VIRILE AGITUR

Student Number

Knox Grammar School

2010

Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time - 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Subject Teachers
Mr M Vuletich
Mrs J Harnwell

Total Marks – 120

- Attempt Questions 1 – 8
- Answer each question in a separate writing booklet
- All questions are of equal value

This paper MUST NOT be removed from the examination room

Number of Students in Course: 29

Number of Writing Booklets Per Student (Eight Page) 8

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Total Marks – 120
Attempt Questions 1 – 8
All Questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Find the indefinite integral for :-

$$\int \frac{1}{1+e^x}\, dx. \qquad 2$$

(b)

(i) Find real numbers a and b such that for all values of t, **1**

$$\frac{1}{(2-t)(1+2t)} = \frac{a}{2-t} + \frac{b}{1+2t}$$

(ii) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ and the identity in part (i) to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta} \quad .$$

(c)

(i) Show that the indefinite integral for :- **2**

$$\int \frac{x^3}{x^2+1}\, dx = \frac{1}{2}x^2 - \frac{1}{2}\log_e(x^2+1) + c\,.$$

(Hint: degree of numerator is greater than degree of denominator)

(ii) By first integrating using parts and then using the result from part i) above evaluate the definite integral:- **2**

$$\int_0^1 x^2 \tan^{-1} x\, dx$$

Question 1 continues on page 3

(d) Find the indefinite integral for :-

$$\int \frac{x+3}{\sqrt{x^2-2x+5}}\,dx$$ **4**

End of Question 1

Question 2 (15 Marks) Use a SEPARATE writing booklet.

a) Given $z = -\sqrt{3} + i$.

(i) Write z in modulus – argument form. **1**

(ii) Hence find z^8 in the form $x + iy$ where x and y are real. **2**

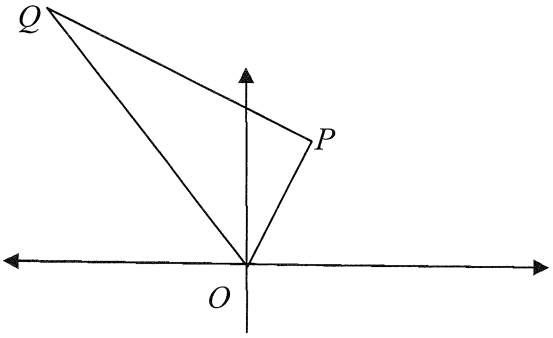
(iii) Find the least positive value of n such that z^n is real. **2**

b) Sketch each of the following regions on a separate Argand diagram.

(i) $|z - 2 - i| \leq 2$. **2**

(ii) $0 \leq \arg\left[(1+i)z\right] \leq \frac{\pi}{2}$. **2**

c)



The diagram shows a complex plane with origin O. The points P and Q represent arbitrary non-zero complex numbers z and w respectively.

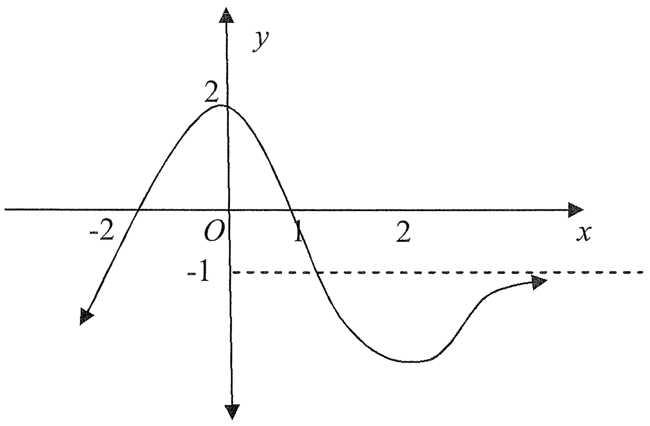
If $|z - w| = |z + w|$, what can be said about the complex number $\frac{w}{z}$? **3**

d) Find all numbers z such that $z^5 = 4 + 4i$, giving your answer in modulus-argument form. **3**

End of Question 2

Question 3 (15 Marks) Use a SEPARATE writing booklet.

a) The diagram shows the graph $y = f(x)$.



Draw separate one- third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f(\frac{1}{x})$ **2**

b) For the curve defined by $3x^2 + y^2 - 2xy - 8x + 2 = 0$, find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$. **3**

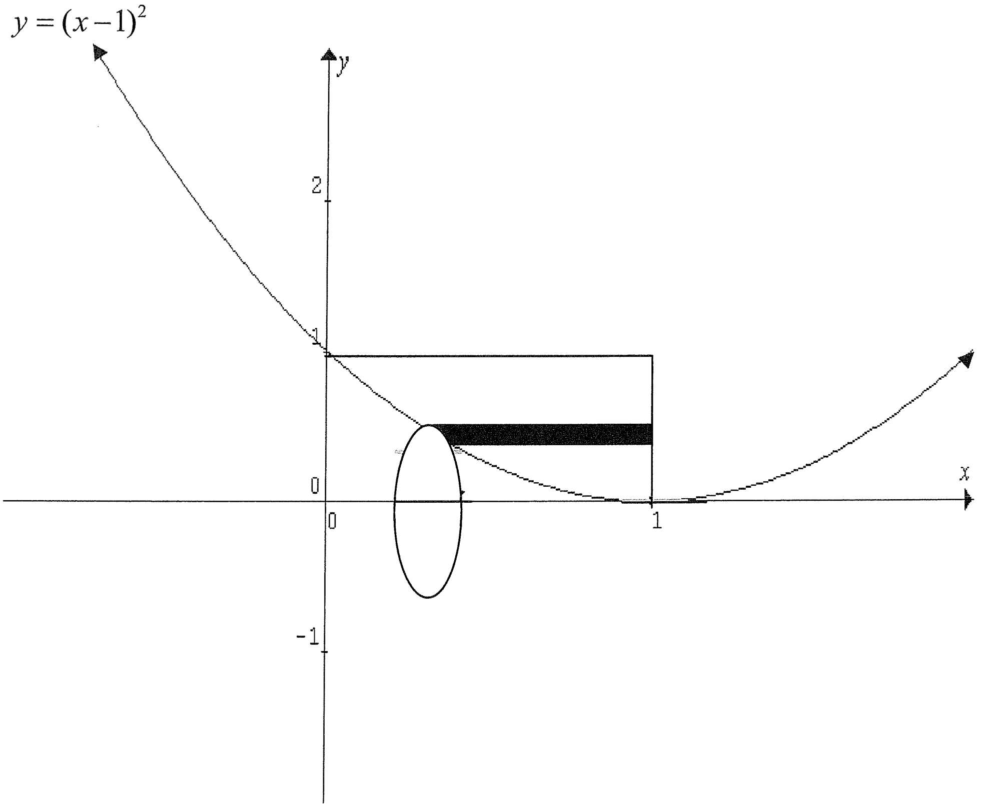
c) If the equation $x^3 + 3mx + n = 0$, where m and n are constants, has a double root, then prove that $n^2 = -4m^3$. **3**

Question 3 continues on page 6

Question 3 (continued)

d)

The region bounded by the curve $y=(x-1)^2$, $x=1$, and $y=1$ is rotated about the x-axis.

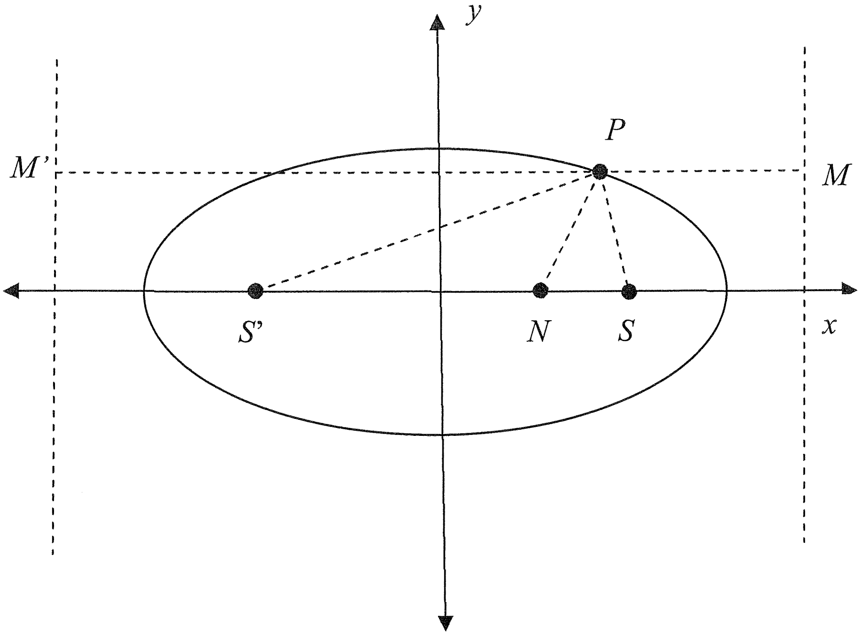


Use the method of cylindrical shells to find the volume (in terms of π) of the solid described above. **3**

End of Question 3

Question 4 (15 Marks) Use a SEPARATE writing booklet.

a)



The point $P(2\cos\theta, \sqrt{3}\sin\theta)$ lies on the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$. The normal at P cuts the x-axis at N. M and M' are points on the directrices perpendicular to P.

(i) Determine the coordinates of the foci S and S' of the ellipse. **2**

(ii) Show that the equation of the normal to the ellipse at P is

$$y-\sqrt{3}\sin\theta=\frac{2\sqrt{3}\sin\theta}{3\cos\theta}(x-2\cos\theta).$$ **2**

(iii) Hence show that N has coordinates $(\frac{1}{2}\cos\theta, 0)$ **1**

(iv) Prove that $\frac{S'P}{SP}=\frac{S'N}{SN}$ and hence, or otherwise, prove that the normal PN bisects the $\angle S'PS$. **4**

Question 4 continues on page 8

Question 4 (continued)

b)

A body of mass m is projected vertically upwards from the ground with speed u_0. The force due to gravity acting on the body is constant but there is a resisting force of magnitude mkv^2 at speed v.

(i) Show that the maximum height H which the body reaches is given by

$H = \frac{1}{2k}\ln(\frac{g+ku_0^2}{g})$, where g is the acceleration due to gravity. **3**

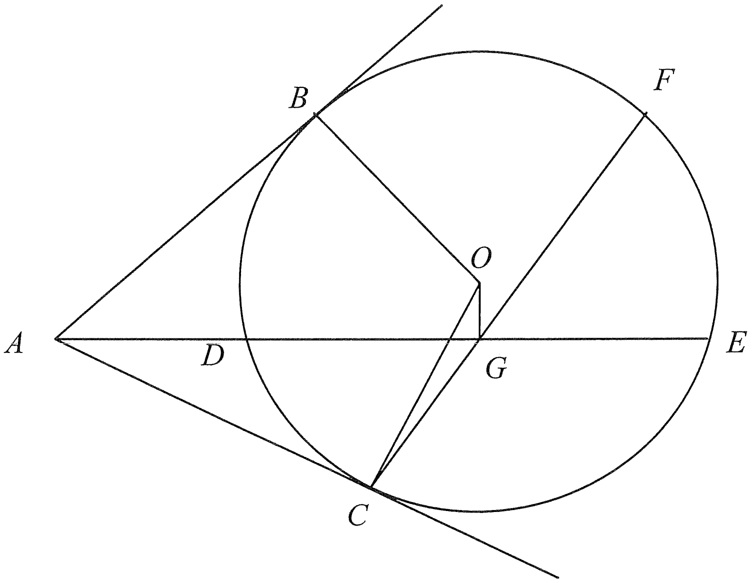
(ii) Show that the speed v_0 with which the body reaches the ground is given by

$2kH = \ln(\frac{g}{g-kv_0^2})$. **3**

End of Question 4

Question 5 (15 Marks) Use a SEPARATE writing booklet.

a) In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C respectively. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.



(i) Copy the diagram onto your answer sheet and prove that $ABOC$ and $AOGC$ are cyclic quadrilaterals. **3**

(ii) Explain why $\angle OGF = \angle OAC$. **1**

(iii) Prove that $BF \parallel AE$. **3**

b) Let $f(x) = \dfrac{x^2 - 1}{x + 2}$.

(i) Find all the asymptotes of $f(x)$. **2**

(ii) Sketch the curve showing asymptotes and the x and y intercepts. **2**

(There is no need to find or label stationary points)

Question 5 continues on page 10

Question 5 (continued)

c) If a polynomial $P(x)$ is divided by $(x-a)(x-b)$ so that the remainder $R(x)$ is obtained.

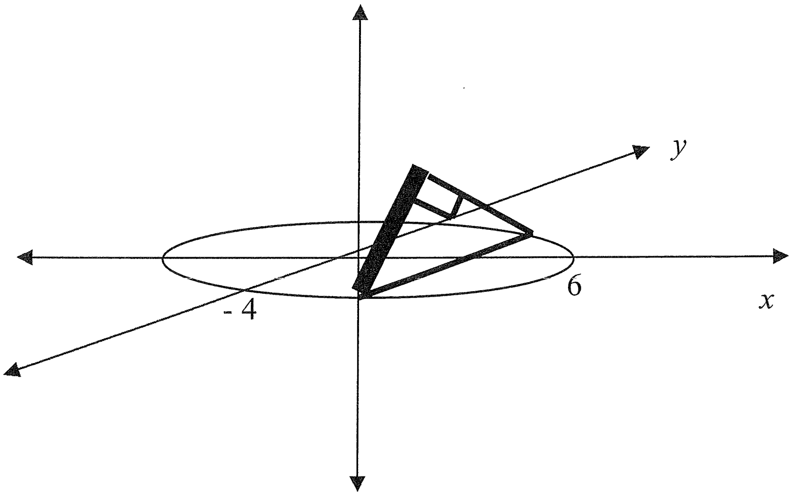
(i) Explain why the remainder is of the form $rx+s$, where r and s are constants. **1**

(ii) Hence show that the remainder $R(x)=(\frac{P(a)-P(b)}{a-b})x+(\frac{aP(b)-bP(a)}{a-b})$. **3**

End of Question 5

Question 6 (15 Marks) Use a SEPARATE writing booklet.

a) The base of a solid is the area of a region bounded by the ellipse whose equation is $4x^2 + 9y^2 = 144$ (note: its cuts the x-axis at 6 and –6 and the y-axis at 4 and –4). Each cross-section of the solid formed by a plane perpendicular to the x and y plane is an isosceles right angled triangle with its hypotenuse in the x and y plane. **4**



Find the volume of this solid.

b) (i) Use de Moivre's Theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **1**

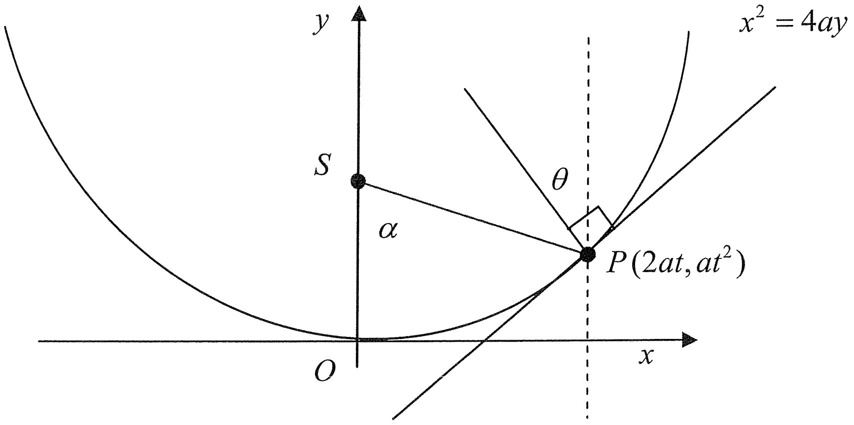
(ii) Deduce that has $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \frac{1}{2}$. **2**

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$. **2**

(iv) Hence evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$. **2**

Question 6 continues on page 12

c) P is the point $(2at, at^2), 0 < t < 1$, on the parabola $x^2 = 4ay$ with focus S. The normal to the parabola at P makes an angle θ with the vertical through P, While the focal chord PS makes an angle α with the vertical.



(i) Show that $\tan\theta = t$ **1**

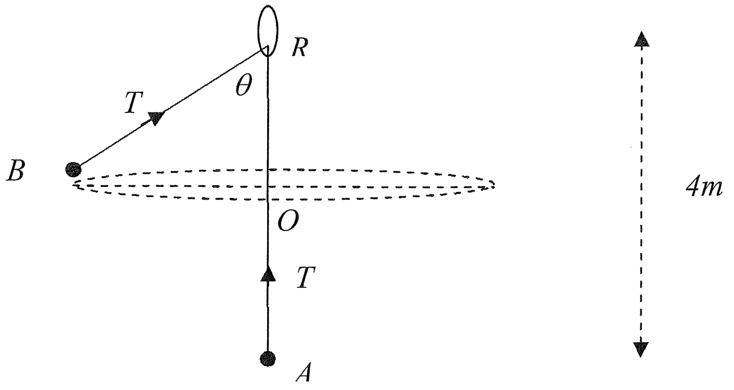
(ii) Find the gradient PS and hence, or otherwise, prove that $\alpha = 2\theta$. **2**

(iii) If l is the distance PS, show that $l\cos^2\theta = a$ **1**

End of Question 6

Question 7 (15 Marks) Use a SEPARATE writing booklet.

(a) Two particles of mass 4kg and 6kg are attached at either end of a light inextensible string of length 7 metres, which pass through a small vertical frictionless ring R. The heavier particle A hangs vertically at a distance of 4 metres below the ring while the other particle B describes a horizontal circle whose centre is O. Let θ be the acute angle which particle B makes with the vertical. Let T be the tension force of the string.



(i) Resolve all forces at A and B. **1**

(ii) Find the distance OR and the radius OB, of the horizontal circle. **3**

(iii) Find the angular velocity, ω, of B about O in revolutions / minute to two decimal

places (use $g = 9.8 \ m^2$) . **3**

b) For positive real numbers a , b and c.

(i) Show that $a + \frac{1}{a} \geq 2$. **1**

(ii) Show that $(a+b)(\frac{1}{a}+\frac{1}{b}) \geq 4$. **2**

(iii) Hence, or otherwise, show that $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9$. **2**

Question 7 continues on page 14

Question 7 (continued)

c) From a set of objects of which two are white and the rest are black, four objects are taken at random without replacement. The probability that both white objects will be chosen is twice the probability that neither white object will be chosen.

Let *n* be the number of objects.

(i) Find the probability in terms of *n* that both white object will be chosen. **1**

(ii) Find the probability in terms of *n* that neither white object will be chosen. **1**

(iii) Hence find the number of objects. **1**

End of Question 7

Question 8 (15 Marks) Use a SEPARATE writing booklet.

a) Consider the roots of $z^n - 1 = 0$. These roots are plotted on an Argand diagram. The points represented by these roots are joined to form a regular n-sided polygon.

(i) Show that the area of this polygon is given by $A_n = \frac{n}{2}\sin\frac{2\pi}{n}$. **2**

(ii) Show that the perimeter of the polygon is given by $P_n = 2n\sin\frac{\pi}{n}$. **2**

(iii) Show that $P_n > 2A_n$ for all positive integers n. **2**

(iv) Prove that the $\lim\limits_{n\to\infty} A_n = \pi$. **2**

(v) Find the $\lim\limits_{n\to\infty} P_n$ **1**

b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

(i) Show that $I_n = (\frac{n-1}{n})I_{n-2}$, for $n \geq 2$. **3**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ **3**

End of Question 8

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Year 12 2010 Extension 2 Mathematics Trial HSC Assessment Task 4

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q1 (a) $\int \frac{1}{1+e^x}\,dx$ $= \int \frac{1+e^x-e^x}{1+e^x}\,dx$ $= \int 1 - \frac{e^x}{1+e^x}\,dx$ ✓ $= x - \ln(1+e^x) + C$ ✓	NOTE: Can be done by parts, but much longer.	$= \int_0^1 \frac{dt}{(2-t)(1+2t)}$ ✓ $= \frac{1}{5}\left\{ \int_0^1 \frac{1}{2-t} + 2\int_0^1 \frac{1}{1+2t} \right\}$ from part (i) $= \left[-\frac{1}{5}\ln(2-t) + \frac{1}{5}\ln(1+2t) \right]_0^1$ $= \left[\frac{1}{5}\ln\left(\frac{1+2t}{2-t}\right) \right]_0^1$ ✓ $= \frac{1}{5}\ln 3 - \frac{1}{5}\ln(\frac{1}{2})$ $= \frac{1}{5}\ln 6$ ✓	
(b) (i) $\frac{1}{(2-t)(1+2t)} = \frac{a}{2-t} + \frac{b}{1+2t}$ $\therefore 1 = a(1+2t) + b(2-t)$ $t=2$ $1 = 5a \Rightarrow a = \frac{1}{5}$ $t=-\frac{1}{2}$ $1 = \frac{5}{2}b \Rightarrow b = \frac{2}{5}$ ✓ (ii) $t = \tan\frac{\theta}{2}$ $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(\tan^2\frac{\theta}{2}+1) = \frac{1}{2}(t^2+1)$ $\therefore \frac{2dt}{t^2+1} = d\theta$ $\theta = \frac{\pi}{2}$ $t = \tan\frac{\pi}{4} = 1$ $\theta = 0$ $t = 0$ $\therefore I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta}$ $= \int_0^1 \frac{2dt}{3(\frac{2t}{1+t^2}) + 4(\frac{1-t^2}{1+t^2})} \times \frac{1}{(t^2+1)}$ ✓ $= \int_0^1 \frac{2\,dt}{6t+4-4t^2}$ $= \int_0^1 \frac{dt}{3t+2-2t^2}$		(c) (i) $\begin{array}{r} x \\ x^2+1\,\overline{)\,x^3} \\ \underline{x^3+x} \\ -x \end{array}$ $\therefore \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$ ✓ $\therefore \int \frac{x^3}{x^2+1}\,dx = \int x - \frac{x}{x^2+1}\,dx$ $= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C$ ✓ (ii) $\int_0^1 x^2 \tan^{-1}x\,dx$ $= \left[\frac{x^3}{3}\tan^{-1}x \right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx$ ✓ $= \frac{\pi}{12} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) \right]_0^1$ $= \frac{\pi}{2} - \frac{1}{3}\left[\frac{1}{2} - \frac{1}{2}\ln 2 \right]$ $= \frac{\pi}{2} + \frac{1}{6}\ln 2 - \frac{1}{6}$ ✓ $= \frac{1}{6}(3\pi + \ln 2 - 1)$	

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Suggested Solution (s) | **Comments**

Q1(d) $\int \frac{x+3}{\sqrt{x^2-2x+5}}\,dx$

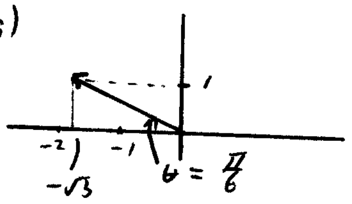
$= \frac{1}{2}\int \frac{2x-2+8}{\sqrt{x^2-2x+5}}\,dx$ ✓

$= \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+5}}\,dx + \frac{1}{2}\int \frac{8}{\sqrt{x^2-2x+5}}\,dx$ ✓

$= \sqrt{x^2-2x+5} + 4\int \frac{dx}{\sqrt{(x-1)^2+2^2}}$ ✓

$= \sqrt{x^2-2x+5} + 4\ln\left[(x-1)+\sqrt{(x-1)^2+4}\right] + C.$ ✓

Q2 (a)



(i) $\arg(z) = \frac{5\pi}{6}$ $|z| = 2$

$\therefore z = 2\,\text{cis}\,\frac{5\pi}{6}$ ✓

(ii) $z^8 = 2^8\,\text{cis}\,\frac{40\pi}{6}$

$= 256\,\text{cis}\,\frac{20\pi}{3}$ ✓ by de Moivre's Thm.

$= 256\,\text{cis}\,\frac{2\pi}{3}$

$= 256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$= -128 + 128\sqrt{3}\,i$ ✓

(iii) $z^n = 2^n\,\text{cis}\,\frac{5n\pi}{6}$

if z^n real $\Rightarrow \sin\frac{5n\pi}{6} = 0$ ✓

$\Rightarrow \frac{5n\pi}{6} = k\pi$ k integer.

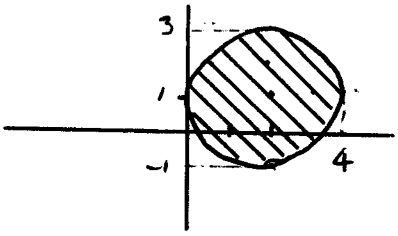
$\Rightarrow \frac{5n}{6} = k$

ie $k = 6$ is least value ✓

(b)

(i) $|z-2-i| \le 2$

$|z-(2+i)| \le 2$

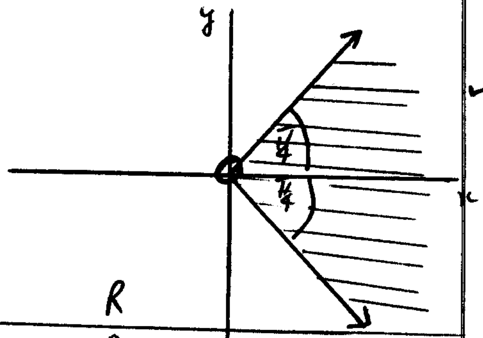


(ii) $0 \le \arg[(1+i)z] \le \frac{\pi}{2}$

$0 \le \arg(1+i) + \arg(z) \le \frac{\pi}{2}$

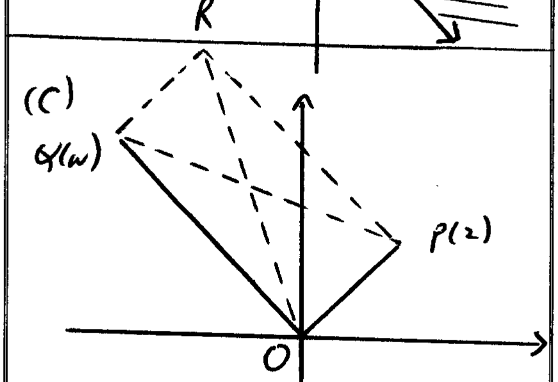
$0 \le \frac{\pi}{4} + \arg(z) \le \frac{\pi}{2}$

$\therefore -\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{4}$ ✓



✓

(c)



If $|z-w| = |z+w|$

$\Rightarrow$ diagonals of quadrilateral OPRQ are equal

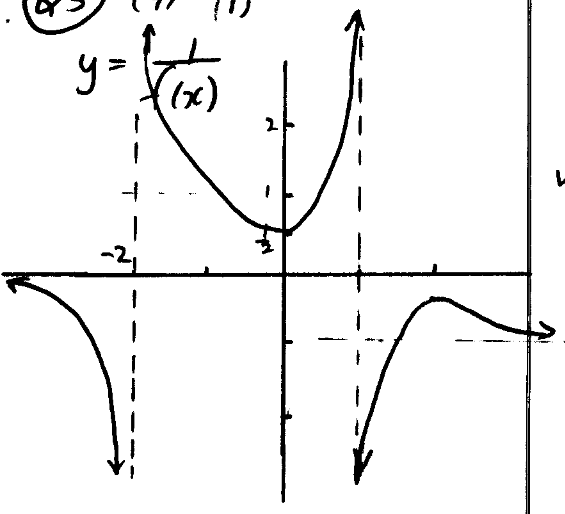
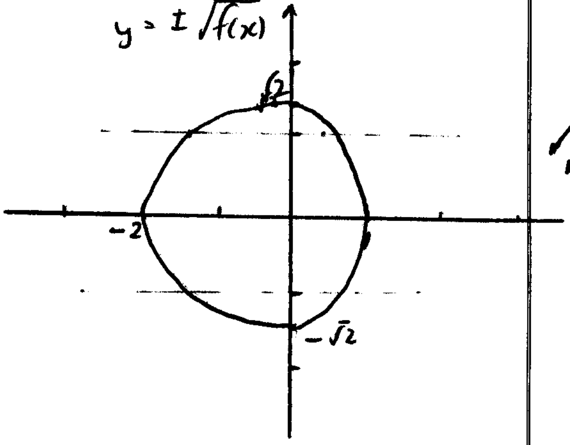
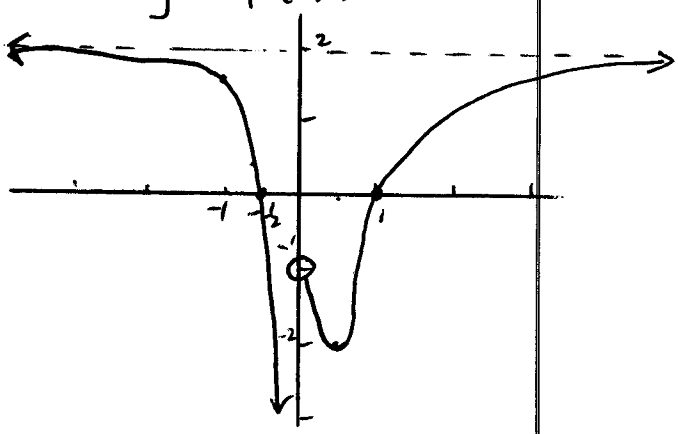
$\Rightarrow$ OPRQ is at least a rectangle ✓

$\Rightarrow \angle QOP = 90^\circ$

$\therefore \arg(w) - \arg(z) = \frac{\pi}{2}.$

$\arg\left(\frac{w}{z}\right) = \frac{\pi}{2}$ ✓

$\Rightarrow \frac{w}{z}$ is purely imaginary ✓

Suggested Solution (s)	**Comments**	**Suggested Solution (s)**	**Comments**
(c) $z^5 = 4+4i = 4\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}+2k\pi\right)$ ✓ Let $z = r\ \text{cis}\ \theta$ $z^5 = r^5\ \text{cis}\ 5\theta$ by de Moivre's Thm $\Rightarrow z^5 = 4\sqrt{2}$ $\Rightarrow z = \sqrt{2}$ $5\theta = \frac{\pi}{4} + 2k\pi \quad k = 0, 1, \ldots 4$ $\theta = \frac{\pi}{20} + \frac{2k\pi}{5} = \frac{\pi + 8k\pi}{20}$ $\therefore k=0 \quad \theta = \frac{\pi}{20} \quad z_1 = \sqrt{2}\ \text{cis}\frac{\pi}{20}$ $k=1 \quad \theta = \frac{9\pi}{20} \quad z_2 = \sqrt{2}\ \text{cis}\frac{9\pi}{20}$ $k=2 \quad \theta = \frac{17\pi}{20} \quad z_3 = \sqrt{2}\ \text{cis}\frac{17\pi}{20}$ $k=3 \quad \theta = \frac{25\pi}{20} \quad z_4 = \sqrt{2}\ \text{cis}\frac{25\pi}{20}$ $k=4 \quad \theta = \frac{33\pi}{20} \quad z_5 = \sqrt{2}\ \text{cis}\frac{33\pi}{20}$ <u>Alternatively using</u> $-\pi \leq \arg z \leq \pi$ $\theta = \frac{\pi + 8k\pi}{20} \quad k = 0, \pm 1, \pm 2$ $k=0 \quad z_1 = \sqrt{2}\ \text{cis}\frac{\pi}{20}$ $k=1 \quad z_2 = \sqrt{2}\ \text{cis}\frac{9\pi}{20}$ $k=-1 \quad z_3 = \sqrt{2}\ \text{cis} -\frac{7\pi}{20}$ $k=2 \quad z_4 = \sqrt{2}\ \text{cis}\frac{17\pi}{20}$ $k=-2 \quad z_5 = \sqrt{2}\ \text{cis} -\frac{15\pi}{20}$ $= \sqrt{2}\ \text{cis} -\frac{3\pi}{4}$	✓ $k=0,1,2,3,4$ ✓ ✓	Q3 (a) (i)  (ii) $y^2 = f(x) \quad \therefore f(x) \geq 0$ $y = \pm\sqrt{f(x)}$  (iii) $y = f\left(\frac{1}{x}\right)$ 	✓✓ ✓✓

Year 12 2010 Extension 2 Mathematics Trial HSC Assessment Task 4

Suggested Solution (s) | **Comments**

Q3 b)

$3x^2+y^2-2xy-8x+2=0$

$6x+2y\frac{dy}{dx}-2(y+x\frac{dy}{dx})-8=0$

$6x+2y\frac{dy}{dx}-2y-2x\frac{dy}{dx}-8=0$

$\frac{dy}{dx}(2y-2x)=8+2y-6x$

$\frac{dy}{dx}=\frac{8+2y-6x}{2(y-x)}$

$=\frac{4+y-3x}{y-x}$ ✓

$\therefore \frac{4+y-3x}{y-x}=2$

$4+y-3x=2y-2x$

$4-x=y$ ✓

$\therefore 3x^2+(4-x)^2-2x(4-x)-8x+2=0$

$6x^2-24x+18=0$

$\Rightarrow (x-3)(x-1)=0$

$x=1,3 \quad y=3,1$

∴ point $(1,3)$ and $(3,1)$ ✓

(c) let $p(x)=x^3+3mx+n=0$

$p'(x)=3x^2+3m$

$p'(x)=0 \Rightarrow x^2=-m$ ✓

$p(x)=0 \Rightarrow x(x^2+3m)=-n$ ✓

$x^2(x^2+3m)^2=n^2$ ✓

$-m(-m+3m)^2=n^2$

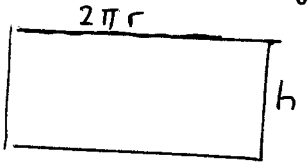
$-m(2m)^4=n^2$

$-4m^3=n^2$

$n^2=-4m^3$

Suggested Solution (s) | **Comments**

(d) $\delta V=2\pi r h\,\delta y.$



$\delta V=2\pi y(1-x)\,dy.$

$=2\pi y(1-(1-\sqrt{y}))\,dy$ ✓

as $(x-1)=\pm y \quad x=1\pm\sqrt{y}$

ie $x=1-\sqrt{y}.$ ✓

$V=2\pi\int_0^1 y(\sqrt{y})\,dy$

$=2\pi\int_0^1 y^{3/2}\,dy$

$=2\pi\left[\frac{2y^{5/2}}{5}\right]_0^1$

$=\frac{4\pi}{5}u^3.$ ✓

Q4 (a) $\frac{x^2}{4}+\frac{y^2}{3}=1$

(i) $\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{3})^2}=1 \Rightarrow a=2,\ b=\sqrt{3}$

$b^2=a^2(1-e^2)$

$\Rightarrow e^2=\frac{1}{4} \Rightarrow e=\frac{1}{2}$ ✓

$S=(ae,0)=(1,0)$

$S'=(-ae,0)=(-1,0)$ ✓

(ii) $\frac{2x}{4}+\frac{2y}{3}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{3x}{4y}$ ✓

at P $\frac{dy}{dx}=\frac{-6\cos t}{4\sqrt{3}\sin t}$

$\therefore$ grad of normal is $\frac{4\sqrt{3}\sin t}{6\cos t}$

$=\frac{2\sqrt{3}\sin t}{3\cos t}$ ✓

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Suggested Solution (s) | **Comments**

∴ Q4 a(ii) continued

∴ Equat of normal is

$$y - \sqrt{3}\sin t = \frac{2\sqrt{3}\sin\theta}{3\cos\theta}(x - 2\cos t)$$

(iii) Let $y=0$ in (ii)

$$-\sqrt{3}\sin t = \frac{2\sqrt{3}\sin t}{3\cos t}(x - 2\cos t)$$

$$\therefore x = \frac{-\sqrt{3}\sin t \times 3\cos t}{2\sqrt{3}\sin t} + 2\cos t \quad ✓$$

$$= 2\cos t - \frac{3}{2}\cos t$$

$$= \frac{1}{2}\cos t$$

∴ N has co-ords $(\frac{1}{2}\cos t, 0)$

(iv) $SP = e\,PM = e(4 - 2\cos t) = 2e(2 - \cos\theta)$

$S'P = e\,PM' = e(4 + 2\cos t) = 2e(2 + \cos\theta)$

as $x = \pm\frac{a}{e} = \pm 4$ are the directrices

$$\therefore \frac{S'P}{SP} = \frac{2 - \cos t}{2 + \cos t}$$

$S'N = 1 - \frac{1}{2}\cos t$

$SN = 1 + \frac{1}{2}\cos t$ ✓

$$\therefore \frac{S'N}{SN} = \frac{1 - \frac{1}{2}\cos\theta}{1 + \frac{1}{2}\cos\theta} = \frac{2 - \cos\theta}{2 + \cos t} = \frac{S'P}{SP}$$

Using the sine Rule in Δ's PNS and PNS'

$$\frac{\sin(\angle NPS)}{SN} = \frac{\sin(\angle SNP)}{SP}$$

$$\therefore \sin(\angle NPS) = \sin(\angle SNP) \times \frac{SN}{SP} \quad ✓$$

and

$$\frac{\sin(\angle S'PN)}{S'N} = \frac{\sin(180^\circ - \angle PNS)}{PS'}$$

$$\therefore \sin(\angle S'PN) = \sin(\angle PNS) \times \frac{S'N}{PS'} \quad ✓$$

as $\frac{S'N}{SN} = \frac{S'P}{SP}$ from above

$$\frac{S'N}{S'P} = \frac{SN}{SP}$$

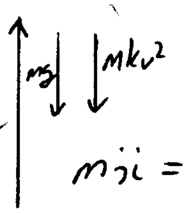
$\therefore \sin(\angle S'PN) = \sin(\angle NPS)$

$\angle S'PN = \angle NPS$

ie PN bisects $\angle S'PS$

(b)

(i)



$$m\ddot{x} = -mg - mkv^2$$

$$\ddot{x} = -g - kv^2 \quad ✓$$

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = -\frac{(g + kv^2)}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\int_0^H dx = -\int_{u_0}^0 \frac{v}{g + kv^2}\,dv$$

$$[x]_0^H = \left[-\frac{1}{2k}\ln(g + kv^2)\right]_{u_0}^0 \quad ✓$$

$$H = -\frac{1}{2k}\left[\ln g - \ln(g + ku_0^2)\right] \quad ✓$$

$$= \frac{1}{2k}\ln\left(\frac{g + ku_0^2}{g}\right)$$

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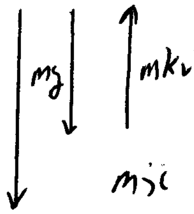
Q4 b (ii)



$m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

$v\frac{dv}{dx} = g - kv^2$ ✓

$\frac{dv}{dx} = \frac{g-kv^2}{v}$

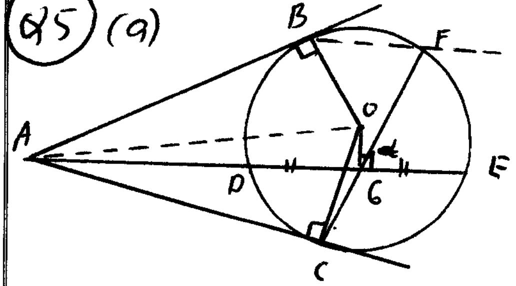
$\int_0^H dx = \int_0^{V_0} \frac{v}{g-kv^2}\, dv$

$[x]_0^H = \left[-\frac{1}{2k}\ln(g-kv^2)\right]_0^{V_0}$ ✓

$\therefore H = \frac{-1}{2k}\left[\ln(g-kv_0^2) - \ln g\right]$ ✓

$2kH = \ln\left(\frac{g}{g-kv_0^2}\right)$

Q5 (a)



(i) ∠ABO = ∠OCA = 90° (tangents perp. to radii) ✓

∴ ABOC is cyclic quad. (opp ∠'s supple.)

OG ⊥ DE ie ∠OGD = 90° ✓

(DG = GD, midpoint of chord perp. to centre)

∴ ∠OGA = ∠OCA = 90°

∴ AOGC is cyclic (either ∠'s in semi circle are 90° OR ∠'s at circumference standing on same arc AO are equal) ✓

(ii) ∠OGF = ∠OAC

(ext ∠. of cyclic quad AOGC = interior opp ∠) ✓

(iii) Let ∠FGE = α

∴ ∠OGF = 90 − α (adj. complem. ∠'s)

∴ ∠OAC = 90 − α from (ii) above

∴ ∠AOC = α (∠ sum of △AOC) ✓

Now △ABO ≡ △AOC (SSS)

OB = OC (equal radii)

AB = AC (tangents from ext. point equal) ✓

OA common

⇒ ∠AOB = ∠AOC = α

⇒ ∠BOC = 2α

⇒ ∠BFG = α (angles at circumference is ½ angle at centre) ✓

∴ ∠BFG = ∠FGE

BF ∥ AE (alt. ∠'s equal)

(b) Let $f(x) = \frac{x^2-1}{x+2} = \frac{(x-1)(x+1)}{x+2}$

(i) $\therefore x \neq -2$

$f(x) = x - 2 + \frac{3}{x+2}$

$\lim_{x\to\pm\infty} f(x) \longrightarrow x-2$

∴ vertical asymptote $x = -2$ ✓

oblique asymptote is $y = x-2$ ✓

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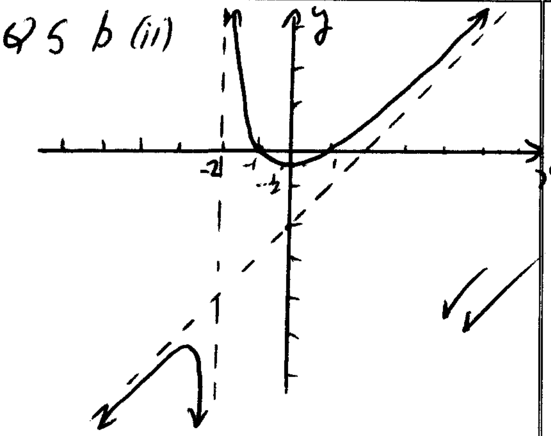
Suggested Solution (s) | **Comments**

Q5 b (ii)



✓✓

C (i) $P(x) = (x-a)(x-b)Q(x) + R(x)$

$\deg R(x) < \deg[(x-a)(x-b)]$

$< \deg 2.$ ✓

$\therefore \deg R(x)$ is at most 1.

ie $R(x) = rx + s$, r, s constants

(ii)

$P(x) = (x-a)(x-b)Q(x) + rx + s$

$P(a) = ar + s$ ··· ①

$P(b) = br + s$ ··· ② ✓

$\therefore P(a) - P(b) = r(a-b)$

$r = \frac{P(a)-P(b)}{a-b}$ ✓

from ① $s = P(a) - ar$

$= P(a) - a\left[\frac{P(a)-P(b)}{a-b}\right]$

$= \frac{aP(a) - bP(a) - aP(a) + aP(b)}{a-b}$

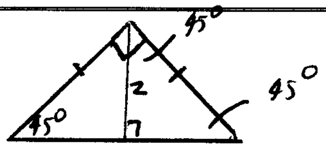
$= \frac{aP(b) - bP(a)}{a-b}.$ ✓

$\therefore R(x) = \left(\frac{P(a)-P(b)}{a-b}\right)x + \frac{aP(b) - bP(a)}{a-b}$

Suggested Solution (s) | **Comments**

Q6

(a)



$\delta V = \frac{1}{2}(2y)z\ \delta x$

but $z = y$.

$\therefore \delta V = y^2\ \delta x$

$= \left(\frac{144 - 9x^2}{9}\right)\delta x$ ✓

$V = \frac{2}{9}\int_0^6 144 - 9x^2$

$= \frac{2}{9}\left[144x - \frac{9x^3}{3}\right]_0^6$ ✓

$= \frac{2}{9}[864 - 288]$

$= \frac{1152}{9} = 128\ u^3$ ✓

(b)

(i)

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ by de Moivre's Thm.

$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$ ✓

$= 4\cos^3\theta - 3\cos\theta$

(ii) $8x^3 - 6x - 1 = 0$

Let $x = \cos\theta$

$\therefore 8\cos^3\theta - 6\cos\theta = 1$ ✓

$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$ ✓

$\therefore \cos 3\theta = \frac{1}{2}$ from (i)

OR — See next page

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Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q6 b (ii) Since $\cos 3\theta = \frac{1}{2}$ $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$ from (i) $8\cos^3\theta - 6\cos\theta = 1$ when $x = \cos\theta$ this becomes $8x^3 - 6x - 1 = 0$ (iii) $\cos 3\theta = \frac{1}{2}$ from (ii) $3\theta = 2n\pi \pm \frac{\pi}{3}$ $n = 0, \pm 1$ $\theta = \frac{\pi}{9}(6n \pm 1)$ $n = 0, \pm 1$ Hence Solution are $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$ (iv) $\left.\begin{array}{l}\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9} \\ \cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}\end{array}\right\}$ Product of roots gives $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$ Hence $\cos\frac{\pi}{9} \times \cos\frac{2\pi}{9} \times \cos\frac{4\pi}{9} = \frac{1}{8}$	✓ ✓ ✓ ✓ ✓ ✓	Hence $\tan\theta = \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $= \frac{2at}{2a} = t$ ie $\tan\theta = t$ (ii) Gradient PS $= \tan(\frac{\pi}{2} + \alpha) = \frac{a(1-t^2)}{-2at}$ ie $-\cot\alpha = \frac{1-t^2}{-2t}$ $\tan\alpha = \frac{2t}{1-t^2} = \tan 2\theta$ $\therefore \alpha = 2\theta$ (iii) PS = distance from P to directrix $y = -a$ $\therefore PS = a + at^2 = a(1+t^2)$ $= a(1+\tan^2\theta)$ $= a\sec^2\theta$ $\therefore l = a\sec^2\theta$ $l\cos^2\theta = a$	✓ ✓ ✓ ✓
(c) (i) The normal at P makes an angle θ with the vertical. $\therefore$ the tangent at P makes an angle of θ with the horizontal			

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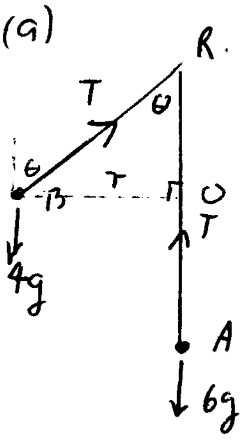
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q7 (a) (i)  At A $T = 6g \cdots ①$ At B Vertically $T\cos\theta = 4g \cdots ②$ Radially $mr\omega^2 = T\sin\theta \cdots ③$ (ii) $\cos\theta = \frac{OR}{3}$ from ① & ② $\cos\theta = \frac{4g}{T} = \frac{4g}{6g} = \frac{2}{3}$ $\Rightarrow OR = 2.$ $\therefore OB = \sqrt{3^2-2^2} = \sqrt{5}$ (iii) $mr\omega^2 = T\sin\theta$ from ③ $= 6g\sin\theta$ from ① $\therefore \omega^2 = \frac{6g\sin\theta}{4\sqrt{5}}$	✓ ✓ ✓ ✓ ✓	$\sin\theta = \frac{OB}{BR} = \frac{\sqrt{5}}{3}$ $\therefore \omega^2 = \frac{6g\sqrt{5}}{12\sqrt{5}} = \frac{g}{2}$ $= \frac{9.8}{2}$ $= 4.9$ $\omega = \sqrt{4.9}$ rad/s $= \sqrt{4.9} \times \frac{60}{2\pi}$ rev/minute $\doteqdot 21.14$ rev/minute (b) (i) $a > 0$ $\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4 \geq 4$ as $\left(a-\frac{1}{a}\right)^2$ is real and ≥ 0 $\therefore \left(a+\frac{1}{a}\right)^2 \geq 4$ $a+\frac{1}{a} \geq 2$ (ii) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 1+1+\frac{a}{b}+\frac{b}{a}$ $= 2 + \frac{a}{b} + \frac{b}{a}$ from (i) $\frac{a}{b} + \frac{b}{a} \geq 2$ replacing a by $\frac{a}{b}$ $\therefore (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 2+2$ ie $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$	✓ ✓ ✓ ✓ ✓

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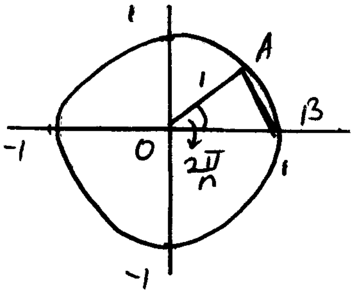
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q7 b (iii) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ $= 1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)$ $+\left(\frac{c}{a}+\frac{a}{c}\right)$ ✓ $\geq 1+1+1+2+2+2$ from (i)&(ii) ✓ ≥ 9 (c) (i) n = number of objects P(both white chosen) = P(2W, 2B) $P(2W,2B) = {}^4C_2\left(\frac{2}{n}\right)\left(\frac{1}{n-1}\right)\left(\frac{n-2}{n-2}\right)\left(\frac{n-3}{n-3}\right)$ $= \frac{12}{n(n-1)}$ ✓ (ii) P(neither white) = P(4B) $P(4B) = \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\left(\frac{n-5}{n-3}\right)$ $= \frac{(n-4)(n-5)}{n(n-1)}$ ✓ (iii) P(2W, 2B) = 2 P(4B) $\therefore \frac{12}{n(n-1)} = \frac{2(n-4)(n-5)}{n(n-1)}$ $6 = n^2-9n+20$ $(n-2)(n-7) = 0$ ie $n = 2$ or 7		Q8 (a) $z^n-1=0$  (i) Area of ΔOAB $= \frac{1}{2}\times 1\times 1\times \sin\frac{2\pi}{n}$ $= \frac{1}{2}\sin\frac{2\pi}{n}$ ✓ ∴ Area of polygon $= n\times\frac{1}{2}\sin\frac{2\pi}{n}$ ✓ ie $A_n = \frac{n}{2}\sin\frac{2\pi}{n}$ (ii) length AB $AB^2 = 1^2+1^2-2\times 1\times 1\times\cos\frac{2\pi}{n}$ $= 2-2\cos\frac{2\pi}{n}$ $= 2-2\cos 2\left(\frac{\pi}{n}\right)$ $= 2-2\left(1-2\sin^2\frac{\pi}{n}\right)$ $= 4\sin^2\frac{\pi}{n}$ ✓ $\therefore AB = 2\sin\frac{\pi}{n}$ ∴ Perimeter $= 2\times\sin\frac{\pi}{n}\times n$ ✓ ie $P_n = 2n\sin\frac{\pi}{n}$ (iii) $P_n - 2A_n = 2n\sin\frac{\pi}{n} - 2\frac{n}{2}\sin\frac{2\pi}{n}$ ✓ $= 2n\sin\frac{\pi}{n} - 2n\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ $= 2n\sin\frac{\pi}{n}\left(1-\cos\frac{\pi}{n}\right)$ > 0 as ✓	

but $n > 2$ ∴ $n = 7$ ✓

$n > 0$, $\sin\frac{\pi}{n} > 0$ & $1-\cos\frac{\pi}{n} > 0$

$\therefore P_n > 2A_n$

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Suggested Solution (s) | **Comments** | **Suggested Solution (s)** | **Comments**

Q8 (a) (iv)

let $n = \frac{1}{h}$

As $n \to \infty$, $h \to 0$

$\lim_{n\to\infty} \frac{n}{2}\sin\frac{2\pi}{n}$

$= \lim_{h\to 0} \frac{1}{2h}\sin 2\pi h$ ✓

$= \pi \lim_{h\to 0} \frac{\sin 2\pi h}{2\pi h}$ ✓

$= \pi$

(v) $\lim_{n\to\infty} P_n = 2\pi$ as ✓

$n\to\infty$ $P_n \to 2A_n$

(b) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}x \sin x \, dx$

$= \left[-\sin^{n-1}x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x \cos^2 x \, dx$ ✓

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x \,(1-\sin^2 x)$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x - \sin^n x \; dx$ } ✓

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$ ✓

$I_n(1+n-1) = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(ii) $\int_0^{\frac{\pi}{2}} \sin^{2n}x \, dx = I_{2n}$

$I_{2n} = \frac{2n-1}{2n} \times I_{2n-2}$ from (i)

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{1}{2} \times I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = \frac{\pi}{2}$

$\therefore I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}.$ ✓

$= \frac{2n}{2n} \times \frac{2n-1}{2n} \times \frac{2n-2}{2n-2} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{2}{2} \times \frac{1}{2} \times \frac{\pi}{2}$ ✓ ✓

$= \frac{(2n)!}{(2n)^2(2n-2)^2(2n-4)^2 \times \cdots \times 2^2} \times \frac{\pi}{2}$

$= \frac{(2n)!}{4n^2 \times 4(n-1)^2 \times 4(n-2)^2 \times \cdots \times 4} \times \frac{\pi}{2}$ ✓

$= \frac{(2n)!}{4^n (n!)^2} \times \frac{\pi}{2}$

$= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$