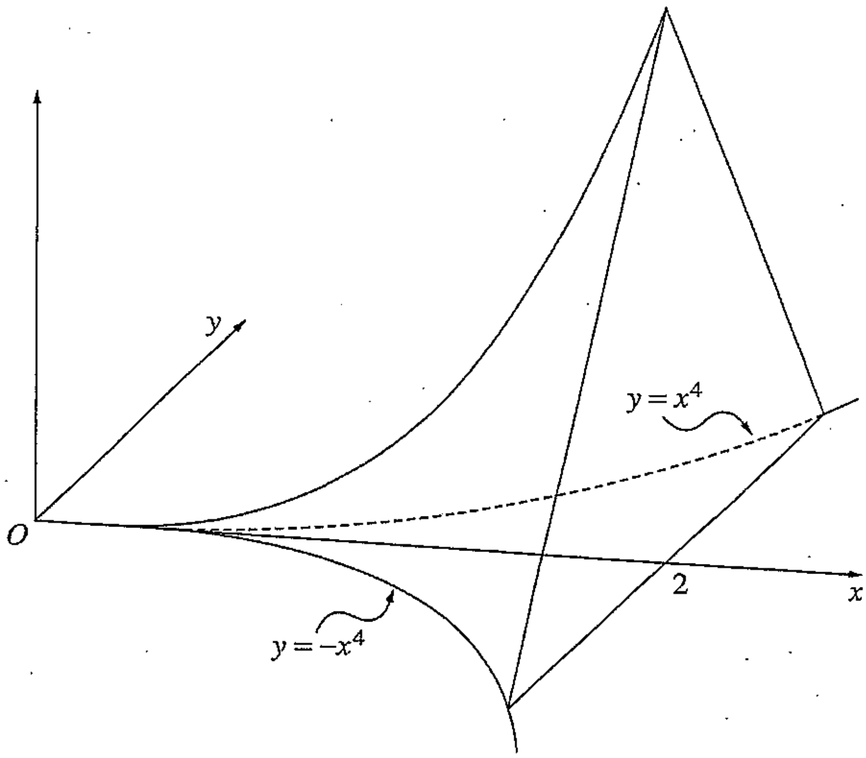
(c) Find the equation, in general form, of the tangent to the curve defined by $x^2 - xy + y^3 = 5$ at the point $(2, -1)$. **2**

Question 3 continues on page 6

Marks

Question 3 (continued)

(c) The base of a solid is the region in the xy plane enclosed by the curves $y = x^4$, $y = -x^4$ and the line $x = 2$. Each cross-section perpendicular to the x-axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at $x = h$ is $\sqrt{3}h^8$. **2**

(ii) Hence find the volume of the solid. **2**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The ellipse E has Cartesian equation $\frac{x^2}{4}+\frac{y^2}{3}=1$.

(i) Write down its eccentricity, the coordinates of its foci S and S' and the equation of each directrix, where S lies on the positive side of the x-axis. **3**

(ii) Sketch E clearly labeling all essential features. **2**

(iii) If P lies on E, then prove that the sum of the distances PS and PS' is independent of P. **2**

(b) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$. **2**

If M is the midpoint of the chord PQ and OM is perpendicular to PQ, express q in terms of p.

(c) (i) Suppose the polynomial $P(x)$ has a a double root a $x=\alpha$. **2**

Prove that $P'(x)$ also has a root at $x=\alpha$.

(ii) The polynomial $A(x)=x^4+ax^2+bx+36$ has a double root at $x=2$. **2**

Find the values of a and b.

(iii) Factorise the polynomial $A(x)$ of part (ii) over the real numbers. **2**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A solid is formed by rotating the region bounded by the curve $y = x(x-1)^2$, the line $y = 0$ and between $x = 0$ and $x = 1$. **3**

Use the method of cylindrical shells to find the exact volume of this solid.

(b) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated around the line $y = 1$. **4**

Using a slicing technique find the exact volume formed.

(c) A particle is moving in the positive direction along a straight line in a medium that exerts a resistance to motion proportional to the cube of the velocity.
No other forces act on the particle, that is, $\ddot{x} = -kv^3$, where k is a positive constant.

At time $t = 0$, the particle is at the origin and has velocity U. At time $t = T$, the particle is at $x = D$ and has velocity V.

(i) Using the identity $\ddot{x} = \frac{dv}{dt}$ show that **3**

$$\frac{1}{V^2} - \frac{1}{U^2} = 2kT.$$

(ii) Using the identity $\ddot{x} = v\frac{dv}{dx}$, show that **3**

$$\frac{1}{V} - \frac{1}{U} = kD.$$

(iii) Hence show that $\frac{D}{T} = \frac{2UV}{U+V}$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Graph $y = \ln x$ and draw the tangent to the graph at $x = 1$. **1**

(ii) By considering the appropriate area under the tangent, deduce that **2**

$$\int_1^{\frac{3}{2}} \ln x \, dx \le \frac{1}{8}.$$

(b) A mass of 2 kg, on the end of a string 0.6 metres long, is rotating as a conical pendulum, with angular velocity 3π radians per second. The acceleration due to gravity is 10 m/s^2.

Let θ be the angle that the string makes with the vertical.

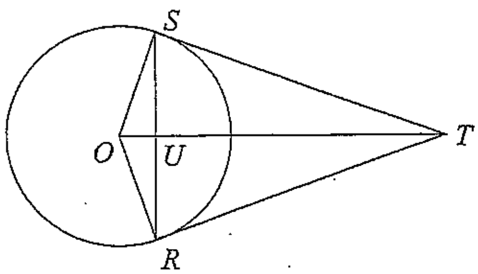
(i) Draw a diagram showing all forces acting on the mass. **1**

(ii) By resolving all forces show that the tension in the string is $10.8\pi^2$ **3**

(iii) Hence, or otherwise, find θ correct to the nearest degree. **1**

(c) Solve for x: $\tan^{-1} 5x - \tan^{-1} 3x = \tan^{-1} \frac{1}{4}$. **3**

(d)



The points R and S lie on a circle with centre O and radius 1. The tangents to the circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular. **4**

By considering ΔSOU and ΔTOS, show that

$$OU \times OT = 1.$$

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let x be a fixed, non-zero number satisfying $x > -1$. **3**

Use the method of mathematical induction to prove that

$$(1+x)^n > 1+nx \text{ for } n = 2, 3, \ldots .$$

(ii) Deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots .$ **1**

(b) (i) Differentiate $\sin^{-1}(u) - \sqrt{1-u^2}$. **2**

(ii) Hence show that **1**

$$\int_0^{\alpha} \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2} \text{ for } 0 < \alpha < 1.$$

(c) A ball of mass 2 kilograms is thrown vertically upward from the origin with an initial speed of 8 metres per second. The ball is subject to a downward gravitational force of 20 newtons and air resistance of $\left(v^2/5\right)$ newtons in the opposite direction to the velocity, v metres per second.

Hence, until the ball reaches its highest point, the equation of motion is:

$$\ddot{y} = -\frac{v^2}{10} - 10 \text{ where } y \text{ metres is its height.}$$

(i) Using the fact that $\ddot{y} = v\frac{dv}{dy}$, show that, while the ball is rising, **3**

$$v^2 = 164e^{-\frac{y}{5}} - 100$$

(ii) Hence find the exact maximum height reached. **1**

(iii) Using the fact that $\ddot{y} = \frac{dv}{dt}$, find how long the ball takes to reach this maximum height. **2**

(iv) How fast is the ball travelling when it returns to the origin? **2**

Marks

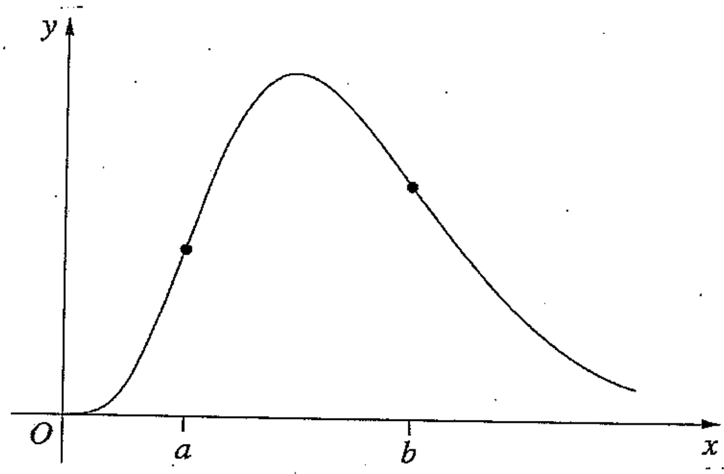
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$. **1**

(ii) Let $I_n=\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}dx$ where $n=0,1,2,\ldots$ **3**

Show that $nI_n=(n-1)I_{n-2}$ for $n=2,3,4,\ldots$

(b) For $x>0$, let $f(x)=x^n e^{-x}$, where n is an integer and $n\geq 2$. **4**



The two points of inflexion of $f(x)$ occur at $x=a$ and $x=b$, where $0<a<b$.

Find a and b in terms of n.

(c) A straight line is drawn through a fixed point $P(a,b)$ in the first quadrant on a number plane. The line cuts the positive part of the x-axis at A and the positive part of the y-axis at B.

(i) If $\angle OAB=\theta$, prove that the length of AB is given by $AB=a\sec\theta+b\operatorname{cosec}\theta$. **2**

(ii) Show that the length of AB will be a minimum if $\cot\theta=\left(\frac{a}{b}\right)^{\frac{1}{3}}$. **3**

(iii) Show that the minimum length of AB is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **2**

End of paper

BLANK PAGE

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

KNOX Trial Ext. 2

2008 → $\boxed{\text{Year 12 Extension 2 Trial HSC}}$

Q1

a) $\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$

$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ ✓

$= -\frac{1}{2}\left(e^{-1}-e^{0}\right)$

$= \frac{1}{2}\left(1-\frac{1}{e}\right)$

$= \frac{e-1}{2e}$ ✓

b) $u = e^x$, $du = e^x\,dx$

$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$ ✓

$= \sin^{-1}u + C$

$= \sin^{-1}(e^x) + C$ ✓

c) $\int \frac{4x^3-2x^2+1}{2x-1}\,dx = \int \frac{2x^2(2x-1)+1}{2x-1}\,dx$ ✓

$= \int 2x^2 + \frac{1}{2x-1}\,dx$ ✓

$= \frac{2x^3}{3} + \frac{1}{2}\ln|2x-1| + C$ ✓

d) (i) $x^2+2x = (ax+b)(x-2) + c(x^2+4)$ ①

$= (a+c)x^2 + (b-2a)x + 4c-2b$

$a+c=1$ ②

$b-2a=2$ ③

Sub. $x=2$ in ① → $8=8c$

$\therefore c=1$ ✓

Sub. $c=1$ in ② → $a=0$

Sub. $a=0$ in ③ → $b=2$ ✓

(ii) $\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx = \int \frac{2}{x^2+4} + \frac{1}{x-2}\,dx$ ✓

$= 2\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + \ln|x-2| + C$

$= \tan^{-1}\frac{x}{2} + \ln|x-2| + C$ ✓

e) $u = x$ $\quad dv = \sec^2x\,dx$

$du = dx$ $\quad v = \tan x\ dx$

$\int_0^{\frac{\pi}{3}} x\sec^2x\,dx = \left[x\tan x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx$ ✓

$= \frac{\pi}{3}\tan\frac{\pi}{3} - \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$ ✓

$= \frac{\pi\sqrt{3}}{3} + \left[\ln(\cos x)\right]_0^{\frac{\pi}{3}}$ ✓

$= \frac{\pi\sqrt{3}}{3} + \left(\ln\frac{1}{2} - \cancel{\ln 1}^{\,0}\right)$

$= \frac{\pi\sqrt{3}}{3} - \ln 2$ ✓

Q2

a) (i) $r = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{4}$

$= 2$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\theta = -\frac{\pi}{6}$

$\therefore \sqrt{3} - i = 2\,\text{cis}\left(\frac{-\pi}{6}\right)$

$= 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

(ii) $(\sqrt{3}-i)^6 = 2^6\,\text{cis}\left(\frac{-\pi}{6}\times 6\right)$

$= 2^6(\cos\pi - i\sin\pi)$

$= -2^6$

$= -64$

b) (i) $(-2i)^3 = -8i^3$

$= 8i$

(ii) $z^3 = 8i$

$z^3 - 8i = 0$

$z^3 + 8i^3 = 0 \rightarrow z^3 + (2i)^3 = 0$

$(z+2i)(z^2 - 2iz + 4i^2) = 0$

$(z+2i)(z^2 - 2iz - 4) = 0$

$z = -2i$ or $z = \frac{2i \pm \sqrt{4i^2 + 16}}{2}$

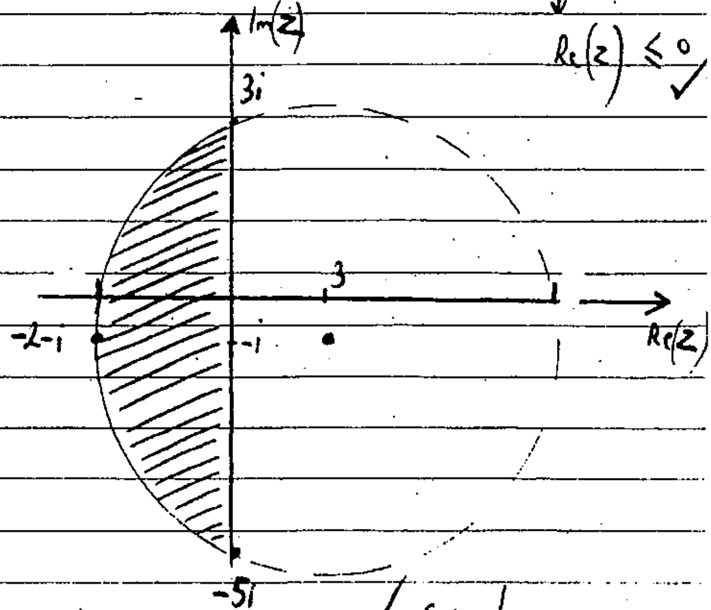
$= \frac{2i \pm \sqrt{12}}{2}$

$= \pm\sqrt{3} + i$

c) $|z - 3 + i| \leq 5$ and $|z+1| \leq |z-1|$

$\downarrow$

$Re(z) \leq 0$



correct shading

d) (i) $wz = \frac{(3+4i)(5+12i)}{65}$

$= \frac{-33 + 56i}{65}$

$= \frac{-33}{65} + \frac{56}{65}i$

$w\bar{z} = \frac{(3+4i)(5-12i)}{65}$

$= \frac{63 - 16i}{65}$

$= \frac{63}{65} - \frac{16}{65}i$

(ii) $|wz| = \sqrt{\left(\frac{33}{65}\right)^2 + \left(\frac{56}{65}\right)^2}$

$= |w|\cdot|z|$

But $1 = \sqrt{\left(\frac{33}{65}\right)^2 + \left(\frac{56}{65}\right)^2}$

$1 = \left(\frac{33}{65}\right)^2 + \left(\frac{56}{65}\right)^2$

$\therefore 65^2 = 33^2 + 56^2 \quad \left(\begin{array}{l} a = 33 \\ b = 56 \end{array}\right)$

Q2 cont'd

Also $\quad |w\bar{z}| = \sqrt{\left(\frac{63}{65}\right)^2 + \left(\frac{16}{65}\right)^2}$

$= |w| \cdot |\bar{z}|$

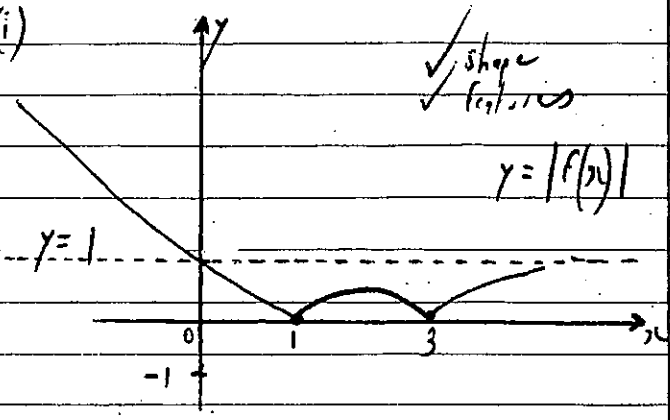
But $|w| = |z| = 1$ and $|\bar{z}| = |z|$

$\therefore \quad 1 = \sqrt{\left(\frac{63}{65}\right)^2 + \left(\frac{16}{65}\right)^2}$

$\therefore \quad 65^2 = 16^2 + 63^2 \quad \begin{pmatrix} a = 16 \\ b = 63 \end{pmatrix}$ ✓

Q3

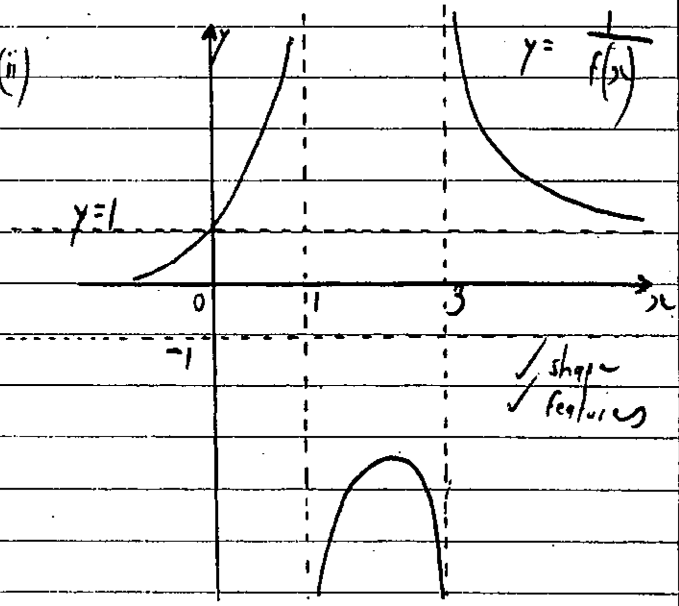
b) (i)



✓ shape
✓ features

$y = |f(x)|$

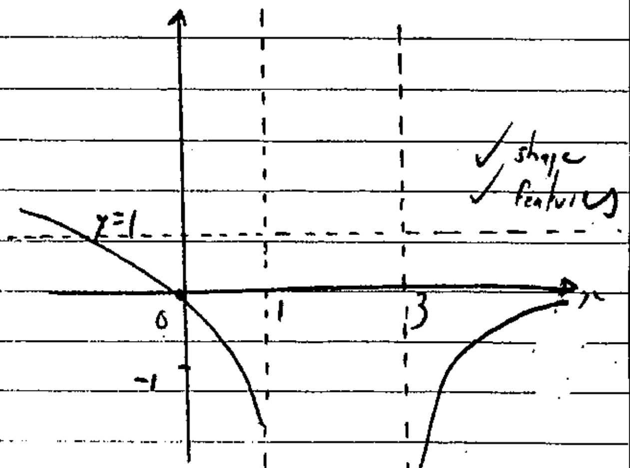
(ii)



$y = \frac{1}{f(x)}$

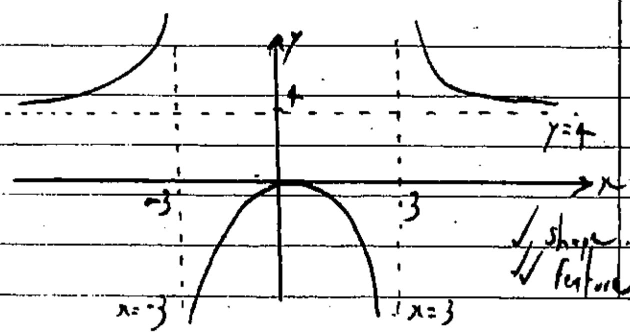
✓ shape
✓ features

(iii)



✓ shape
✓ features

e)



✓ shape
✓✓ features / asymptotes

c) $x^2 - xy + y^3 = 5$

$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{y - 2x}{3y^2 - x}$ ✓

At $(2, -1)$, $\frac{dy}{dx} = \frac{-1-4}{3-2}$

$= -5$

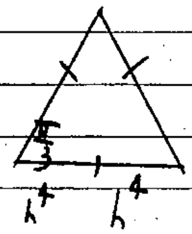
$= m_{tangent}$

$\therefore \quad y + 1 = -5(x - 2)$

$y = -5x + 9$ or

$5x + y - 9 = 0$ ✓

d) (i)



$\text{Area} = \frac{1}{2}(2h^4)^2 \sin\frac{\pi}{3}$ ✓

$= \frac{1}{2}(4h^8) \cdot \frac{\sqrt{3}}{2}$ ✓

$= \sqrt{3}h^8$

(ii) $\text{Volume} = \int_0^2 \sqrt{3}x^8 \, dx$

$= \frac{\sqrt{3}}{9}\left[x^9\right]_0^2$ ✓

$= \frac{512\sqrt{3}}{9}$ units^3 ✓

Q4

a) (i) $\frac{x^2}{2^2} + \frac{y^2}{(\sqrt{3})^2} = 1$

$b^2 = a^2(1-e^2)$

$3 = 4(1-e^2)$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$ ✓

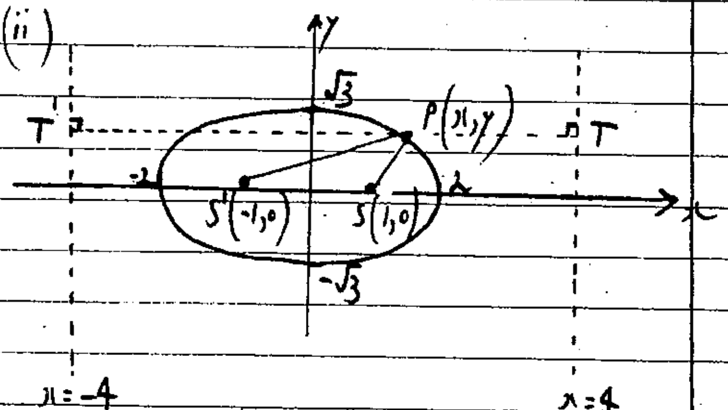
$S = (\pm ae, 0)$

ie $S = (1,0)$ and $S' = (-1,0)$ ✓

Directrix at $x = \pm \frac{a}{e}$

ie $x = \pm 4$ ✓

(ii)



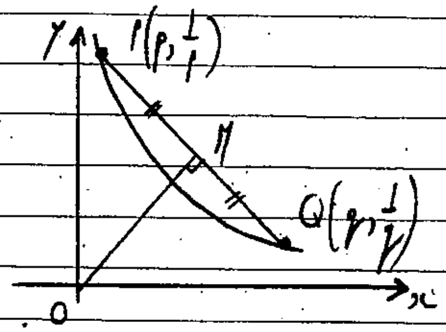
✓ shape
✓ features

(iii) Now $PS = \frac{1}{2}PT$
$= \frac{1}{2}(4-x)$

and $PS' = \frac{1}{2}PT'$
$= \frac{1}{2}(4+x)$ ✓

$\therefore PS + PS' = \frac{1}{2}(4-x) + \frac{1}{2}(4+x)$ ✓
$= 4$ which is independent of P

b)



$M_{PQ} = \left(\frac{p+q}{2}, \frac{\frac{1}{p}+\frac{1}{q}}{2}\right)$
$= \left(\frac{p+q}{2}, \frac{p+q}{2pq}\right)$ ✓

$m_{PQ} = \frac{\frac{1}{q}-\frac{1}{p}}{q-p}$
$= \frac{p-q}{pq(q-p)}$
$= -\frac{1}{pq}$

$m_{OM} = \frac{\frac{p+q}{2pq}}{\frac{p+q}{2}}$
$= \frac{1}{pq}$

since $OM \perp PQ$ then

$\frac{1}{pq} \times \frac{-1}{pq} = -1$ ✓

$1 = (pq)^2$

ie $pq = 1$ ($pq > 0$)

$\therefore q = \frac{1}{p}$

Q4 cont'd

c) (i) If $P(x)$ has a double root at $x=\alpha$

then $P(x) = (x-\alpha)^2 Q(x)$, $Q(x) \neq 0$

$\therefore P'(x) = Q(x) \cdot 2(x-\alpha) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$ ✓

$\therefore P'(\alpha) = (\alpha-\alpha)\left[2Q(\alpha) + (\alpha-\alpha)Q'(x)\right]$ ✓

$= 0$

$\therefore x=\alpha$ is also a root of $P'(x)$

(ii) $A(x) = x^4 + ax^2 + bx + 36$

$A'(x) = 4x^3 + 2ax + b$

$A(2) = A'(2) = 0$ since $x=2$ is a double root

$\therefore$ $16 + 4a + 2b + 36 = 0$

$4a + 2b = -52$

$2a + b = -26$ ①

and $32 + 4a + b = 0$

$4a + b = -32$ ②

② − ① $\rightarrow$ $2a = -6$

$a = -3$ ✓

$b = -20$ ✓

(iii) $A(x) = x^4 - 3x^2 - 20x + 36$

$= (x-2)^2 Q(x)$

$= (x-2)^2 (kx^2 + Lx + m)$ ✓ (method)

By inspection, $m = 9$, $k = 1$

When $x = 1$, $1 - 3 - 20 + 36 = (1)(1 + L + 9)$

$14 = 10 + L$

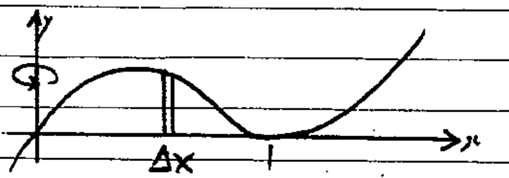
$4 = L$

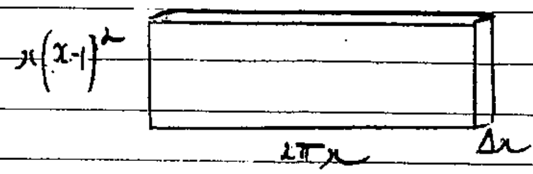
$\therefore A(x) = (x-2)^2(x^2 + 4x + 9)$ over $\mathbb{R}$. ✓

or by Long Division

Q5

a)

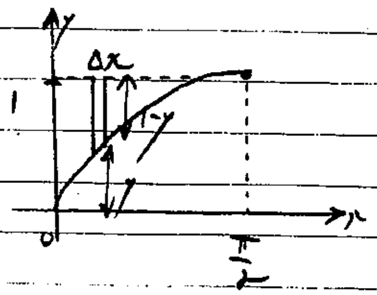




$$\Delta V = 2\pi x \cdot x(x-1)^2 \cdot \Delta x$$
$$= 2\pi x^2 (x-1)^2 \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x^2 (x-1)^2 \Delta x$$
$$= 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx$$
$$= 2\pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$$
$$= 2\pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right)$$
$$= \frac{\pi}{15} \text{ units}^3$$

b)



Area each slice $= \pi (1-y)^2$
$= \pi (1-\sin x)^2$

Volume each slice $= \pi (1-\sin x)^2 \Delta$

$(\cos 2x = 1 - 2\sin^2 x)$

$$\therefore V = \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \sin^2 x \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} - 2\sin x - \frac{1}{2}\cos 2x\right) dx$$
$$= \pi \left[\frac{3x}{2} + 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \pi \left(\frac{3\pi}{4} - 2\right)$$
$$= \pi \left(\frac{3\pi - 8}{4}\right) \text{ units}^3$$

c) (i) $\frac{dv}{dt} = -kv^3$

$$\frac{dt}{dv} = -\frac{1}{k} v^{-3}$$
$$t = -\frac{1}{k} \int v^{-3} \, dv$$
$$= -\frac{1}{k} \cdot \frac{-1}{2v^2} + C$$
$$= \frac{1}{2kv^2} + C$$

At $t = 0$, $v = U \rightarrow 0 = \frac{1}{2kU^2} + C$

$$\therefore t = \frac{1}{2k}\left(\frac{1}{v^2} - \frac{1}{U^2}\right)$$

At $t = T$, $v = V \rightarrow T = \frac{1}{2k}\left(\frac{1}{V^2} - \frac{1}{U^2}\right)$

$$\text{ie } \frac{1}{V^2} - \frac{1}{U^2} = 2kT$$

(ii) $v \frac{dv}{dx} = -kv^3$

$$\frac{dv}{dx} = -kv^2$$
$$\frac{dx}{dv} = -\frac{1}{k} v^{-2}$$

Q5 cont'd

$$x = \int -\frac{1}{k} v^{-2}\, dv$$
$$= \frac{1}{kv} + c$$

At $x=0$, $v=U \rightarrow 0 = \frac{1}{kU} + c$

ie $x = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{U}\right)$

At $x=D$, $v=V \rightarrow D = \frac{1}{k}\left(\frac{1}{V} - \frac{1}{U}\right)$

$\therefore kD = \frac{1}{V} - \frac{1}{U}$

iii) $$\frac{kD}{kT} = \frac{\frac{1}{V} - \frac{1}{U}}{\frac{1}{2}\left(\frac{1}{V^2} - \frac{1}{U^2}\right)}$$

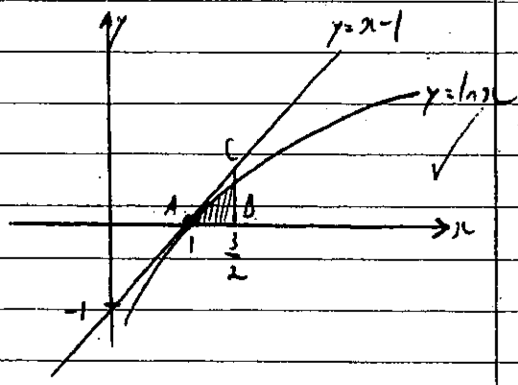
$$= \frac{\frac{U-V}{\cancel{VU}}}{\frac{U^2-V^2}{2V^{\cancel{2}}U^{\cancel{2}}}}$$

$$= \frac{2VU\,\cancel{(U-V)}}{\cancel{(U-V)}(U+V)}$$

$$\therefore \frac{D}{T} = \frac{2UV}{U+V}$$

Q6

(a) (i)



(ii) $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

At $x = 1$, $\frac{dy}{dx} = 1$

Tangent is $y = x - 1$

Shaded Area $= \int_1^{\frac{3}{2}} \ln x$

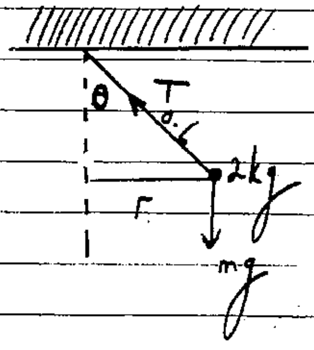
$< \text{area of } \Delta ABC$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$

$\therefore \int_1^{\frac{3}{2}} \ln x \, dx < \frac{1}{8}$

b) (i)



(ii) Vertical $\rightarrow$ $T\cos\theta = 2g = 20$

Horizontal $\rightarrow$ $T\sin\theta = 2r\omega^2$.

$\therefore T \cdot \frac{r}{0.6} = 2r(3\pi)^2$

$= 2(0.6)9\pi^2$

$= 10.8\pi^2$

(iii) $\cos\theta = \frac{20}{T}$

$= \frac{20}{10.8\pi^2}$

$\theta = 79^\circ$

c) $\tan^{-1}5x - \tan^{-1}3x = \tan^{-1}\frac{1}{4}$

$\frac{5x - 3x}{1 + 15x^2} = \frac{1}{4}$

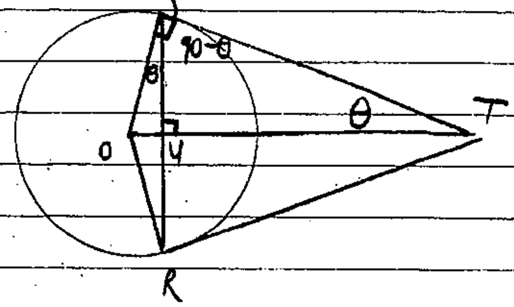
$1 + 15x^2 = 8x$

$15x^2 - 8x + 1 = 0$

$(5x - 1)(3x - 1) = 0$

$x = \frac{1}{5}$ or $\frac{1}{3}$

d)



Let $S\hat{T}U = \theta$

$S\hat{U}T = 90^\circ$ ($OT \perp SR$)

$\therefore U\hat{S}T = 90^\circ - \theta$ ($\angle$ sum of ΔSUT)

$O\hat{S}T = 90^\circ$ (radius $OS \perp$ tangent ST)

$\therefore O\hat{S}U = \theta$

$S\hat{O}U = 90^\circ - \theta$ ($\angle$ sum of ΔSOU)

$\therefore \Delta SOU \,|||\, \Delta TOS$ (equiangular)

$\therefore \frac{OU}{OS} = \frac{OS}{OT}$ (corr. sides in same ratio)

$OU \cdot OT = OS^2 \rightarrow OU \cdot OT = 1$ ($OS = 1$)

Q7

a) (i) Step (1) → For $n=2$,

$LHS = (1+x)^2$

$= 1+2x+x^2$

$RHS = 1+2x$ ✓

Since $x^2>0$, $1+2x+x^2 > 1+2x$.

$\therefore (1+x)^n > 1+nx$ when $n=2$

Step (2) → Assume the statement is true for $n=k$

ie $(1+x)^k > 1+kx$

Step (3) → Prove the statement is true for $n=k+1$

ie $(1+x)^{k+1} > 1+(k+1)x$

$LHS = (1+x)^{k+1}$

$= (1+x)(1+x)^k$

$> (1+x)(1+kx)$ from assumption and since $x>-1 \rightarrow 1+x>0$ ✓

$= 1+kx+x+kx^2$

$> 1+(k+1)x$, since $kx^2>0$ ✓

Step (4) → If the statement is true for $n=2$ and $n=k+1$ it is true for $n=3,4,\ldots$ by mathematical induction

(ii) Let $x=\frac{-1}{2n}$. If $n\geq 2$ then this satisfies conditions in part (i), $x>-1$.

$\therefore (1+x)^n > 1+nx$

$\therefore \left(1-\frac{1}{2n}\right)^n > 1-\frac{n}{2n}$

$= \frac{1}{2}$ ✓

$\therefore \left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n=2,3,\ldots$

(b) (i) Let $y=\sin^{-1}(u)-\sqrt{1-u^2}$, $-1\leq u\leq$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} - \frac{1}{2}\left(1-u^2\right)^{-\frac{1}{2}}\cdot -2u$, ✓ $-1\leq u\leq$

$= \frac{1}{\sqrt{1-u^2}} + \frac{u}{\sqrt{1-u^2}}$

$= \frac{1+u}{\sqrt{1-u^2}} \rightarrow \frac{1+u}{\sqrt{1+u}\cdot\sqrt{1-u}}$

$= \sqrt{\frac{1+u}{1-u}}$ ✓

(ii) $\int_0^{\alpha} \sqrt{\frac{1+u}{1-u}}\,du$

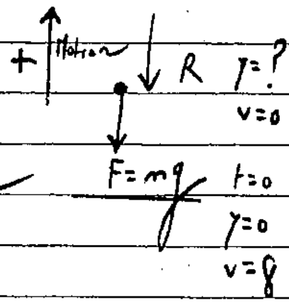
$= \left[\sin^{-1}u - \sqrt{1-u^2}\right]_0^{\alpha}$

$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} - \left(0-\sqrt{1}\right)$ ✓

$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} + 1$, $-1\leq\alpha$

and therefore for $0<\alpha<$

(c) (i) $v\frac{dv}{dy} = -10 - \frac{v^2}{10}$



$\frac{dv}{dy} = \frac{-10}{v} - \frac{v}{10}$

$= -\left(\frac{100+v^2}{10v}\right)$ ✓

$\frac{dy}{dv} = \frac{-10v}{v^2+100}$

$y = -10\int_8^v \frac{v}{v^2+100}\,dv$ ✓

$= -5\left[\ln\left(v^2+100\right)\right]_8^v$

$\frac{-y}{5} = \ln\left(\frac{v^2+100}{164}\right)$ ✓

$\frac{v^2+100}{164} = e^{-\frac{y}{5}}$ ✓

$\therefore v^2 = 164e^{-\frac{y}{5}} - 100$

Q7 cont'd

c) (ii) Max Height → $v=0$

$$y = -5\ln\left(\frac{100}{164}\right)$$

$$= -5\ln 1{\cdot}64 \text{ m}$$

(iii) $\frac{dv}{dt} = -10 - \frac{v^2}{10} \rightarrow \frac{-100-v^2}{10}$

$$\frac{dt}{dv} = \frac{-10}{v^2+100}$$

$$t = -10\int_8^0 \frac{1}{v^2+100}\,dv$$

$$= 10\int_0^8 \frac{1}{v^2+100}\,dv$$

$$= \cancel{10}\cdot\frac{1}{\cancel{10}}\left[\tan^{-1}\frac{v}{10}\right]_0^8$$

$$= \tan^{-1}\left(\frac{4}{5}\right) - 0$$

$\therefore\ t = \tan^{-1}\left(\frac{4}{5}\right)$ seconds

(iv) + ↓ Motion

R

$F=mg$

$t=0$, $y=0$, $v=0$

$t=t$, $y=5\ln 1{\cdot}64$, $v=?$

$$m\ddot{y} = F - R$$

$$2\ddot{y} = 20 - \frac{v^2}{5}$$

$$\ddot{y} = 10 - \frac{v^2}{10}$$

$$v\frac{dv}{dy} = 10 - \frac{v^2}{10}$$

$$\frac{dv}{dy} = \frac{10}{v} - \frac{v}{10}$$

$$\frac{dy}{dv} = \frac{10v}{100-v^2}$$

$$\left[y\right]_0^{5\ln 1{\cdot}64} = 10\int_0^v \frac{v}{100-v^2}\,dv$$

$$5\ln 1{\cdot}64 = \frac{-5}{\cancel{10}\cdot\frac{1}{2}}\left[\ln(100-v^2)\right]_0^v$$

$$-\ln 1{\cdot}64 = \ln(100-v^2) - \ln 100$$

$$\ln(100-v^2) = \ln\left(\frac{100}{1{\cdot}64}\right)$$

$$100 - v^2 = \frac{100}{1{\cdot}64}$$

$$v^2 = 100 - \frac{100}{1{\cdot}64}$$

$$= \frac{64}{1{\cdot}64}$$

$$v = \frac{8}{\sqrt{1{\cdot}64}} \text{ since } v>$$

$\therefore$ Ball is travelling at $6{\cdot}25$ m/s (2dp) when returning to origin.

Q8

a) (i) $LHS = (1-x^2)^{\frac{n-3}{2}} - (1-x^2)^{\frac{n-1}{2}}$

$= (1-x^2)^{\frac{n-3}{2}}\left[1-(1-x^2)^{\frac{2}{2}}\right]$ ✓

$= (1-x^2)^{\frac{n-3}{2}} x^2$

$= RHS$

Hence shown

(ii) Using IBP,

$I_n = \int_0^1 (1-x^2)^{\frac{n-1}{2}} \frac{d}{dx}(x)\, dx$

$u = (1-x^2)^{\frac{n-1}{2}}$ $\quad dv = 1$

$du = \frac{n-1}{2}(1-x^2)^{\frac{n-3}{2}} \cdot -2x\, dx$ $\quad v = x$

$I_n = \int_0^1 (1-x^2)^{\frac{n-1}{2}} dx$

$= \left[x(1-x^2)^{\frac{n-1}{2}}\right]_0^1 - \frac{n-1}{2}\int_0^1 -2x^2(1-x^2)^{\frac{n-3}{2}} \cdot dx$ ✓

$= (n-1)\int_0^1 x^2(1-x^2)^{\frac{n-3}{2}} dx$

$= (n-1)\int_0^1 (1-x^2)^{\frac{n-3}{2}} dx - (n-1)\int_0^1 (1-x^2)^{\frac{n-1}{2}} dx$ ✓ from (i)

$I_n = (n-1)I_{n-2} - (n-1)I_n$ ✓

$\therefore (n-1)I_n + I_n = (n-1)I_{n-2}$

$\therefore nI_n = (n-1)I_{n-2}$

Hence shown

b) $f(x) = x^n e^{-x}$

$f'(x) = (e^{-x})(nx^{n-1}) + (x^n)(-e^{-x})$ ✓

$= x^{n-1}e^{-x}(n-x)$

Stationary point occurs between 2 points of inflexion at $x = n$

$f''(x) = \left[e^{-x}(n-x)\right] n-1 \cdot x^{n-2} + x^{n-1} \cdot \left[(n-x) - e^{-x} + e^{-x}(-1)\right]$ ✓

$= \left[ne^{-x} - xe^{-x}\right] n-1 \cdot x^{n-2} + x^{n-1} \cdot \left[-ne^{-x} + xe^{-x} - e^{-x}\right]$

$= n(n-1)x^{n-2}e^{-x} - (n-1)e^{-x} \cdot x^{n-1} - nx^{n-1}e^{-x} + x^n e^{-x} - e^{-x} \cdot x^{n-1}$

$= n(n-1)x^{n-2}e^{-x} - 2nx^{n-1}e^{-x} + x^n e^{-x}$

$= e^{-x}x^{n-2}\left[n(n-1) - 2nx + x^2\right]$ ✓

$= e^{-x}x^{n-2}\left[x^2 - 2nx + n(n-1)\right]$

$= 0$ when $x^2 - 2nx + n(n-1)$

since $a, b > 0$

$(x-n)^2 = n$

$x - n = \pm\sqrt{n}$

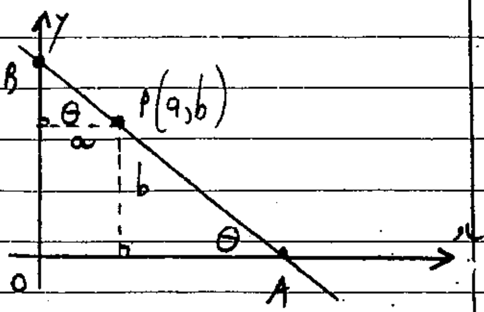
$x = n \pm \sqrt{n}$ ✓

$\therefore a = n - \sqrt{n}$ and $b = n + \sqrt{n}$

c) p/o

Q8 cont'd

c)



(i) $\sin\theta = \frac{b}{AP}$

$AP = \frac{b}{\sin\theta}$

$= b\,\text{cosec}\,\theta$

Similarly $\cos\theta = \frac{a}{PB}$

$PB = a\sec\theta$

$\therefore AB = a\sec\theta + b\,\text{cosec}\,\theta$

(ii) $\frac{d}{d\theta}(AB) = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta$

$= 0$ when

$a\sec\theta\tan\theta = b\,\text{cosec}\,\theta\cot\theta$

$\frac{a}{b} = \frac{\text{cosec}\,\theta\cot\theta}{\sec\theta\tan\theta}$

$= \frac{\cot\theta}{\tan^2\theta}$

$= \cot^3\theta$

$\therefore \cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$

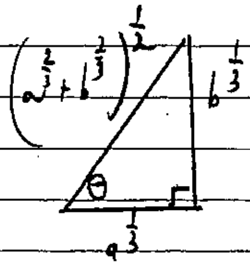
$\frac{d^2}{d\theta^2}AB = a\sec^3\theta + a\sec\theta\tan^2\theta + b\,\text{cosec}\,\theta\cot^2\theta + b\,\text{cosec}^3\theta$

since $0 \leq \theta \leq \frac{\pi}{2}$ and all trig. ratios > 0 then

$\frac{d^2}{d\theta^2}AB > 0$ ∪

so $\cot\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$ minimises AB

iii. $\cot\theta = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \rightarrow \theta$ is acute



$r^2 = a^{\frac{2}{3}} + b^{\frac{2}{3}}$

$r = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}$

$\therefore \sec\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}$

$\text{cosec}\,\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

$\therefore$ Minimum length of AB

$= a.\frac{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} + b.\frac{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}$

$= \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)$

$= \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Hence shown