Teacher's Name ____________________

Student Number

Knox Grammar School

2014

Trial Higher School Certificate
Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time - 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Subject Teachers
Mr I Bradford
Mr D Sedgman

Setter

Mr M Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 34

Total Marks – 100

Section I 10 Marks

- Answer Questions 1 to 10
- Use the Multiple Choice Answer Sheet

Section II 90 Marks

- Answer Questions 11 to 16
- All questions are worth 15 marks
- Answer each question in a separate Writing Booklet.

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Section I

10 Marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. Which of these ellipses has foci (0, ± 3)?

A) $8x^2 + y^2 = 8$ B) $5x^2 + 4y^2 = 20$

C) $16x^2 + 25y^2 = 400$ D) $25x^2 + 16y^2 = 400$

2. Find $\int \cot x\, dx$

A) $-\operatorname{cosec}^2 x + c$ B) $-\ln(\operatorname{cosec} x) + c$ C) $\frac{1}{2}\cot^2 x + c$ D) $\ln(\sec x) + c$

3. The speed of a particle moving in a horizontal circle with radius 6 cm is $48\pi\ \text{cm s}^{-1}$.

How many revolutions per second does this particle make?

A) 4 B) 240 C) 480 D) 480π

4. The polynomial equation $x^3 - 2x^2 + 3 = 0$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

A) – 2 B) – 1 C) – 8 D) 8

5. Evaluate $\displaystyle\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$.

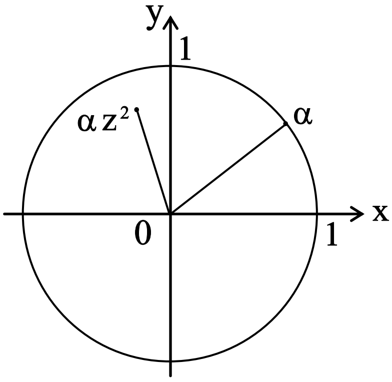
A) $-\frac{\pi}{4} + \tan^{-1} e$ B) $\ln\left(\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}\right)$ C) $\frac{\pi}{4}$ D) $\ln(1+\sqrt{2})$

6. What is the simplest expression of

$$\operatorname{Arg}\left(\frac{1}{1+i}\right)+\operatorname{Arg}\left(\frac{1}{(1+i)^2}\right)+\operatorname{Arg}\left(\frac{1}{(1+i)^3}\right)+\ \ldots+\operatorname{Arg}\left(\frac{1}{(1+i)^{20}}\right)\ ?$$

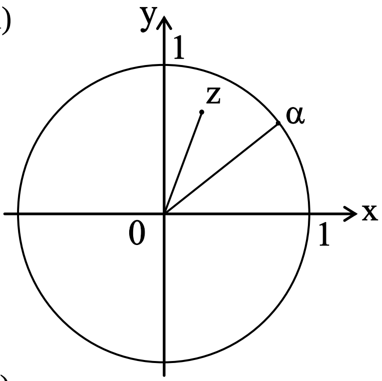
A) $-\frac{\pi}{2}$ B) $-\frac{\pi}{4}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{2}$

7. The Argand diagram below shows the complex numbers α and αz^2.

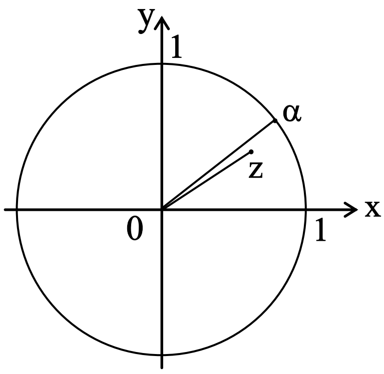


Which of the following best represents the positions of z and α?

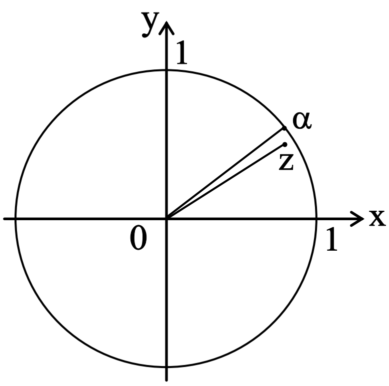
A)



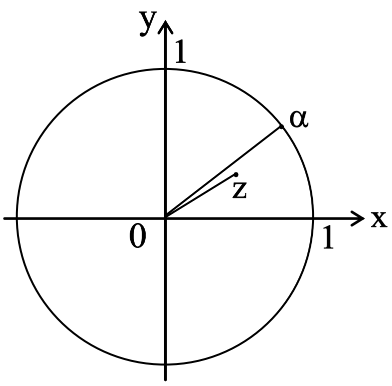
B)



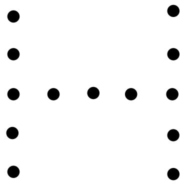
C)



D)



8. The diagram shows a shape made by 13 points.



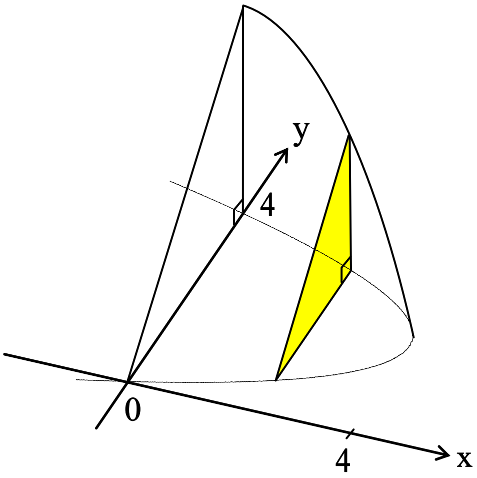
How many triangles can be made with these points as vertices?

A) ${}^{13}C_3 - 3\,{}^{5}C_3 - 3$ B) ${}^{13}C_3 - 2\,{}^{5}C_3 - 4$

C) ${}^{13}C_3 - 3\,{}^{5}C_3 - 4$ D) ${}^{13}C_3 - 3\,{}^{5}C_3 - 5$

9. The base of a solid is the region bounded by the parabola $x = 4y - y^2$ and the y axis.

Vertical cross sections are right angled isosceles triangles perpendicular to the x axis as shown.

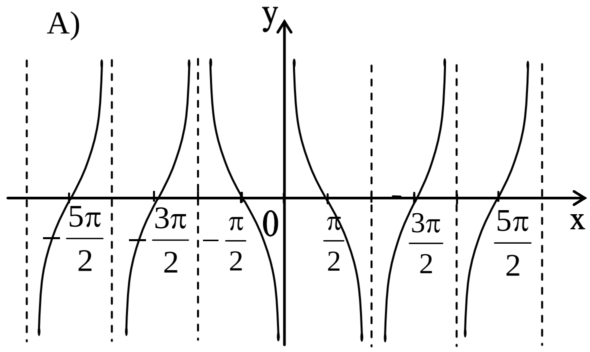


Which integral represents the volume of this solid?

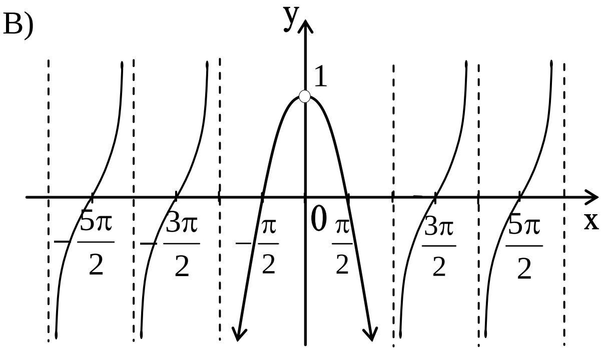
A) $\int_0^4 2\sqrt{4-x}\,dx$ B) $\int_0^4 \pi(4-x)\,dx$ C) $\int_0^4 (8-2x)\,dx$ D) $\int_0^4 (16-4x)\,dx$

10. Which of the following shows the graph of y = x cot x?

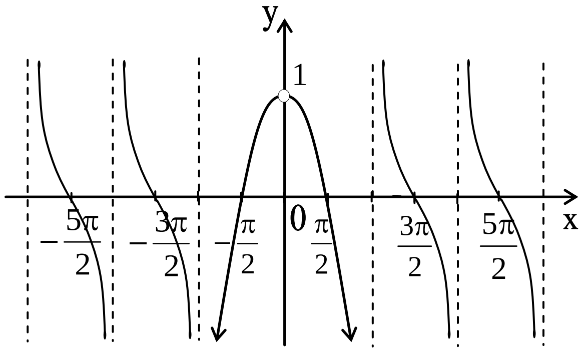
A)



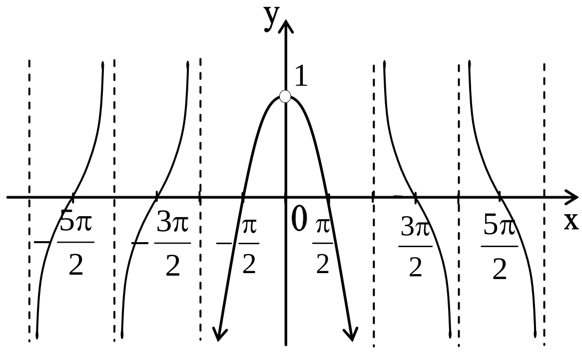
B)



C)



D)



End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The complex number z is given by $z=-\sqrt{3}+i$.

(i) Express z in modulus argument form. **2**

(ii) Hence show that $z^7+64z=0$ **2**

(b) Find values *A*, *B* and *C* such that:

$$\frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)}=\frac{A-Bx}{\left(2-2x+x^2\right)}-\frac{Cx}{\left(2-x^2\right)}$$ **3**

(c) Factorise $z^2+4iz+5$ over the complex field. **1**

(d) Using the substitution $x=2\sin\theta$, show that **4**

$$\int_{-1}^{\sqrt{3}}\frac{x^2}{\sqrt{4-x^2}}\,dx=\pi-\sqrt{3}$$

(e) Sketch the region in the Argand diagram defined by $z\overline{z}+2(z+\overline{z})\le 0$ **3**

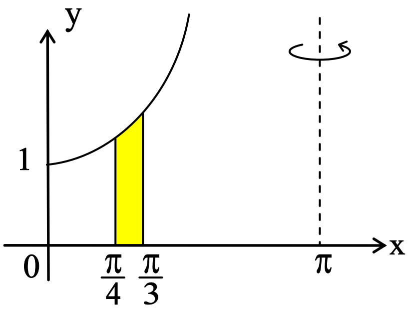
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int \frac{dx}{1+\sin x+\cos x}$. **3**

(b) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where *m* and *n* are real. **3**
It is known that $1-i$ is a root of the equation. Find the values of *m* and *n*.

(c) The area bounded by the curve $y = \sec^2 x$, the *x*-axis, **4**
$x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ is rotated about the line $x = \pi$ to form a solid.

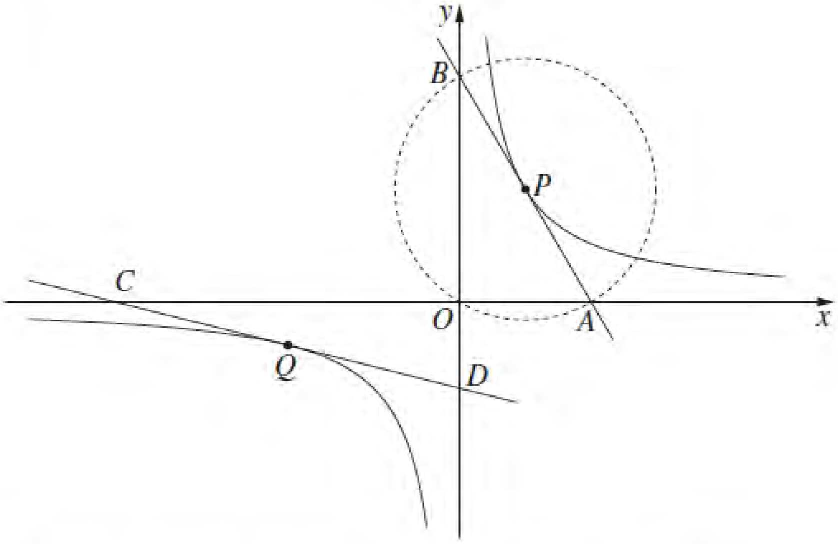


Use the method of cylindrical shells to find the volume of the solid.

Question 12 continues on page 8

(d) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$, where $|p| \neq |q|$, lie on the rectangular hyperbola with equation $xy = c^2$.

The tangent to the hyperbola at P intersects the x-axis at A and the y-axis at B. Similarly, the tangent to the hyperbola at Q intersects the x-axis at C and the y-axis at D.



(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$. **2**

(ii) Show that A, B and O are on a circle with centre P. **2**

(iii) Prove that BC is parallel to PQ. **1**

End of Question 12

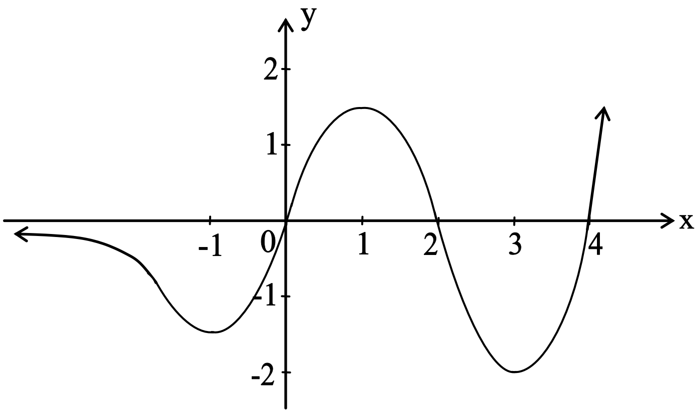
Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Let $I_n = \int_0^1 x^n \sqrt{1-x^3}\,dx$ for $n \geq 2$. Show that: **3**

$$I_n = \frac{2n-4}{2n+5} \times I_{n-3} \text{ for } n \geq 5$$

(ii) Hence find I_8 **2**

(b) The graph of a certain function $y = f\left(x\right)$ is shown below.



Sketch the following curves:

(i) $y = \dfrac{1}{f(x)}$ **2**

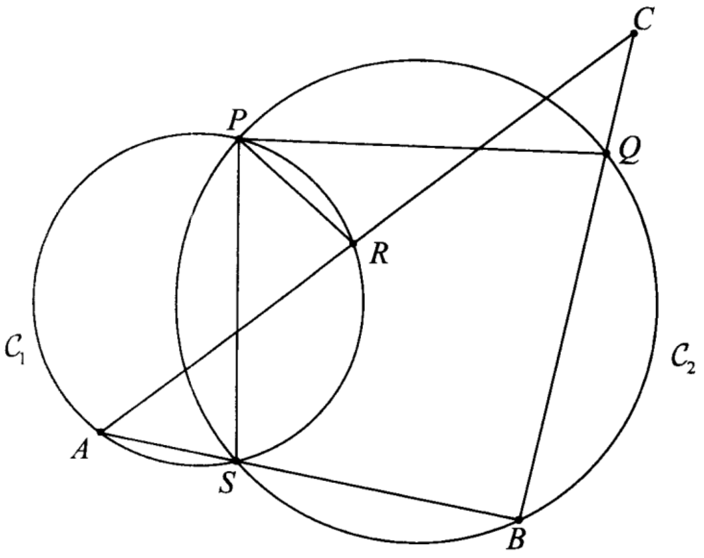
(ii) $y = \ln\left[1 - f(x)\right]$ **2**

(iii) $y^2 = 1 + f(x)$ **2**

Question 13 continues on page 10

Question 13 (continued)

(c) Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ meet at P and S. Points A and R lie on $\mathcal{C}_1$ and points B and Q lie on $\mathcal{C}_2$. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



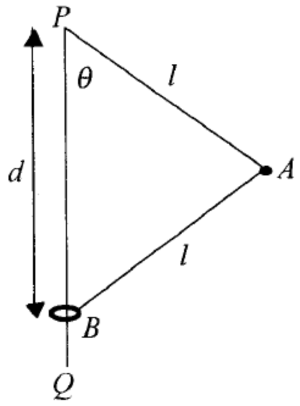
(i) Prove that $\angle PRA = \angle PQB$. **2**

(ii) Prove that the points P, R, Q and C are concyclic. **2**

End of Question 13

Question 14 **(15 marks)** Use a SEPARATE writing booklet

(a)



PQ is a smooth vertical rod. Particle *A* of mass *m* is attached to a point *P* by a string of length l and *A* is also attached by a second string of length l to a smooth ring *B* of mass *M* which is free to slide on the rod *PQ* without friction. *A* is set in motion in a horizontal circle about *PQ* with constant angular velocity ω. *B* is in equilibrium.

T_1 and T_2 are the tensions in the strings *AP* and *AB* respectively when *AP* makes an angle θ with the vertical.

(i) Draw diagrams showing the forces acting on each of *A* and *B*. **1**

(ii) Hence show that $T_1 - T_2 = \dfrac{mg}{\cos\theta}, T_1 + T_2 = ml\omega^2 \; and \; T_2 = \dfrac{Mg}{\cos\theta}$. **2**

(iii) Deduce that $d = \dfrac{2g}{\omega^2}\left(1 + 2\dfrac{M}{m}\right)$. **2**

Question 14 continues on page 12

Question 14 (continued)

(b) A particle's resistance to motion in a medium is proportional to mv^2 where m is the particle's mass and v is its velocity at time t.

(i) Initially the particle is projected downwards in the medium where the speed of projection is equal to the terminal velocity V_T. **1**

Show that $V_T^2 = \frac{g}{k}$ where k is the constant of proportionality.

The particle is now projected vertically <u>upwards</u> in the same medium.

(ii) Show that $x = \frac{V_T^2}{2g} \ln\left(\frac{2V_T^2}{V_T^2 + v^2}\right)$. **2**

(iii) Hence show that $H = \frac{V_T^2 \ln 2}{2g}$ where H represents the particle's maximum height above its point of projection. **1**

(iv) Show that during the particle's ascent $v = V_T \tan\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$. **2**

(v) Hence show that $\frac{2V_T^2}{V_T^2 + v^2} = 1 + \sin\left(\frac{2g}{V_T}t\right)$. **2**

(vi) If T is the time taken to achieve *half* its maximum height, show that **2**

$T = \frac{V_T}{2g}\sin^{-1}\left(\sqrt{2} - 1\right)$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) (i) If $0 \le a \le b$ show that $\dfrac{a}{1+a} \le \dfrac{b}{1+b}$ **2**

(ii) Hence or otherwise show that $\dfrac{a}{1+a} \le \dfrac{b}{1+b} + \dfrac{c}{1+c}$ where $a \le b+c$ and $a, b, c \ge 0$ **2**

(b) An urn contains 5 balls numbered from 1 to 5. A ball is chosen at random and its number is noted. The ball is then returned to the urn. This is done a total of five times.

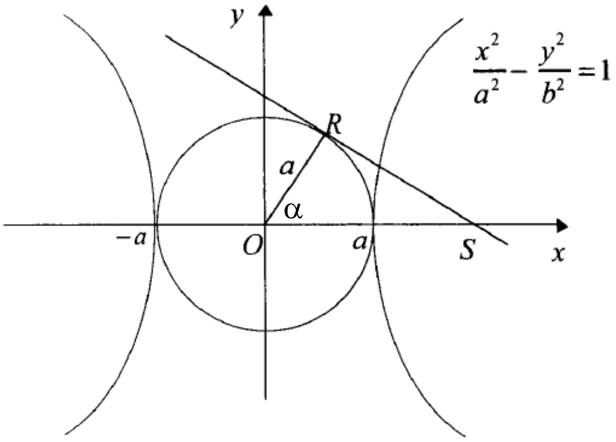
(i) What is the probability that each ball is selected exactly once? **1**

(ii) What is the probability that at least one ball is not selected? **1**

(iii) What is the probability of obtaining 1,1,2,3,4 in any order? **1**

(iv) What is the probability that exactly one of the balls is not selected? **2**

(c)



S is the focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a \ne b$, which lies on the positive x-axis. R is a point on the auxiliary circle of the hyperbola such that R lies in the first quadrant and SR is tangent to the auxiliary circle. The eccentricity of the hyperbola is e and $\angle ROS = \alpha$.

(i) Show that R lies on a directrix of the hyperbola. **1**

(ii) Show that SR has equation $y = -\dfrac{1}{\sqrt{e^2-1}}\left(x - ae\right)$. **1**

(iii) If SR meets the hyperbola at the point $\left(a\sec\theta, b\tan\theta\right)$, show that **2**

$$e^2(2-e^2)\sec^2\theta - 2e\sec\theta + \left\{e^2 + (e^2-1)^2\right\} = 0.$$

(iv) By considering this as a quadratic equation in $\sec\theta$, deduce that SR intersects the hyperbola in two distinct points P and Q, lying on the same branch of the hyperbola if $e^2 < 2$ and lying on opposite branches if $e^2 > 2$. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) The *n*th Fermat number, F_n, is defined by $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, 3...$, where 2^{2^n} means 2 raised to the power of 2^n. **4**

Prove by mathematical induction, that for all positive integers:

$$F_0 \times F_1 \times F_2 \times ... \times F_{n-1} = F_n - 2$$

(b) A coin is tossed six times. What is the probability that there will be more tails on the first three of the six throws than on the last three throws? **3**

(c) Let $x = \cos\theta + i\sin\theta$ for $0 < \theta < 2\pi$, and let *n* be a positive integer.

(i) Show that $x^k + \frac{1}{x^k} = 2\cos k\theta$, for any positive integer *k*. **2**

(ii) Show that **3**

$$\left(x + \frac{1}{x}\right)^{2n} = \left(x^{2n} + \frac{1}{x^{2n}}\right) + \binom{2n}{1}\left(x^{2n-2} + \frac{1}{x^{2n-2}}\right) + \binom{2n}{2}\left(x^{2n-4} + \frac{1}{x^{2n-4}}\right) + \dots + \binom{2n}{n-1}\left(x^2 + \frac{1}{x^2}\right) + \binom{2n}{n}.$$

(iii) Deduce that **1**

$$2^{2n-1}\cos^{2n}\theta = \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \dots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}.$$

(iv) Hence show that $\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n-1}}\binom{2n}{n}$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2014 Year 12 Mathematics Extension 2 HSC Task 4 Trial HSC Solutions

Suggested Solution (s)

Comments

1. D 2. B 3. A 4. B 5. D
6. A 7. C 8. C 9. C 10. D

1. D

$25x^2 + 16y^2 = 400$ is equivalent to
$\frac{x^2}{16} + \frac{y^2}{25} = 1$ which has its foci on the y axis.
$e^2 = 1 - \frac{16}{25} = \frac{9}{25}$, so $e = \frac{3}{5}$
focal length is ae, $ae = 5 \times \frac{3}{5} = 3$
So foci are $(0, \pm 3)$

2. B

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$
$$= \ln(\sin x) + c$$
$$= \ln\left(\frac{1}{\operatorname{cosec} x}\right) + c$$
$$= \ln 1 - \ln(\operatorname{cosec} x) + c$$
$$= - \ln(\operatorname{cosec} x) + c$$

3. A

As $v = r\omega$ then
$48\pi = 6\omega$ that is $\omega = 8\pi$ rad/s
Now, the angular velocity is 8π and
as each revolution is 2π radians then
the particle makes 4 revolutions per second.

4. B

$x^3 - 2x^2 + 3 = 0$ that is $x^3 - 2x^2 + 0x + 3 = 0$
This means $\alpha + \beta + \gamma = 2$
$\alpha\beta + \beta\gamma + \alpha\gamma = 0$ and $\alpha\beta\gamma = -3$
Now as α is a solution of the equation then
$\alpha^3 - 2\alpha^2 + 3 = 0$, so $\alpha^3 = 2\alpha^2 - 3$
Similarly, $\beta^3 = 2\beta^2 - 3$ & $\gamma^3 = 2\gamma^2 - 3$
Therefore,
$\alpha^3 + \beta^3 + \gamma^3 = 2(\alpha^2 + \beta^2 + \gamma^2) - 9$
But $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 +$

$2(\alpha\beta + \beta\gamma + \alpha\gamma)$
So $(2)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(0)$
that is $\alpha^2 + \beta^2 + \gamma^2 = 4$
Hence, $\alpha^3 + \beta^3 + \gamma^3 = 2(4) - 9 = -1$

Suggested Solution (s)

Comments

5. D

$u = \ln x, \therefore du = \frac{dx}{x}$

$$\int_1^e \frac{dx}{x\sqrt{1+\ln^2 x}}.$$
$$= \int_0^1 \frac{du}{\sqrt{1+u^2}}. = \left[\ln(u + \sqrt{1+u^2}) \right]_0^1$$
$$= \ln(1+\sqrt{2}) - \ln 1$$
$$= \ln(1+\sqrt{2})$$

6. A

$\operatorname{Arg}\left(\frac{1}{1+i}\right) = \operatorname{Arg} 1 - \arg(1+i) = 0 - \frac{\pi}{4} = -\frac{\pi}{4}$

$\operatorname{Arg}\left(\frac{1}{(1+i)^2}\right) = \operatorname{Arg} 1 - \arg(1+i)^2 = 0 - 2 \times \frac{\pi}{4} = -\frac{\pi}{2}$

$\operatorname{Arg}\left(\frac{1}{(1+i)^{20}}\right) = \operatorname{Arg} 1 - \arg(1+i)^{20} = 0 - 20 \times \frac{\pi}{4} = -\frac{20\pi}{4}$

Let $S = \operatorname{Arg}\left(\frac{1}{1+i}\right) + \operatorname{Arg}\left(\frac{1}{(1+i)^2}\right) + \ldots + \operatorname{Arg}\left(\frac{1}{(1+i)^{20}}\right)$

$$= -\frac{\pi}{4} - \frac{2\pi}{4} - \frac{3\pi}{4} \ldots\ldots\ldots - \frac{20\pi}{4}$$
$$= -\frac{\pi}{4}(1 + 2 + 3 + \cdots \ldots + 20)$$

(This is an Arithmetic Series with 20 terms,
the first is a= 1 and the last being 20,
so the sum is $\frac{20}{2}(1 + 20) = 210$)

Therefore, $S = -\frac{\pi}{4}(210) = -52\frac{1}{2}\pi$.
But an Arg is always between $-\pi$ *and* π
Hence, S = 26 revolutions $-\frac{\pi}{2} = -\frac{\pi}{2}$

[redacted]

7. C

$Arg\,(az^2) = Arg\,(a) + 2Arg\,(z)$

So option A is not correct as $Arg\,(az^2)$ in the diagram is incorrect.

$Mod\,(az^2) = mod\,(a) \times (mod\,(z))^2$

and $mod\,(a) = 1$ this means

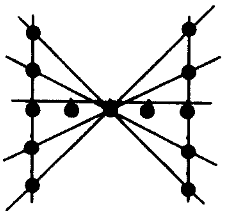
$Mod\,(az^2) = (mod\,(z))^2$ but $Mod\,(az^2) < 1$

Hence $(mod\,(z))^2 < 1$

and so $(mod\,(z))^2 < mod\,(z)$

Only the diagram in option C satisfies this condition.

8. C



13 points can form combinations of 3 in ${}^{13}C_3$ ways.

But 5 points in a straight line cannot form a triangle, and there are 3 such lines. So $3 \times {}^5C_3$ must be subtracted from the total.

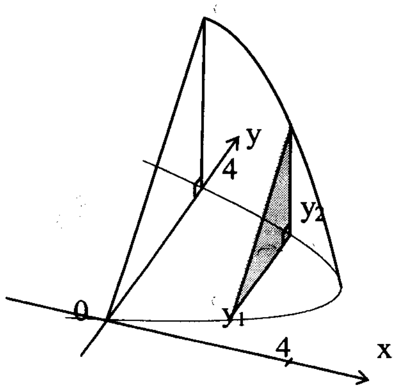
Also 3 points in a straight line cannot form a triangle, and there are 4 such lines.

So $4 \times {}^3C_3 = 4$ must be subtracted from the total,

Hence, the total number of triangles that can be formed is

${}^{13}C_3 - 3\,{}^5C_3 - 4$.

9. C



The area of 1 cross section $= \frac{1}{2}(y_2 - y_1)^2$

Find an expression for y_1 and y_2 in terms of x.

$x = 4y - y^2$ so $y^2 - 4y + x = 0$

Hence $y = \frac{4 \pm \sqrt{16 - 4x}}{2}$

$= \frac{4 \pm 2\sqrt{4 - x}}{2}$

$y = 2 \pm \sqrt{4 - x}$

So $y_2 = 2 + \sqrt{4 - x}$ and $y_1 = 2 - \sqrt{4 - x}$

Hence $y_2 - y_1 = 2\sqrt{4 - x}$

The area of a cross section $= \frac{1}{2}\left(2\sqrt{4 - x}\right)^2$

$= 2(4 - x)$

So the volume $= \int_0^4 (8 - 2x)\,dx$

10. D

The graph of $y = x \cot x$ can be found using the "multiplication of ordinates" method

Alternatively as $y = \cot x$ and $y = x$ are both odd functions, $y = x \cot x$ must be even.

From the four options provided only D represents an even function.

Suggested Solution (s) **Comments**

Question 11

a) i) $2\,\text{cis}\,\frac{5\pi}{6}$ ✓ ✓

ii) $z^7 + 64z$

$= 2^7\text{cis}\frac{35\pi}{6} + 64\left(2\,\text{cis}\frac{5\pi}{6}\right)$ ✓ (DMT)

$= 128\left(\text{cis}\left(-\frac{\pi}{6}\right) + \text{cis}\left(\frac{5\pi}{6}\right)\right)$

$= 128\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 128\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} - \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ ✓

$= 128 \times 0$

$= 0$

as $\sin\theta = \sin(180-\theta)$
$\cos\theta = \cos(-\theta)$
$\sin(-\theta) = -\sin\theta$
$\cos(180-\theta) = -\cos\theta$

(b) Let $8(1-x) \equiv (A-Bx)(2-x^2) - Cx(2-2x+x^2)$

when,

$x=0$ $8 \equiv 2A$ $\therefore A = 4$ ✓

$x=1$ $0 = 4 - B - C$

$\therefore B + C = 4$ —— ①

$x=-1$ $16 = 4 + B + 5C$

$B + 5C = 12$ —— ② ✓

①−② $-4C = -8$ $\therefore C = 2$, $B = 2$ ✓

$\therefore A = 4, B = 2, C = 2$

(c) $z^2 + 4iz + 5$

$= (z+5i)(z-i)$ ✓

$\begin{matrix} z & & -i \\ & \times & \\ z & & 5i \end{matrix}$

(d) $x = 2\sin\theta$ $x=-1$, $\theta = -\frac{\pi}{6}$

$\frac{dx}{d\theta} = 2\cos\theta$ $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$ ✓

So $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta}} . 2\cos\theta . d\theta$

$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cancel{\cos}\theta} . \cancel{\cos}\theta \; d\theta$ ✓

$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2}\cos 2\theta \; d\theta$ ✓

$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$ ✓ evaluation

$= 2\left[\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} - \left(-\frac{\pi}{6} - \frac{1}{2}\sin\left(-\frac{\pi}{6}\right)\right]\right.$

$= 2\left[\frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right]$

$= \pi - \sqrt{3}$

(e) $z\bar{z} + 2(z+z) \leq 0$

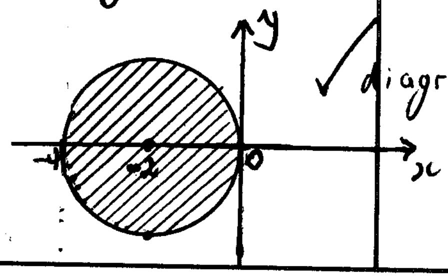
If $z = x + iy$

$(x+iy)(x-iy) + 2(x+iy+x-iy) \leq 0$

$x^2 + y^2 + 4x \leq 0$ ✓

$x^2 + 4x + 4 + y^2 \leq 4$

$(x+2)^2 + y^2 \leq 4$ ✓



✓ diagram

Suggested Solution (s) | **Comments** | **Suggested Solution (s)** | **Comments**

Question 12

(a) $t = \tan\left(\frac{x}{2}\right)$

$$\therefore \quad dx = \frac{2\,dt}{1+t^2} \quad ✓$$

$$\int \frac{dx}{1+\sin x+\cos x}$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} \quad ✓$$

$$= \int \frac{2}{1+t^2+2t+1-t^2}\,dt$$

$$= \int \frac{\cancel{2}\,1}{\cancel{2}(1+t)}\,dt$$

$$= \ln\lvert 1+t\rvert + C \quad ✓$$

or $= \ln\left\lvert 1+\tan\frac{x}{2}\right\rvert + c$

(b) Since coeff's are real

$1+i$ is also a root.

so if $(1-i)(1+i)\alpha = -6$ ✓

product of roots

$2\alpha = -6$

$\alpha = -3$

is third root

Sum: $1-i+1+i-3 = -m$

$-1 = -m$

$m = 1$ ✓

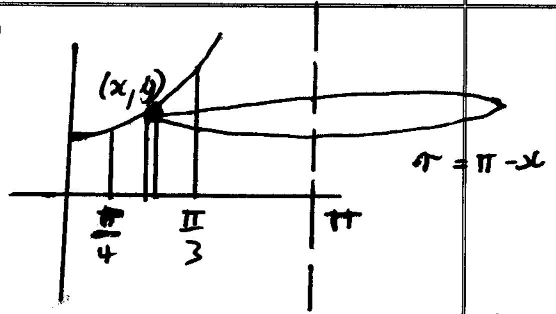
and $P(-3) = 0$

so $-27+9-3n+6 = 0$

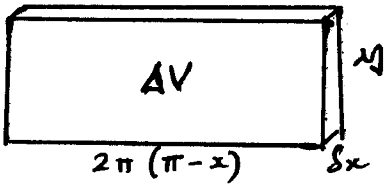
$3n = -12$

$n = -4$ ✓

(c)



Using method of cylindrical shells:



$$\therefore \Delta V = 2\pi(\pi - x)\,y\,\delta x$$

$$V = 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\pi - x)\cdot\sec^2 x\,dx \quad ✓$$

$$= 2\pi^2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx - 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x\sec^2 x\,dx \quad ✓$$

$$= 2\pi^2\Big[\tan x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - 2\pi\left[\Big[x\tan x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\,dx\right]$$

$$= 2\pi^2(\sqrt{3}-1) - 2\pi\left(\frac{\sqrt{3}\pi}{3} - \frac{\pi}{4}\right) \quad ✓$$

$$- 2\pi\Big[\ln(\cos x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{4\sqrt{3}\pi^2}{3} - \frac{3\pi^2}{2} + \pi\ln 2 \quad ✓$$

or equivalent

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Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
(d) i) $xy = c^2$ $\frac{d}{dx}(xy) = 0$ $\therefore x \cdot \frac{dy}{dx} + y = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ at $x = cp$, $y = \frac{c}{p}$ $M_T = -\frac{c}{p} \times \frac{1}{cp}$ $= -\frac{1}{p^2}$ ✓ $\therefore y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$ $p^2 y - cp = -x + cp$ $\therefore x + p^2 y = 2cp$ ii) at A, $y = 0$ $\therefore x = 2cp$ $\therefore A(2cp, 0)$ at B, $x = 0$ $\therefore y = \frac{2c}{p}$ $\therefore B\left(0, \frac{2c}{p}\right)$ Midpoint of AB is $x = \frac{0 + 2cp}{2} = cp$ $y = \frac{\frac{2c}{p} + 0}{2} = \frac{c}{p}$ which are coordinates of P $\therefore PA = PB$ and since $\angle POB = 90°$ A, B, O are on a circle centre P. (∠ in semi circle is 90°)	✓ ✓ ✓	iii) Similarly. $QD = QC$ Thus $\frac{AP}{AB} = \frac{DQ}{QC} = \frac{1}{2}$ $\therefore AD \parallel PQ \parallel BC$ (Intercepts on transversals cut by parallel lines are in the same ratio).	✓

Suggested Solution (s) | **Comments**

Question 13

(a) (i) Let $I_n = \int_0^1 x^{n-2} \cdot x^2(1-x^3)^{\frac{1}{2}}\,dx$

where

$u = x^{n-2}$ $\quad v' = x^2(1-x^3)^{\frac{1}{2}}$

$u' = (n-2)x^{n-3}$ $\quad v = -\frac{2}{9}(1-x^3)^{\frac{3}{2}}$

$$\therefore I_n = \left[-\frac{2}{9}x^{n-2}(1-x^3)^{\frac{3}{2}}\right]_0^1 + \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)^{\frac{3}{2}}\,dx$$ ✓

$$= \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1-x^3)(1-x^3)^{\frac{1}{2}}\,dx$$

$$= \frac{2}{9}(n-2)\left\{\int_0^1 x^{n-3}(1-x^3)^{\frac{1}{2}}\,dx - \int_0^1 x^{n}(1-x^3)^{\frac{1}{2}}\,dx\right\}$$

$$= \frac{2}{9}(n-2)\left\{I_{n-3} - I_n\right\}$$ ✓

$$\therefore I_n + \frac{2}{9}(n-2)I_n = \frac{2}{9}(n-2)I_{n-3}$$

$$9I_n + (2n-4)I_n = 2(n-2)I_{n-3}$$

$$(2n+5)I_n = (2n-4)I_{n-3}$$

$$\therefore I_n = \frac{2n-4}{2n+5}I_{n-3}$$ ✓ Clear progress to

ii) $I_2 = \int_0^1 x^2(1-x^3)^{\frac{1}{2}}\,dx$

$= -\frac{2}{9}\left[(1-x^3)^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{9}$ ✓

Suggested Solution (s) | **Comments**

So $I_5 = \frac{10-4}{10+5}\cdot I_2$

$= \frac{6}{15}\times\frac{2}{9}$

$= \frac{4}{45}$

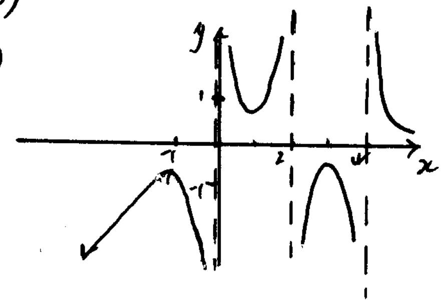
$I_8 = \frac{16-4}{16+5}\cdot I_5$

$= \frac{12}{21}\times\frac{4}{45}$

$= \frac{16}{315}$ ✓

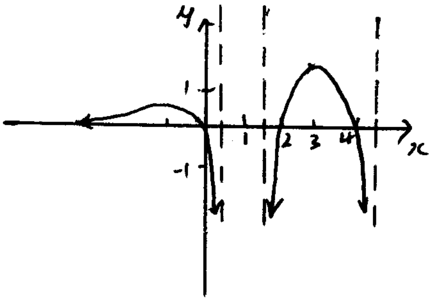
(b)

(i)



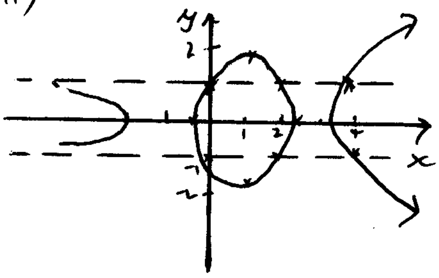
✓ asymptotes
✓ shape

(ii)



✓ asymptotes
✓ shape.

(iii)



✓ all asymptotes
✓ shape.

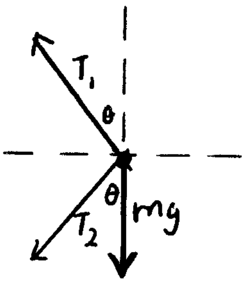
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Question 13 (Continued)			
(c)(i) Let $\angle PRA = \alpha$			
$\therefore \angle PSA = \alpha$ ($\angle$'s at circum. subtended by arc PA are equal)	✓		
$\therefore \angle PSB = 180 - \alpha$ (Adjacent supplementary $\angle$'s on a straight line)			
$\therefore \angle PQB = \alpha$ (Opp. $\angle$'s of cyclic quad. are supplementary)	✓		
$\therefore \angle PRA = \angle PQB$			
ii) $\angle PRC = 180 - \alpha$ (Adjacent supplementary $\angle$'s on a straight line) $\angle PQC = 180 - \alpha$ (Adjacent supplementary $\angle$'s on a straight line)	✓		
Since $\angle PRC = \angle PQC$ we have equal $\angle$s subtended by PC, which means P, R, Q and C are concyclic.	✓		

Suggested Solution (s) | **Comments**

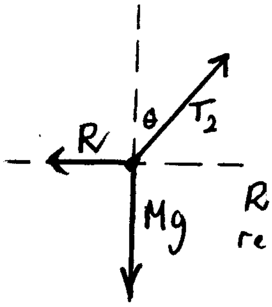
Question 14

a)

i) Forces on A ✓

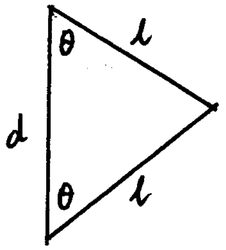


Forces on B



R is reaction force

Dimensions Diagram



ii) Sum of Radial forces at A $= mr\omega^2$

$\therefore T_1\sin\theta + T_2\sin\theta = mr\omega^2$

$\therefore T_1 + T_2 = \dfrac{mr\omega^2}{\sin\theta}$ but $\sin\theta = \dfrac{r}{l}$

$\therefore T_1 + T_2 = ml\omega^2$ ....(i) $\therefore l = \dfrac{r}{\sin\theta}$ ✓

Vertical forces at A are balanced

$\therefore T_1\cos\theta = T_2\cos\theta + mg$

$\therefore T_1 - T_2 = \dfrac{mg}{\cos\theta}$ ....(ii)

Vertical Forces at B are balanced ✓

$\therefore T_2\cos\theta = Mg$ ......(iii)

$T_2 = \dfrac{Mg}{\cos\theta}$

iii) Subtracted (i) and (ii) $2T_2 = ml\omega^2 - \dfrac{mg}{\cos\theta}$ ✓

$\therefore 2\left(\dfrac{Mg}{\cos\theta}\right) = ml\omega^2 - \dfrac{mg}{\cos\theta}$

$2Mg = ml\omega^2\cos\theta - mg$

$\therefore ml\omega^2 = \dfrac{g(m+2M)}{\cos\theta}$ ✓

$l\cos\theta = \dfrac{g}{\omega^2}\left(1 + \dfrac{2M}{m}\right)$

but $\cos\theta = \dfrac{\frac{d}{2}}{l}$ $l\cos\theta = \dfrac{d}{2}$

$\therefore d = \dfrac{2g}{\omega^2}\left(1 + \dfrac{2M}{m}\right)$

Suggested Solution (s)	Comments

(b) (i) Downwards

$\uparrow mkv^2$

$\downarrow +$ $\quad \downarrow mg$

$\therefore m\ddot{x} = mg - mkv^2$ is resultant force

$\ddot{x} = g - kv^2$

At terminal velocity $\ddot{x} = 0$.

$\therefore kV_T^2 = g$

$V_T^2 = \frac{g}{k}$

ii) Upwards

$\uparrow +$ $\quad \downarrow mg \quad \downarrow mkv^2$

$\therefore m\ddot{x} = -mg - mkv^2$ is resultant force

$$\therefore v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dx}{dv} = \frac{-v}{g + kv^2}$$

$$x = \frac{-1}{2k}\int \frac{2kv}{g + kv^2}\,dv$$

$$\therefore x = -\frac{1}{2k}\ln(g + kv^2) + C$$

at $x = 0, v = V_T$ $\quad \therefore C = \frac{1}{k}\ln(g + kV_T^2)$

So $\quad x = \frac{1}{2k}\ln\frac{g + kV_T^2}{g + kv^2} \qquad g = kV_T^2$

$\quad = \frac{1}{2k}\ln\frac{kV_T^2 + kV_T^2}{kV_T^2 + kv^2} \qquad$ ad $\frac{1}{k} = \frac{V_T^2}{g}$

$\quad = \frac{V_T^2}{2g}\ln\left(\frac{2V_T^2}{V_T^2 + v^2}\right)$

Suggested Solution (s) | **Comments**

iii) At maximum height $x = H$, $v = 0$

$$\therefore \quad H = \frac{V_T^2}{2g}\ln\left(\frac{2V_T^2}{V_T^2 + 0}\right) \checkmark$$

$$= \frac{V_T^2}{2g}\ln\left(2V_T^2\right)$$

iv) Starting with $\ddot{x} = -(g + kv^2)$

$$\frac{dt}{dv} = \frac{-1}{k\left(\frac{g}{k} + v^2\right)}$$

$$\therefore \quad t = -\frac{1}{k}\int \frac{1}{V_T^2 + v^2}\,dv$$

$$= -\frac{1}{k}\cdot\frac{1}{V_T}\tan^{-1}\left(\frac{v}{V_T}\right) + C$$

at $t = 0$ $v = V_T$ $\therefore$ $C = \frac{1}{kV_T}\tan^{-1}(1) = \frac{\pi}{4kV_T}$

So $t = \frac{1}{kV_T}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{v}{V_T}\right)\right)$

$$\therefore \quad \tan^{-1}\left(\frac{v}{V_T}\right) = \frac{\pi}{4} - kV_Tt$$

$v = V_T\tan\left(\frac{\pi}{4} - kV_Tt\right)$ but $kV_T = \frac{g}{V_T}$

$\therefore$ $v = V_T\tan\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$ as required

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Suggested Solution (s) | **Comments**

(v) Algebraically and using trig identities

$$\frac{v}{V_T} = \tan\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$$

$$\frac{v^2}{V_T^2} = \tan^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$$

so $$1 + \frac{v^2}{V_T^2} = 1 + \tan^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$$

$$\therefore 1 + \frac{v^2}{V_T^2} = \sec^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right) \checkmark$$

$$\frac{V_T^2 + v^2}{V_T^2} = \sec^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$$

reciprocals

$$\left.\begin{aligned}\frac{V_T^2}{V_T^2+v^2} &= \cos^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\\ &= \frac{1}{2} + \frac{1}{2}\cos\left(2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right)\\ \therefore \frac{2V_T^2}{V_T^2+v^2} &= 1 + \cos\left(\frac{\pi}{2} - \frac{2g}{V_T}t\right)\end{aligned}\right\} \checkmark$$

$$= 1 + \sin\left(\frac{2gt}{V_T}\right) \text{ as required.}$$

Alternatively,

$$x = \int V_T \tan\left(\frac{\pi}{4} - \frac{g}{V_T}t\right) dt$$

$$= V_T \int \frac{\sin\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)}{\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)} dt$$

$$= V_T \times \frac{-V_T}{g} \times -\int \frac{-\sin\left(\frac{\pi}{4} - \frac{g}{V_T}t\right) \cdot \frac{-g}{V_T}}{\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)} dt$$

Suggested Solution (s)	Comments
∴ $x = \frac{V_T^2}{g}\ln\left[\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right] + C$	
at $t=0$, $c=0$ ∴ $C = -\frac{V_T^2}{g}\ln\left(\frac{1}{\sqrt{2}}\right)$	
∴ $x = \frac{V_T^2}{g}\ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right]$ ✓	
Equating expressions ~~for~~ displacement	
$\frac{V_T^2}{2g}\ln\left[\frac{2V_T^2}{V_T^2+v^2}\right] = \frac{V_T^2}{g}\ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right]$	
log laws: $\ln\left[\frac{2V_T^2}{V_T^2+v^2}\right]^{\frac{1}{2}} = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right]$	✓ correct process
equating objects and squaring $\frac{2V_T^2}{V_T^2+v^2} = 2\cos^2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)$	
$= 2\left(\frac{1}{2} + \frac{1}{2}\cos\left[2\left(\frac{\pi}{4} - \frac{g}{V_T}t\right)\right]\right)$	
$= 1 + \cos\left(\frac{\pi}{2} - \frac{2g}{V_T}t\right)$	
$= 1 + \sin\frac{2g}{V_T}t$ as required	
vi) Noting $\frac{1}{2}H = \frac{V_T^2\ln 2}{4g}$ ✓	
∴ $\cancel{V_T^2}\frac{\ln 2}{4\cancel{g}} = \frac{\cancel{V_T^2}}{2\cancel{g}}\ln\left(\frac{2V_T^2}{V_T^2+v^2}\right)$ from part i)	
∴ $\frac{1}{2}\ln 2 = \ln\left(\frac{2V_T^2}{V_T^2+v^2}\right)$	
∴ $\frac{2V_T^2}{V_T^2+v^2} = 2^{\frac{1}{2}} = 1 + \sin\left(\frac{2g}{V_T}t\right)$ ✓ from part v)	
rearranging gives $T = \frac{V_T}{2g}\sin^{-1}(\sqrt{2}-1)$	

Suggested Solution (s)	Comments

Question 15

(a) i) Since $a \le b$

$$a + ab \le b + ab \quad ✓$$
$$a(1+b) \le b(1+a)$$
$$\therefore \frac{a}{1+a} \le \frac{b}{1+b} \quad ✓$$

ii) Replacing b with $b+c$ from the inequality in part i)

$$\frac{a}{1+a} \le \frac{b+c}{1+b+c} \quad ✓$$
$$\le \frac{b}{1+b+c} + \frac{c}{1+b+c}$$
$$\le \frac{b}{1+b} + \frac{c}{1+c}$$

progress ✓

(b) i) $P(E) = \frac{5}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{5!}{5^5} = \frac{24}{625}$ ✓

ii) $P(E) = 1 - \frac{5!}{5^5} = \frac{601}{625}$ ✓ iii) $\frac{5!}{2!} / 5^5 = \frac{12}{625}$ ✓

iv) Given that there can be also double 2's, 3's or 4's ✓ when one ball is not selected, there are $4 \times \frac{5!}{2!}$ ways of leaving out the five. As each of the other numbers can be left out, there are $5 \times 4 \times \frac{5!}{2!}$ ✓ permutations of exactly one ball not being selected. $\therefore P(E) = \left(20 \times \frac{5!}{2!}\right) \div 5^5 = \frac{48}{125}$

(c) i) Since $S(ae, 0)$ $\sec\alpha = \frac{OS}{OR} = \frac{ae}{a} = e$ ✓ At R, $x = a\cos\alpha = \frac{a}{e}$ $\therefore R$ lies on directrix

ii) $m_{OR} = \tan\alpha = \sqrt{\sec^2\alpha - 1} = \sqrt{e^2-1}$ ✓ $\therefore m_{RS} = \frac{-1}{\sqrt{e^2-1}}$

$\therefore SR$ is $y = \frac{-1}{\sqrt{e^2-1}}(x - ae)$

iii) $x = a\sec\theta$, $y = b\tan\theta$ ✓ $\therefore b\sqrt{e^2-1} = -(a\sec\theta - ae)$

squaring: $b^2(e^2-1)\tan^2\theta = a^2(\sec^2\theta - 2e\sec\theta + e^2)$ and since $\frac{b^2}{a^2} = (e^2-1)$ for hyperbola

$(e^2-1)^2(\sec^2\theta - 1) = \sec^2\theta - 2e\sec\theta + e^2$ ✓ Rearranging leads to quadratic.

iv) If SR is not a tangent to the hyperbola, the quadratic above will have two distinct real roots. ✓

Product of Roots $= \frac{e^2 + (e^2-1)^2}{e^2(2-e^2)}$

So roots have same sign if $(2-e^2) > 0$ ie, $e^2 < 2$
and opposite sign if $(2-e^2) < 0$ ie, $e^2 > 2$ ✓
$\therefore$ Same branch as P is $e^2 < 2$
$\therefore$ opposite branch to P if $e^2 > 2$

Suggested Solution (s)	Comments

Question 16

(a) When $n=1$, $L.H.S = F_0$ $\quad R.H.S = F_1 - 2$

$$= 2^{(2^0)}+1 \qquad = 2^{(2^1)}+1-2 \quad ✓$$

$$= 3 \qquad = 3$$

$\therefore$ statement is true when $n=1$.

Assume the statement is true for $n=k$, some fixed positive integer

ie. $F_0 \times F_1 \times F_2 \times \dots \times F_{k-1} = F_k - 2$

When $n=k+1$, $L.H.S = F_0 \times F_1 \times F_2 \times \dots \times F_{n-1}$

$$= F_0 \times F_1 \times F_2 \times \dots \times F_{k-1} \times F_k$$

$$= (F_k - 2) \times F_k \quad \text{by assumption} \quad ✓$$

$$= (F_k)^2 - 2F_k$$

$$= (2^{2^k}+1)^2 - 2(2^{2^k}+1)$$

$$= 2^{2\times 2^k} + 2\times 2^{2^k} + 1 - 2\times 2^{2^k} - 2 \quad ✓$$

$$= 2^{2^{k+1}} + 1 - 2$$

$$= (2^{2^n}+1) - 2$$

$$= F_n - 2 \quad \text{as required} \quad ✓$$

If statement is true for $n=k$, it has been proved true for $n=k+1$, Since true for $n=1$, then proved true for $n=2,3,4\dots$

(b) Considering the first three and last three throws separately, the are: Equal tails, More Tails, Less Tails ✓

Taking $P(\text{Equal Tails}) = P(1H) + P(2H) + P(3H) + P(0H)$

$$= 9\left(\tfrac{1}{2}\right)^6 + 9\left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^6$$

$$= \frac{20}{64} \quad ✓$$

$\therefore P(\text{More tails in first 3 throws}) = \frac{1}{2} \times \left(1 - \frac{20}{64}\right)$

$$= \frac{11}{32} \quad ✓$$

(c) (i)

By DeMoivre's theorem,

$$x^k = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

$$x^{-k} = (\cos\theta + i\sin\theta)^{-k} = \cos(-k\theta) + i\sin(-k\theta) = \cos k\theta - i\sin k\theta$$ ✓

$$\begin{aligned} x^k + x^{-k} &= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta \\ &= 2\cos k\theta \end{aligned}$$ ✓

(ii)

$$\begin{aligned}\left(x+\frac{1}{x}\right)^{2n} &= \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-2} + \binom{2n}{2}x^{2n-4} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2n-2}\frac{1}{x^{2n-4}} + \binom{2n}{2n-1}\frac{1}{x^{2n-2}} + \binom{2n}{2n}\frac{1}{x^{2n}} \\ &= \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-2} + \binom{2n}{2}x^{2n-4} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2}\frac{1}{x^{2n-4}} + \binom{2n}{1}\frac{1}{x^{2n-2}} + \binom{2n}{0}\frac{1}{x^{2n}}\end{aligned}$$ ✓ expansion

since $\binom{2n}{2n-k} = \binom{2n}{k}$ ✓ use of identity.

$$\left(x+\frac{1}{x}\right)^{2n} = \binom{2n}{0}\left(x^{2n} + \frac{1}{x^{2n}}\right) + \binom{2n}{1}\left(x^{2n-2} + \frac{1}{x^{2n-2}}\right) + \binom{2n}{2}\left(x^{2n-4} + \frac{1}{x^{2n-4}}\right) + \ldots + \binom{2n}{n-1}\left(x^2 + \frac{1}{x^2}\right) + \binom{2n}{n}$$

Note $\binom{2n}{0} = 1$

(iii)

Using the result from part i, $x^k + \frac{1}{x^k} = 2\cos k\theta$ in the identity from part ii:

$$(2\cos\theta)^{2n} = \binom{2n}{0}(2\cos 2n\theta) + \binom{2n}{1}(2\cos(2n-2)\theta) + \binom{2n}{2}(2\cos(2n-4)\theta) + \ldots + \binom{2n}{n-1}(2\cos 2\theta) + \binom{2n}{n}$$ ✓

Dividing both sides by 2:

$$2^{2n-1}\cos^{2n}\theta = \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \ldots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}$$

(iv)

$$\int_0^{2\pi} 2^{2n-1}\cos^{2n}\theta\,d\theta = \int_0^{2\pi} \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \ldots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}\,d\theta$$

Since $\int_0^{2\pi}\cos k\theta\,d\theta = 0$ for all even integers k, all the integrals on the right hand side are zero except for the constant term. ✓

$$\begin{aligned}\int_0^{2\pi} 2^{2n-1}\cos^{2n}\theta\,d\theta &= \int_0^{2\pi}\frac{1}{2}\binom{2n}{n}\,d\theta \\ &= \left[\frac{1}{2}\binom{2n}{n}\theta\right]_0^{2\pi} \\ &= \pi\binom{2n}{n}\end{aligned}$$

✓

Dividing both sides by 2^{2n-1} ,

$$\int_0^{2\pi}\cos^{2n}\theta\,d\theta = \frac{\pi}{2^{2n-1}}\binom{2n}{n}.$$