Northern Beaches Secondary College
Manly Selective Campus

2010
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators and templates may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1- 8
- All questions are of equal value

Marks

Question 1 *(Answer in a separate booklet)* ***(15)***

(a) Find $\displaystyle\int \frac{1}{\sqrt{4x^2 - 25}}\, dx$ *(2)*

(b) Find $\displaystyle\int \tan^4 x \sec^2 x \, dx$ *(2)*

(c) Find $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \, dx$ *(3)*

(d) Use the substitution $u = x$-2 to evaluate $\displaystyle\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{\sqrt{(x-1)(3-x)}}\, dx$ *(3)*

(e) (i) Write $\dfrac{3x + 2}{x^2 + 5x + 6}$ as a sum of partial fractions. *(2)*

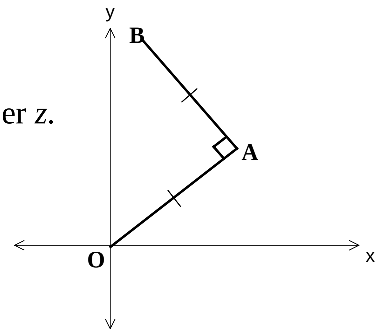
(ii) Hence evaluate $\displaystyle\int_0^2 \frac{3x + 2}{x^2 + 5x + 6}\, dx$ *(3)*

Marks

Question 2 *(Answer in a separate booklet)* ***(15)***

(a) Find the value of z^{10} in the form x + iy when $z = \sqrt{2} - \sqrt{2}\,i$ *(2)*

(b) The complex vector *z* is represented in the accompanying diagram by the point A.
The triangle OAB is a right angled isosceles triangle.



(i) Express the point B in terms of the complex number *z*. *(1)*

(ii) Let M be the midpoint of AB.
What complex number corresponds to M? *(1)*

(iii) Give an example of the vector C which would make a trapezium? *(1)*

(c) (i) Write down the value of i^6 . *(1)*

(ii) Hence or otherwise plot and label the sixth roots of -1 about the unit circle. *(2)*

(iii) Find the roots of $x^4 - x^2 + 1 = 0$ in modulus-argument form. *(2)*

(d) Given the locus of z is $|z - 2 - 2i| = 1$

(i) Sketch the locus of z on the argand diagram. *(1)*

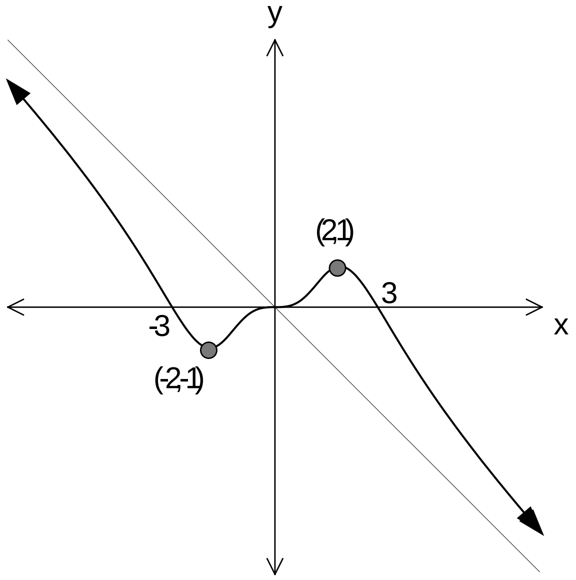
(ii) Find the maximum value of arg z. *(2)*

(iii) Find the maximum value of mod z. *(2)*

Marks

Question 3 *(Answer in a separate booklet)* ***(15)***

(a) The graph of $y = f(x)$ is shown. The line $y = -x$ is an oblique asymptote to the curve.



Use separate half page graphs, to sketch

(i) $f(-x)$ *(1)*

(ii) $f(|x|)$ *(1)*

(iii) $\dfrac{x}{f(x)}$ *(3)*

(b) Given that $1 + i$ is a zero of $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$, factorise $P(x)$ fully over the field of the complex numbers. *(2)*

(c) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equation with roots:

(i) α^2, β^2 and γ^2 *(1)*

(ii) $\alpha^3\beta\gamma, \alpha\beta^3\gamma$ and $\alpha\beta\gamma^3$ *(2)*

(d) For the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$,

(i) find the eccentricity. *(1)*

(ii) find the coordinates of the foci. *(1)*

(iii) find the equations of the directrices. *(1)*

(iv) find the equations of the asymptotes. *(1)*

(v) if P is on the hyperbola and S and S' are it's foci, then given PS=2, find PS'. *(1)*

Marks

Question 4 *(Answer in a separate booklet)* **(15)**

(a) Using the method of cylindrical shells, find the volume generated by revolving the region bounded by $y = \log_e x$, the x-axis and the interval $1 \le x \le e$ about the y axis. *(4)*

(b) The ellipse $\dfrac{(x-4)^2}{9} + \dfrac{y^2}{4} = 1$ is rotated about the y axis to form a donut shape.

(i) By taking slices perpendicular to the axis of rotation, show that the volume of a slice is $8\pi\sqrt{36 - 9y^2}\ \delta y$ *(2)*

(ii) Find the volume of the solid *(2)*

(c) Calculate the modulus and argument of the sum of the roots of the equation *(2)*

$$(3 + 4i)z^2 + (2 - i)z + (8 - 2i) = 0$$

(d) PQ is a chord of a rectangular hyperbola $xy = c^2$

(i) Show that PQ has equation $x + pqy = c(p+q)$ where P and Q have parameters p and q respectively. *(2)*

(ii) If PQ has a constant length k^2, show that

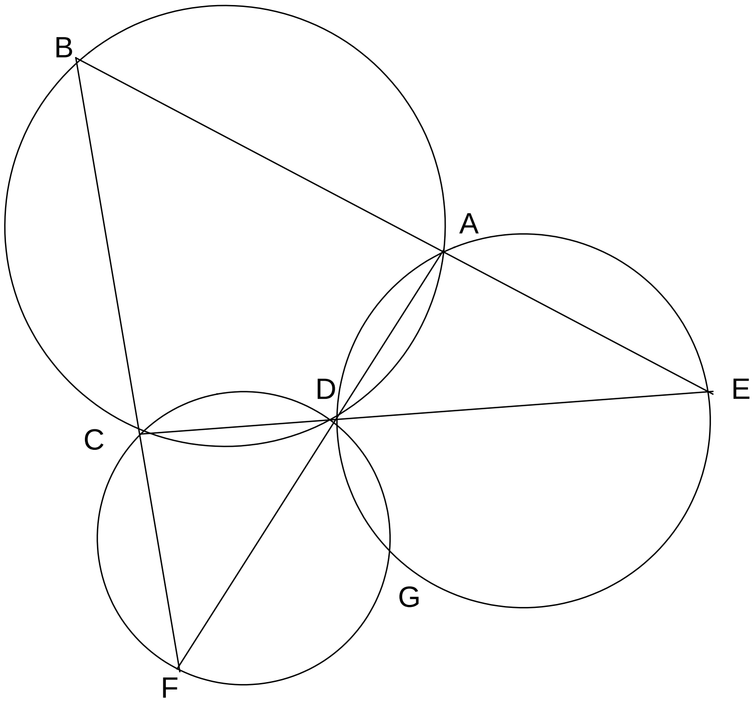
$$c^2[(p+q)^2 - 4pq]\,(p^2q^2 + 1) = k^4p^2q^2$$

and find the locus of R, the midpoint of PQ, in Cartesian form. *(3)*

Marks

Question 5 *(Answer in a separate booklet)* ***(15)***

(a) ABCD is a cyclic quadrilateral. BA and CD are both produced and intersect at E. BC and AD produced intersect at F. The circles EAD and FCD intersect at G as well as at D. Prove the points E, G and F are collinear. *(4)*



(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for integers n, $n \geq 0$

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$ for $n \geq 2$. *(2)*

(ii) Deduce that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$. *(2)*

(iii) Evaluate I_4 *(2)*

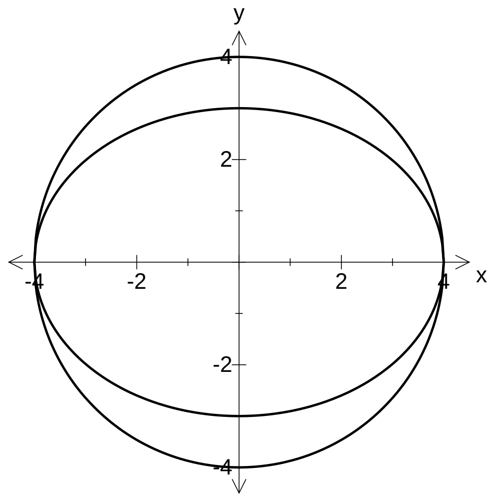
(c) (i) If α is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(\alpha) = 0$ *(2)*

(ii) Find all roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$ if two of the roots are equal. *(3)*

Marks

Question 6 ***(Answer in a separate booklet)*** ***(15)***

(a) The diagram shows an ellipse with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the larger of its auxiliary circles. The coordinates of a point P on the ellipse are $(4\cos\theta, 3\sin\theta)$ where $\theta \neq 0$ or π.



A straight line l parallel to the y axis intersects the x axis at N and the ellipse and the auxiliary circle at the points P and Q respectively.

(i) Find the equations of the tangent to the ellipse at P and to the auxiliary circle at Q. *(4)*

(ii) The tangents at P and Q intersect at point R. Show that R lies on the x axis. *(2)*

(iii) Prove that ON*OR is independent of the positions of P and Q. *(1)*

(b) On Tuesday morning at 5 am, a truck crashes into a harbour. The rescue team and their equipment can only work effectively when the depth of water is no more than 7 m. At low tide, the depth of the water is 5 metres and at high tide, the depth is 10 metres. Low tide occurs at 4 am and high tide at 10.15 am. Assume the movement of the tide is simple harmonic motion.

(i) Find the period and amplitude of the motion. *(2)*

(ii) If the deadline for the rescue operation is 6 pm on Wednesday evening, find the periods of time between 5 am and 6 pm during which the rescue team can work effectively. *(4)*

(c) Given a real polynomial Q(x) show that if α is a root of Q(x)-x=0, then α is also a root of Q(Q(x))-x=0. *(2)*

Marks

Question 7 *(Answer in a separate booklet)* ***(15)***

(a) For the curve $x^2y^2 - x^2 + y^2 = 0$

(i) state any x and y-intercepts. *(1)*

(ii) demonstrate why $|y| < 1$. *(1)*

(iii) demonstrate why $|y| \le |x|$. *(1)*

(iv) use implicit differentiation to show $\frac{dy}{dx} = \frac{x(1 - y^2)}{y(1 + x^2)}$. *(2)*

(v) state the co-ordinates of any critical points (where the derivative is undefined). *(1)*

(vi) hence, explain why the curve has no stationary points. *(1)*

(vii) state the horizontal asymptote(s). *(1)*

(viii) sketch the curve. *(1)*

(b) By expanding $(\cos\theta + i\sin\theta)^3$ it can be shown that

$$\cot 3\theta = \frac{t^3 - 3t}{3t^2 - 1} \text{ where } t = \cot\theta$$

(i) solve $\cot 3\theta = -1$ for $0 \le \theta \le 2\pi$ *(2)*

(ii) Hence show that $\cot\frac{\pi}{12} \cdot \cot\frac{5\pi}{12} \cdot \cot\frac{9\pi}{12} = -1$ *(2)*

(iii) Write down a cubic equation with roots $\tan\frac{\pi}{120}$, $\tan\frac{5\pi}{12}$ and $\tan\frac{9\pi}{12}$. *(2)*

(Express your answer as a polynomial equation with positive integer coefficients).

Marks

Question 8 *(Answer in a separate booklet)* ***(15)***

(a) The series $\frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \ldots = \sum_{n=1}^{\infty} \frac{n^3}{2^n}$ is not geometric and, as such, it is not a routine matter to decide whether or not it converges to a finite sum. Let $y_n = \frac{n^3}{2^n}$

(i) Show that $\frac{y_n}{y_{n-1}} = \frac{1}{2} \times \left(\frac{n}{n-1}\right)^3$ and hence show that this ratio is greater than 1 when $2 \le n \le 4$ but less than 1 when $n \ge 5$. *(3)*

(ii) Show that $\frac{y_n}{y_{n-1}} \le 0.98$ for $n \ge 5$. *(2)*

(iii) Given that $y_4 = 4$, deduce that $y_n \le 4 \times (0.98)^{n-4}$ for $n \ge 4$ and write down the value of $\lim_{n \to \infty} y_n$. *(2)*

(b) Show that the derivative of the function $y = x^x$ for $x > 0$ is $(\log_e x + 1)\, x^x$. *(2)*

(c) Find all solutions $(0 \le \theta < \pi)$ in radians of the equation $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{3}{4}$. *(3)*

(d) A triangle ABC is right-angled at A and it has sides of lengths a, b and c units (the side opposite angle A is a etc). A circle of radius r units is drawn so that the sides of the triangle are tangents to the inscribed circle.

Prove that $r = \frac{1}{2}(c + b - a)$. *(3)*

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

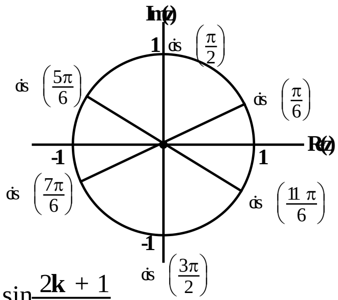
Question 1:

(a)	*Let u=2x* $du = 2dx$ $\int \frac{1}{\sqrt{4x^2-25}}\,dx$ $= \int \frac{1}{\sqrt{u^2-5^2}}\frac{du}{2}$ $= \frac{1}{2}\ln\left(u+\sqrt{u^2-5^2}\right)+c$ $= \frac{1}{2}\ln\left(2x+\sqrt{4x^2-25}\right)+c$ Alternatively $\int \frac{1}{\sqrt{4x^2-25}}\,dx$ $= \int \frac{1}{\sqrt{4\left(x^2-\frac{25}{4}\right)}}\,dx$ $= \frac{1}{2}\int \frac{1}{\sqrt{x^2-\left(\frac{5}{2}\right)^2}}\,dx$ $= \frac{1}{2}\ln\left(x+\sqrt{x^2-\frac{25}{4}}\right)+c_1$ NB These solutions only differ by a constant of integration as $\frac{1}{2}\ln\left(x+\sqrt{x^2-\frac{25}{4}}\right)+c_1$ $= \frac{1}{2}\ln\left(x+\sqrt{\frac{4x^2-25}{4}}\right)+c_1$ $= \frac{1}{2}\ln\left(\frac{2x+\sqrt{4x^2-25}}{2}\right)+c_1$ $= \frac{1}{2}\ln\left(2x+\sqrt{4x^2-25}\right)-\frac{1}{2}\ln 2+c_1$ $\therefore c = c_1-\frac{1}{2}\ln 2$	*2 marks – correct solution*
(b)	$\int \tan^4 x\sec^2 x\,dx$ $= \int \tan^4 x\,d(\tan x)$ $= \frac{\tan^5 x}{5}+c$	*2 marks – correct solution* *1 mark – applying the standard integral.*

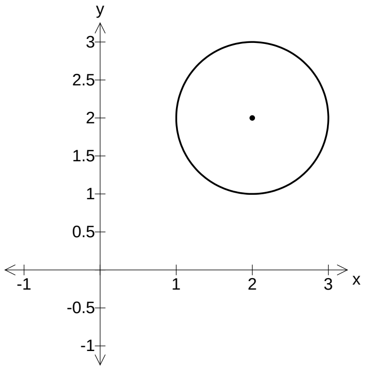
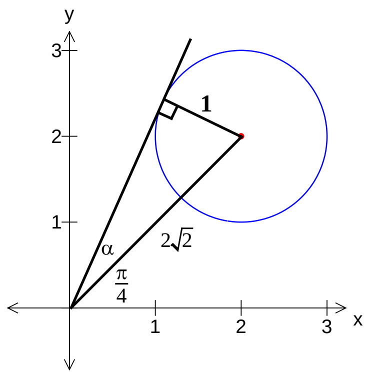
(c)	$\int_0^{\frac{\pi}{2}} x\sin x\, dx$ $= \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, dx$ $= (0 - -0) + \left[\sin x\right]_0^{\frac{\pi}{2}}$ $= 0 + (1 - 0)$ $= 1$	*3 marks – correct solution.* *2 marks – correctly integrating sinx.* *1 mark – correct substitution into integration by parts process.*
(d)	$u = x - 2$ $u + 1 = x - 1$ $1 - u = 3 - x$ when $x = \frac{3}{2},\ u = -\frac{1}{2}$ when $x = \frac{5}{2},\ u = \frac{1}{2}$ $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{(1+u)(1-u)}}\, du$ $= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\, du$ $= [\sin^{-1} u]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= \frac{\pi}{6} - -\frac{\pi}{6}$ $= \frac{\pi}{3}$	*3 marks – correct solution.* *2 marks – correctly integrating.* *1 mark – correct change of variable.*

(e) i	$\dfrac{3x+2}{(x+3)(x+2)} \equiv \dfrac{A}{x+3} + \dfrac{B}{x+2}$ $3x + 2 = A(x+2) + B(x+3)$ $When\ x = \text{-}2,\ \ -4 = B\ \therefore B = \text{-}4$ $When\ x = \text{-}3,\ \ -7 = -A\ \therefore A = 7$ $\dfrac{3x+2}{(x+3)(x+2)} \equiv \dfrac{7}{x+3} - \dfrac{4}{x+2}$	*2 marks –correct solution* *1 marks – finding A or B.*
(e) ii	$\displaystyle\int_0^2 \frac{3x+2}{x^2+5x+6}\,dx$ $\displaystyle = \int_0^2 \frac{7}{x+3} - \frac{4}{x+2}\,dx$ $= \Big[7\ln(x+3) - 4\ln(x+2)\Big]_0^2$ $= 7\ln 5 - 4\ln 4\ - 7\ln 3 + 4\ln 2$ $= 7\ln\left(\dfrac{5}{3}\right) - 4\ln 2$	*3 marks – correct solution.* *2 marks – correct substitution.* *1 mark – correct integrand.*

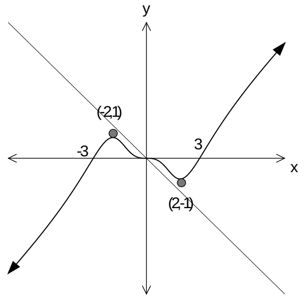
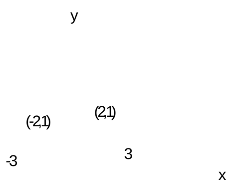
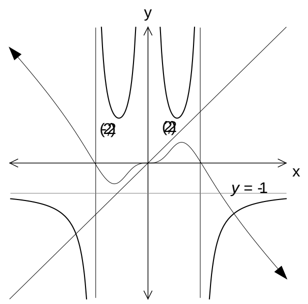
Question 2

(a)	$z = \sqrt{2} - \sqrt{2}\ i = 2\ \text{cis}\left(-\frac{\pi}{4}\right)$ $\therefore z^{10} = 2^{10}\ \text{cis}\left(\frac{-10\pi}{4}\right) = -2^{10}\ i = -1024\ i$	*2 marks – correct answer* *1 mark – correct mod and arg for z*
(b)(i)	*(O – A)(-i) = P – A* *P = A + Ai* *= z(1 + i)*	*1 mark – correct vector for P*
(ii)	$\frac{(O - A)i}{2} = M - A$ $\therefore M = z\left(\frac{i}{2} + 1\right)$	*1 mark – correct vector for M*
(iii)	*OC must be // AB* *k(A – O) I = C – O (k ≠ 1)* *C = kzi – eg 2zi*	*1 mark – correct vector*
(c(i))	$i^6 = i^4 * i^2$ $= (1)(-1) = -1$	*1 mark – correct value*
(ii)	$z^6 = -1 + 0 \times i$ $= (\cos\theta + \mathbf{i}\sin\theta)^6$ $= \cos 6\theta + \mathbf{i}\sin 6\theta$ $\therefore 6\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, \ldots$ $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$  $\therefore$ sixth roots are $\cos\frac{2\mathbf{k}+1}{6}\pi + \mathbf{i}\sin\frac{2\mathbf{k}+1}{6}$ for k = 0, 1, 2, 3, 4,5	*2 marks – correct solutions and correct diagram appropriately labelled* *1 mark - correct solutions but incompletely labelled*
(c)(iii)	$z^6 + 1 = 0$ $(z^2)^3 + (1)^3 = 0$ $(z^2 + 1)(z^4 - z^2 + 1) = 0$ But $z^2 + 1 = 0$ has roots $z = \pm i$ $\therefore z^4 - z^2 + 1 = 0$ has roots $z = \text{cis}\frac{\pi}{6}, \text{cis}\frac{5\pi}{6}, \text{cis}\frac{7\pi}{6}$ and $\text{cis}\frac{11\pi}{6}$	*2 marks – correct roots* *1 mark – recognition of the basic factorisation*

Question 2 (continued)

(d) (i)		*1 mark – correct diagram showing the locus (only)*
(ii)	Max arg when the tangent is on left side of the circle) - NOT through point (1, 2). $\therefore \quad \sin\alpha = \frac{1}{2\sqrt{2}}$ $\therefore \quad \alpha = 20^\circ 42'$ argument of vector to centre is $\frac{\pi}{4}$ $\therefore$ max arg $= \frac{\pi}{4} + \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) = 65^\circ 42'$ 	*2 marks – correct argument* *1 mark – correct argument for vector to centre*
(iii)	Max modulus is when the vector extends through the centre to the other side of the circle. So modulus vector is $2\sqrt{2}$ and radius is 1. Hence max modulus is $2\sqrt{2} + 1$	*2 marks – correct modulus* *1 mark – indication of where the modulus would be*

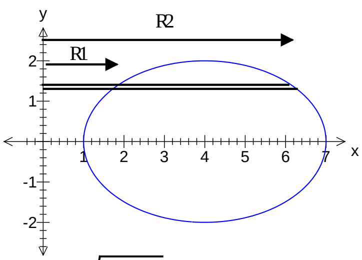
Question 3:

(a)i		*1 mark –correct graph.*
(a)ii		*1 mark –correct graph.*
(a)iii		*3 mark –correct graph.* *1 for both branches above the x-axis* *1 for both branches below x-axis* *1 for horizontal asymptote at y=-1 and for turning points at(+-2,2)*

(b)	As (1+i) is a zero, (1-i) is also a zero. $\therefore\ (x-(1+i))(x-(1-i))$ is a factor $=\ (x^2-2x+2)$ so other factors maybe found by factor theorem or polynomial division $P(1) = 0\ \therefore\ x-1$ is a factor $P(-2) = 0 \therefore\ x+2$ is a factor $\therefore P(x) = (x-1)(x+2)(x-1-i)(x-1+i)$	*2 mark – correct factorisation* *1 mark- correct application of "complex roots of a polynomial, with real coefficients, occur in conjugate pairs"*
(c) i	$x^3 + x^2 - 2x - 3 = 0$ has roots $x = \alpha,\ \beta$ and γ So the equation with roots $X = \alpha^2,\ \beta^2$ and γ^2 will have $\sqrt{X} = \alpha,\ \beta$ and γ so $x = \sqrt{X}$ will satisfy the original equation. $(\sqrt{X})^3 + (\sqrt{X})^2 - 2(\sqrt{X}) - 3 = 0$ $X^{\frac{3}{2}} + X - 2X^{\frac{1}{2}} - 3 = 0$ $X^{\frac{3}{2}} - 2X^{\frac{1}{2}} = 3 - X$ $X^3 - 4X^2 + 4X = 9 - 6X + X^2$ $X^3 - 5X^2 + 10X - 9 = 0$	*1 mark correct equation.*
(c) ii	$\alpha\beta\gamma = 3$ $\therefore \alpha^3\beta\gamma,\ \alpha\beta^3\gamma,\ \alpha\beta\gamma^3$ $= \alpha\beta\gamma\,\alpha^2,\ \alpha\beta\gamma\,\beta^2,\ \alpha\beta\gamma\,\gamma^2$ $= 3\alpha^2,\ 3\beta^2,\ 3\gamma^2$ which are 3 times the roots of $x^3 - 5x^2 + 10x - 9 = 0$ So substitute $\frac{x}{3}$ into $x^3 - 5x^2 + 10x - 9 = 0$ Alternatively, substitute $\sqrt{\frac{x}{3}}$ into $x^3 + x^2 - 2x - 3 = 0$ $\left(\frac{x}{3}\right)^3 - 5\left(\frac{x}{3}\right)^2 + 10\left(\frac{x}{3}\right) - 9 = 0$ $\frac{x^3}{27} - \frac{5x^2}{9} + \frac{10x}{3} - 9 = 0$ $x^3 - 15x^2 + 90x - 243 = 0$	***2 marks correct polynomial*** ***1 mark for obtaining*** $3\alpha^2\ 3\beta^2\ 3\gamma^2$

(d) i	$b^2 = a^2(e^2 - 1)$ $25 = 16(e^2 - 1)$ $e^2 = 1 + \frac{25}{16}$ $e = \pm\frac{\sqrt{41}}{4}$ but, as for a hyperbola, e>0 then $e = \frac{\sqrt{41}}{4}$	*1 mark*
(d) ii	$(\pm ae, 0) = (\pm\sqrt{41}, 0)$	*1 mark*
(d) iii	$x = \pm\frac{a}{e} = \pm\frac{16}{\sqrt{41}}$	*1 mark*
(d) iv	$y = \pm\frac{b}{a}x$ $y = \pm\frac{5}{4}x$	*1 mark*
(d) v	|PS –PS’|=2a |2-PS’|=8 PS’=10 or -6, but PS’ is a distance PS’=10	*1 mark*

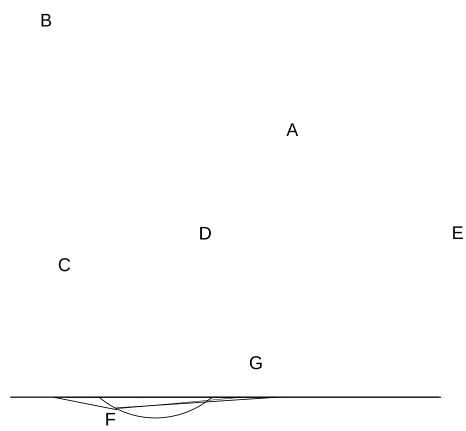
Question 4

(a)	Area of shell $= \pi\{(x+\delta x)^2 - x^2\}$ $\therefore$ Vol $= \pi\, 2x\, dx\, y$ after letting $\delta x^2 \to 0$ or $2\pi xy\,\delta x = 2\pi x \ln x\, \delta x$ $\therefore$ Vol $= \lim_{\delta x \to 0} \sum_{1}^{e} 2\pi x \ln x\, \delta x$ $= 2\pi \int_1^e x \ln x\, dx$ $= 2\pi \left[\ln x \times \frac{x^2}{2}\right]_1^e - 2\pi \int_1^e \frac{x}{2} dx$ $= 2\pi\left[\frac{e^2}{2} - 0\right] - 2\pi\left[\frac{e^2}{4} - \frac{1}{4}\right]$ $= \frac{\pi}{2}[e^2 + 1]$	*4 marks – correct volume* *3 marks – application of integration by parts but subsequent error* *2 marks – correct statement for V* *1 mark – correct development for δV.* *NOTE: (x-1) is not necessary because the integral goes from 1 to e.*
(b) (i)	 $\delta V = \pi(R_2{}^2 - R_1{}^2)\,\delta y$ $x = 4 \pm 3\sqrt{1 - \frac{y^2}{4}}$ $\therefore R_2 + R_1 = 8$ and $R_2 - R_1 = 6\sqrt{1 - \frac{y^2}{4}}$ $\delta V = \pi\, 8 \times \frac{6}{2} \times \sqrt{4 - y^2}\,)$ $= 8\pi\sqrt{36 - 9y^2}$	*2 marks – correct demonstration* *1 mark – correct approach with sufficient progress*
(ii)	$V = \lim_{\delta y \to 0} \sum_{-2}^{2} 24\pi\sqrt{4 - y^2}\ \delta y$ $= 24\pi \int_{-2}^{2} \sqrt{4 - y^2}\ dy$ $= 24\pi \times \frac{\pi \times 2^2}{2} = 48\pi^2$	*2 marks – correct volume calculated* *1 mark – correct approach using any appropriate method (eg trig substitution).* *Always make your life easier by looking for the" circle approach" when you can.*

Question 4 *(continued)*

(c)	Sum of roots $= \dfrac{-(2-\mathbf{i})}{3+4\mathbf{i}}$ $= \dfrac{(\mathbf{i}-2)}{3+4\mathbf{i}} \times \dfrac{3-4\mathbf{i}}{3-4\mathbf{i}}$ $= \dfrac{11\mathbf{i}-2}{25}$ $\therefore$ Modulus $= \sqrt{\dfrac{11^2 + 2^2}{25^2}} = \dfrac{1}{\sqrt{5}} = 0.447$ Argument $= \tan^{-1}\left(-\dfrac{\frac{11}{25}}{\frac{2}{25}}\right) = \tan^{-1}\left(-\dfrac{11}{2}\right) = 100°18'$	*2 marks – correct answers* *1 mark – sum of roots expressed correctly*
(d)(i)	Let the points be $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ $\therefore \dfrac{y-\frac{c}{p}}{\frac{c}{q}-\frac{c}{p}} = \dfrac{x - cp}{cq - cp}$ $\dfrac{\frac{py-c}{p}}{\frac{c(p-q)}{pq}} = \dfrac{x-cp}{c}$ $\dfrac{(py-c)q}{c} = -\dfrac{x-cp}{c}$ $\therefore x + pqy = c(p+q)$	*2 marks – correct proof* *1 mark – correct approach with progress but minor error*
(ii)	$PQ^2 = (cp-cq)^2 + \left(\dfrac{c}{p} - \dfrac{c}{q}\right)^2 = k^4$ $= c^2(p-q)^2 + \dfrac{c^2}{p^2q^2}(q-p)^2$ $\therefore \mathbf{c}^2(\mathbf{p}-\mathbf{q})^2\left(1 + \dfrac{1}{\mathbf{p}^2\mathbf{q}^2}\right) = \mathbf{k}^4$ $\therefore c^2[(p+q)^2 - 4pq](p^2q^2+1) = k^4p^2q^2$ R is $X = \dfrac{cp+cq}{2} = \dfrac{c(p+q)}{2}$ $Y = \dfrac{cq+cp}{2pq}$ $\therefore$ eliminating (p+q) $X = pqY$ Substituting the X = eqn for (p+q) and the last for pq into the previously proved equation gives: $c^2\left[\dfrac{4x^2}{c^2} - \dfrac{4x}{y}\right]\left[\dfrac{x^2}{y^2} + 1\right] = k^4\dfrac{x^2}{y^2}$ $[4x^2y - 4xc^2][x^2+y^2] = k^4x^2y$ $4(xy-c^2)(x^2+y^2) = k^4xy$	*3 marks – final equation for locus correct* *2 marks – values for x and y calculated correctly and some appropriate progress made from there* *1 mark – first equation proved correctly*

Question 5:

(a) i	 Construct chords DG, GF and GE as shown $\angle DGF = \angle BCD$(Exterior angle of cyclic quadrilateral equals interior opposite angle) $\angle DGE = \angle BAD$(Exterior angle of cyclic quadrilateral equals interior opposite angle) $\angle BCD + \angle BAD = 180$(opposite angles of cyclic quadrilateral are supplementary) $\therefore \angle DGF + \angle DGE = 180$ Hence F, G and E are collinear, as angles on a straight line are supplementary.	*4 marks – 1mark for each correct step with a reason.*

(b) i	$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx$ $u = \sin^{n-1} x \qquad dv = \sin x$ $du = (n-1)\sin^{n-2} x\cos x dx \qquad v = -\cos x$ $I_n = \left[\sin^{n-1} x \times -\cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x\cos x \times -\cos x dx$ $= [0-0] + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x dx$	*2 marks – correct solution* *1 mark – correct choice of u and dv.*
(b) ii	$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x dx$ $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x) dx$ $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x dx$ $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx$ $I_n = (n-1)I_{n-2} - (n-1)I_n$ $I_n + (n-1)I_n = (n-1)I_{n-2}$ $nI_n = (n-1)I_{n-2}$ $I_n = \frac{n-1}{n} I_{n-2}$	*2 marks – correct proof.* *1 marks – for obtaining* $(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x dx$
(b) iii	$I_4 = \frac{3}{4} I_2$ $= \frac{3}{4}\left(\frac{1}{2} I_0\right)$ $= \frac{3}{8}\int_0^{\frac{\pi}{2}} \sin^0 x dx$ $= \frac{3}{8}\left[x \right]_0^{\frac{\pi}{2}}$ $= \frac{3\pi}{16}$	*2 marks – correct solution.* *1 mark – correct primitive*
(c) i	Let $P(x) = (x-\alpha)^n Q(x)$ $P'(x) = n(x-\alpha)^{n-1} Q(x) + (x-\alpha)^n Q'(x)$ $P'(x) = (x-\alpha)^{n-1}[nQ(x) + (x-\alpha)Q'(x)]$ $\therefore \quad P'(\alpha) = 0$ and al is a root of $P'(x)$	*2 marks – correct proof.* *1 mark for defining P(x) as* Let $P(x) = (x-\alpha)^n Q(x)$

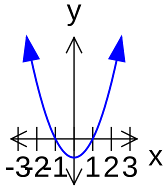
(c) ii	$P'(x) = 54x^2 + 6x - 28$ $P'(x) = 2(27x^2 + 3x - 14)$ $P'(x) = 2(9x + 7)(3x - 2)$ $x = -\frac{7}{9} \text{ or } \frac{2}{3}$ $P\left(\frac{2}{3}\right) = 18\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 - 28\left(\frac{2}{3}\right) + 12 = 0$ $\therefore \quad P'\left(\frac{2}{3}\right) = P\left(\frac{2}{3}\right) = 0$ and x=2/3 is the double root. $\therefore \quad (3x - 2)^2(ax - \beta) = 18x^3 + 3x^2 - 28x + 12$ $(9x^2 - 12x + 4)(ax - \beta) = 18x^3 + 3x^2 - 28x + 12$ so by inspection of the coefficients $(3x - 2)^2(2x + 3) = 18x^3 + 3x^2 - 28x + 12$ the roots are 2/3 and -3/2	*3 marks – correct solution.* *2 mark –* $P'\left(\frac{2}{3}\right) = P\left(\frac{2}{3}\right) = 0$ *1 mark-* $x = -\frac{7}{9} \text{ or } \frac{2}{3}$

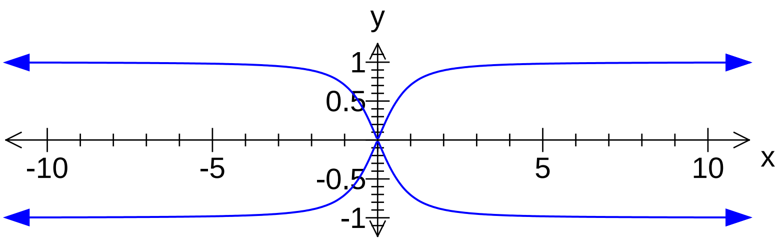
Question 6

(a)(i)	At P $x = 4\cos\theta \quad y = 3\sin\theta$ $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ $= \frac{-3\cos\theta}{4\sin\theta}$ $\therefore y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$ $\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ At Q $x = 4\cos\theta \quad y = 4\sin\theta$ $\therefore \frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta}$ $\mathbf{y - 4\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - 4\cos\theta)}$ $\frac{x\cos\theta}{4} + \frac{y\sin\theta}{4} = 1$	*For each equation:* *2 marks – equation correct* *1 mark – gradient correct*
(ii)	*Solving simultaneously gives* $x\cos\theta = 4$ $\therefore$ substituting back in $16 + 4y\sin\theta = 16$ $\therefore y = 0$ (so the x - axis)	*2 marks –correct demonstration* *1 mark – correct approach to solving simultaneous equations*
(iii)	$ON = 4\cos\theta$ $OR = \frac{4}{\cos\theta}$ $\therefore ON \times OR = 16$ which is independent of θ and so of the positions of P and Q	*1 mark – correct demonstration*
(b) (i)	*Amplitude = 2.5 m* *Period = 12.5 hours*	*2 marks – both amplitude and period correct* *1 mark – either amplitude and period correct*

(ii)	$n = \frac{4\pi}{25}$ $\therefore x = 2.5\cos\left(\frac{4\pi t}{25} + \varepsilon\right)$ $t = 0,\ x = -2.5$ (so at low tide) $\therefore \varepsilon = \pi$ $\therefore x = 2.5\cos\left(\frac{4\pi t}{25} + \pi\right) = -2.5\cos\left(\frac{4\pi t}{25}\right)$ $x = -0.5\ t = \frac{25}{4\pi}\cos^{-1}(0.2)$ $\therefore t =$ 2 hrs 43 mins after 4 am i.e at 6 hrs 43 mins and 2hr 43 mins before 4:30 pm - so 1:47 pm so crew can work from 5 am to 6:43 am and from 1:47 pm to 6:00 pm	*4 marks – correct time intervals determined* *3 marks – correct expression for time* *2 marks – correct value for initial phase determined* *1 mark – appropriate format for a cos equation stated*
(c)	If α is a root $Q(\alpha) - \alpha = 0$ $\therefore Q(\alpha) = \alpha$ $\therefore$ when α is substituted $Q(Q(x)) - x = Q(Q(\alpha)) - \alpha$ $= Q(\alpha) - \alpha = 0$ $\therefore \alpha$ is a root	*2 marks – correct demonstration* *1 mark – some appropriate substitution made.*

Question 7:

(a) i	When $x = 0$ $0 - x^2 - 0 = 0$ $\therefore \quad x = 0$ So x-y intercept *is* $(0, 0)$	*1mark*
ii	$x^2y^2 - x^2 + y^2 = 0$ rearranges to $y^2 = \dfrac{x^2}{1 + x^2}$ and since $x^2 \leq 1 + x^2$ $y^2 < 1$ $y^2 - 1 < 0$ $(y - 1)(y + 1) < 0$ $-1 < y < 1$ $\lvert y \rvert < 1$  	*1mark*
iii	The equation may also be rearranged to $\dfrac{y^2}{x^2} = 1 - y^2.$ As $y^2 \geq 0$ $-y^2 \leq 0$ then $1 - y^2 \leq 1$ $\therefore \quad \dfrac{y^2}{x^2} \leq 1$ $y^2 \leq x^2$ $\sqrt{y^2} \leq \sqrt{x^2}$ $\lvert y \rvert \leq \lvert x \rvert$	*1mark*
iv	$2xy^2 + x^2\, 2y\dfrac{dy}{dx} - 2x + 2y\dfrac{dy}{dx} = 0$ $\dfrac{dy}{dx}(2x^2y + 2y) = 2x - 2xy^2$ $\dfrac{dy}{dx} = \dfrac{2x - 2xy^2}{2x^2y + 2y}$ $\dfrac{dy}{dx} = \dfrac{2x(1 - y^2)}{2y(1 + x^2)}$ $\dfrac{dy}{dx} = \dfrac{x(1 - y^2)}{y(1 + x^2)}$	*2 marks for correct demonstration* *1 mark for implicit differentiation*
v	Critical points occur when $\dfrac{dy}{dx}$ is undefined at y=0. Therefore the critical point is (0,0).	*1mark*

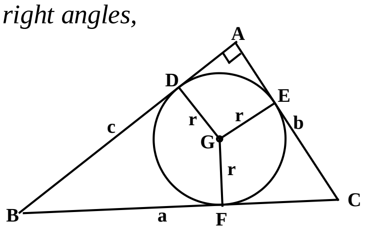
vi	Possible stationary points occur when $\frac{dy}{dx} = 0$, $0 = \frac{x(1-y^2)}{y(1+x^2)}$ $x = 0$ or $y = \pm 1$ But these values do not result in stationary points as they are either at the critical point or outside the range determined above. So there can be no stationary points.	*1mark*
vii	y=+-1 Use limit as x approaches plus or minus infinity.	*1 mark*
viii	Note the relation can be rewritten in form $y^2 = \frac{x^2}{1+x^2}$ and all curves of form $y^2 = f(x)$ have symmetry about x-axis because $y = \pm\frac{x}{\sqrt{1+x^2}}$. Also it is odd. 	*1 mark*
b i	$\cot 3\theta = -1$ $\tan 3\theta = -1$ $3\theta = -\frac{\pi}{4} + k\pi$ where k is an integer $3\theta = \frac{-\pi + 4k\pi}{4}$ $\theta = \frac{-\pi(1-4k)}{12}$ so for $0 \le \theta \le 2\pi$ when k=1,2,3,4,5,6 $\theta = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$	*2 marks for correct angles.* *1 mark for some of the correct angles.*

	If $\cot 3\theta = -1$ and given $\cot 3\theta = \dfrac{t^3 - 3t}{3t^2 - 1}$ then $$-1 = \frac{t^3 - 3t}{3t^2 - 1}$$ $$1 - 3t^2 = t^3 - 3t$$ $$0 = t^3 + 3t^2 - 3t - 1$$ 3 unique solutions of this polynomial are $$t = \cot\frac{\pi}{4},\ \cot\frac{7\pi}{12} \text{ and } \cot\frac{11\pi}{12}$$ By the product of the roots of $$0 = t^3 + 3t^2 - 3t - 1$$ $$\cot\left(\frac{\pi}{4}\right)\cot\left(\frac{7\pi}{12}\right)\cot\left(\frac{11\pi}{12}\right) = 1$$ and since $\cot(-\theta) = -\cot\theta$ then $$-\cot\left(\frac{3\pi}{4}\right) \times -\cot\left(\frac{5\pi}{12}\right) \times -\cot\left(\frac{\pi}{12}\right) = 1$$ $$-\cot\left(\frac{9\pi}{12}\right) \times \cot\left(\frac{5\pi}{12}\right) \times \cot\left(\frac{\pi}{12}\right) = 1$$ $$\cot\left(\frac{\pi}{12}\right) \times \cot\left(\frac{5\pi}{12}\right) \times \cot\left(\frac{9\pi}{12}\right) = -1$$	*2 marks for correct demonstration.* *1 mark for a <u>correct</u> statement regarding the product of the roots.*
	$$0 = \left(\frac{1}{t}\right)^3 + 3\left(\frac{1}{t}\right)^2 - 3\left(\frac{1}{t}\right) - 1$$ $$0 = 1 + 3t - 3t^2 - t^3$$	*1 mark for correct demonstration*

Question 8:

(a) (i)	$f(n) = \frac{y_n}{y_{n-1}} = \frac{n^3}{2^n} \times \frac{2^{n-1}}{(n-1)^3}$ $= \frac{1}{2}\left(\frac{n}{n-1}\right)^3$ $f'(n) = \frac{3}{2}\left(\frac{n}{n-1}\right)^2 \times \frac{(n-1)\times 1 - n \times 1}{(n-1)^2}$ $= \frac{-3n^2}{2(n-1)^4}$ For $n \geq 5$ $f'(n) < 0$ so it is a decreasing function $f(2) = 4$ $f(3) = \frac{27}{16}$ and $f(4) = \frac{32}{27}$ but $f(5) = \frac{125}{128} < 1$ $\therefore$ ratio > 1 for $2 \leq n \leq 4$ but < 1 for $n \geq 5$	*3 marks – correct demonstration or interpretation of pattern* *2 marks – progress with the derivative interpretation or examination of values* *1 mark – correct simplification of ratio and than an attempt to apply it* *NOTE: you need a strategy to show that the ratio does not start to increase.*
(ii)	We know that $f(5) = \frac{125}{128} = 0.977 < 0.98$ and the function is decreasing $\therefore$ always less than 0.98	*2 marks – correct argument* *1 mark – some justification as to why it should be less than 0.98*
(iii)	*The ratio of successive terms in this series is less than 0.98 (from part (ii)). Hence the series will decrease more rapidly than a geometric series which has a common ratio of 0.98.* *So* $y_n \leq 4 \times (0.98)^{n-4}$ for $n \geq 4$ *as $y_4 = 4$. But for the GS, the nth term approaches 0 because $(0.98)^{n-4}$ approaches 0. So the limit of y_n is zero.*	*2 marks – correct argument* *1 mark – correct approach but logic error*
(b)	$y = x^x$ $\therefore \ln y = x \ln x$ $y = e^{x \ln x}$ $\frac{d}{dx}(x \ln x) = \ln x + 1$ $\therefore \frac{dy}{dx} = e^{x\ln x} \times (\ln x + 1)$ $= (\ln x + 1)\, x^x$	*2 marks – correct demonstration* *1 mark – transformation to base e*

Question 8 (continued)

(c)	$\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \dfrac{3}{4}$ $\dfrac{(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{\sin\theta + \cos\theta} = \dfrac{3}{4}$ $\therefore 1 - \sin\theta\cos\theta = \dfrac{3}{4} (\sin\theta \neq -\cos\theta)$ $\therefore \sin 2\theta = \dfrac{1}{2}$ $2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ $\theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$	*3 marks – two correct solutions* *2 marks – relevant progress to a statement for 2θ.* *1 mark – numerator factored correctly*
(d)	*As radii meet the tangents at right angles,*  *<ADG = <AEG = 90°* *As <DAE = 90° (given)* *and AD = AE (tangents from an external point are equal)* *AEGD is a square.* *So AD = AE = r* *So BD = c-r = BF* *EC = b-r = CF* *So BC = a = b-r+c-r* $\therefore r = \dfrac{1}{2}(c + b - a)$	*3 marks – correct proof* *2 marks – appropriate work towards solution* *1 mark – identification that AD and AE = r*