Northern Beaches Secondary College

Manly Selective Campus

2013 HSC –Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours .
- Write using blue or black pen.
- Board-approved calculators and templates may be used.
- All necessary working should be shown in every question.
- Multiple choice questions to be completed on the special answer page.
- Each free response questions to be completed in separate booklets.
- If using more than one booklet per question, number booklet "1 of __"

Total marks – 100 marks

- Attempt Questions 1-16
- Multiple Choice – answer question on answer sheet provided.
- Multiple Choice – 1 mark per question
- Short Answer questions – marks as indicated.

MULTIPLE CHOICE SECTION.

Answer the following questions on the answer sheet provided.

Q1. What is the number of asymptotes on the graph $y = \dfrac{x^2}{x^2 - 1}$?

A. 1

B. 2

C. 3

D. 4

Q2. The value of $\displaystyle\int_0^1 \frac{\cos^{-1}}{\sqrt{1-x^2}}\,dx$ is

A. $-\dfrac{\pi^2}{8}$

B. $\dfrac{\pi^2}{8}$

C. $-\dfrac{\pi^2}{4}$

D. $\dfrac{\pi^2}{4}$

Q3. The base of the solid is the circle $x^2 + y^2 = 1$. Every cross section of the solid taken perpendicular to the *x*-axis is a right-angled, isosceles triangle with its hypotenuse lying in the base of the solid.

Which of the following is an expression for the volume *V* of the solid?

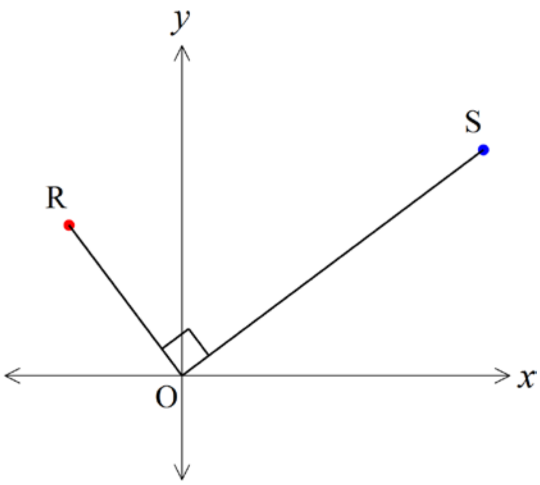
A. $\displaystyle\int_{-1}^{1} \left(1 - x^2\right) dx$

B. $\displaystyle 2\int_{-1}^{1} \left(1 - x^2\right) dx$

C. $\displaystyle 4\int_{-1}^{1} \left(1 - x^2\right) dx$

D. $\displaystyle 8\int_{-1}^{1} \left(1 - x^2\right) dx$

Q4. In the Argand diagram below, the points R and S represent complex numbers ω and z respectively where $\angle SOR = 90^\circ$. The distance OS is $2a$ units and the distance OR is a units.



Which of the following is correct?

A. $\omega = 2iz$

B. $\omega = \mathbf{i}\,\overline{\omega}$

C. $\omega = -\dfrac{iz}{2}$

D. $\omega = -\dfrac{z}{2i}$

Q5. *Px)* is a polynomial of degree 4. Which of the following statements must be false?

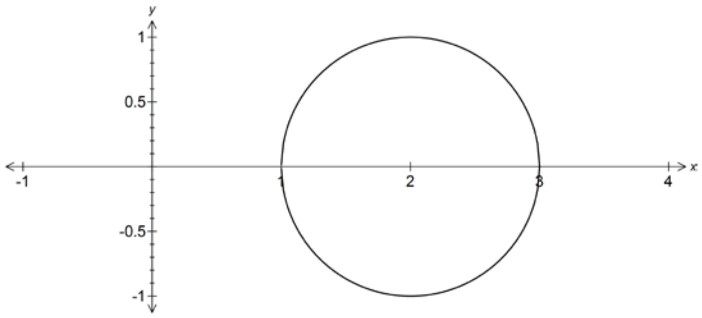
A. *P(x)* has no real roots

B. *P(x)* has 1 real root and 3 non-real roots

C. *P(x)* has 2 real roots and 2 non-real roots.

D. *P(x)* has 4 real roots.

Q6. In the diagram, the circle $(x-2)^2 + y^2 = 1$ is drawn. The circle is rotated around the line $x = 1$. Using the method of cylindrical shells, the volume of the solid is given by the expression.



A. $V = 2\pi \int_1^3 (x-1)\sqrt{1+(x-2)^2}\, dx$

B. $V = 4\pi \int_1^3 (x-1)\sqrt{1+(x-2)^2}\, dx$

C. $V = 2\pi \int_1^3 (x-1)\sqrt{1-(x-2)^2}\, dx$

D. $V = 4\pi \int_1^3 (x-1)\sqrt{1-(x-2)^2}\, dx$

Q7. What is the multiplicity of the root $x = 1$ of the equation

$3x^5 - 5x^4 + 5x - 3 = 0$

A. 1

B. 2

C. 3

D. 4

Q8. Which of the following graphs is the locus of the point *P* representing the complex number *z* moving in an Argand digram such that $|z - 2i| = 2 + Im z$?

A. A straight line

B. A parabola

C. A circle

D. A hyperbola

Q9. Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ where $n \geq 2$. Which of the following is the correct expression for I_n.

A. $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

B. $I_n = \left(\frac{n+1}{n}\right) I_{n-2}$

C. $I_n = \left(\frac{n}{n-1}\right) I_{n-2}$

D. $I_n = \left(\frac{n}{n+1}\right) I_{n-2}$

Q10. The first derivative of the implicit function $x\sin y + y\cos x = 1$ is:

A. $\frac{y\sin x}{x\cos y}$

B. $\left(\frac{y}{x}\right)\tan x$

C. $\frac{y\sin x - x\cos y - \sin y}{\cos x}$

D. $\frac{y\sin x - \sin y}{x\cos y + \cos x}$

FREE RESPONSE SECTION – <u>answer each question in a separate booklet</u>

Question 11 START A NEW BOOKLET **15 marks**

a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x . \sin^3 x \, dx$ (2)

b) Find $\int_2^5 \frac{2dx}{x^2 - 4x + 13}$ (3)

c) (i) Find real numbers A, B and C such that $\frac{5}{x^2(2-x)} \equiv \frac{Ax + B}{x^2} + \frac{C}{2-x}$ (2)

(ii) Hence, or otherwise, find $\int \frac{20}{x^2(2-x)} dx$ (3)

d) Use the substitution $x = \sin\theta$ to find $\int \frac{x^2}{\sqrt{1-x^2}} dx$ (3)

e) Evaluate $\int_0^1 x \sin^{-1} x \, dx$ (2)

Question 12 START A NEW BOOKLET **15 marks**

a) The cubic equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α , β and γ.

(i) Find the cubic equation with roots α^2 , β^2 and γ^2 (2)

(ii) Hence evaluate $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$. (2)

b) (i) Given that ω is one of the complex roots of $z^3 = 1$, show that $1 + \omega + \omega^2 = 0$. (2)

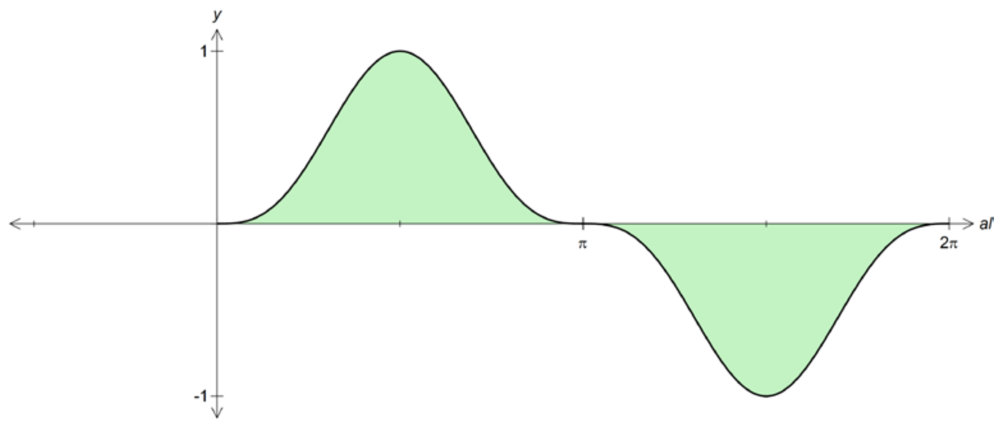
(ii) Hence, or otherwise, evaluate $1 + \omega^4 + \omega^8$ (1)

c) Consider the complex number $z = \cos\alpha + i\sin\alpha$

(i) Prove that $z^n - \frac{1}{z^n} = 2i\sin n\alpha$ (2)

(ii) Expand $\left(z - \frac{1}{z}\right)^3$ and hence prove $\sin^3\alpha = \frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha$. (2)

d) Below is a sketch of $y = \sin^3\alpha$ for $0 \le \alpha \le 2\pi$. Find the area of the shaded region. (2)



e) Sketch the region of the Argand diagram for which the complex number $z = x + iy$ satisfies $Re(z) > 0$ and $|z + 3i| \le 2$. (2)

Question 13 START A NEW BOOKLET **15 marks**

a) $P\left(p, \frac{1}{p}\right)$ and $Q\left(q, \frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$.
M is the midpoint of PQ.

(i) Show that the chord PQ has equation $x + pqy = p + q$ (2)

(ii) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the chord PQ from the origin $O(0,0)$ is always $\sqrt{2}$ units , show that $(p+q)^2 = 2\left[1 + p^2q^2\right]$ (1)

(iii) Hence find the equation of the locus of M. State the restrictions on the domain of the locus. (3)

b) The ellipse E has the equation $4x^2 + 9y^2 = 36$

(i) Write down

(α) the eccentricity of E (1)

(β) the coordinates of the positive focus S (1)

(γ) the equation of the positive directrix (1)

(δ) the length of the major axis (1)

(ii) Draw a clear sketch of the ellipse E. Show on your sketch the directrices, the coordinates of the foci and the vertices. (2)

c) Find the values of the real numbers p and q such that $1 - i$ is a root of the equation $x^3 + px + q = 0$. (3)

Question 14 START A NEW BOOKLET **15 marks**

a) Find $\int \sqrt{1 + \sin 2x}\ dx$ (2)

b) Use De Moivrés' theorem to show that

(i) (α) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (1)

(β) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ (1)

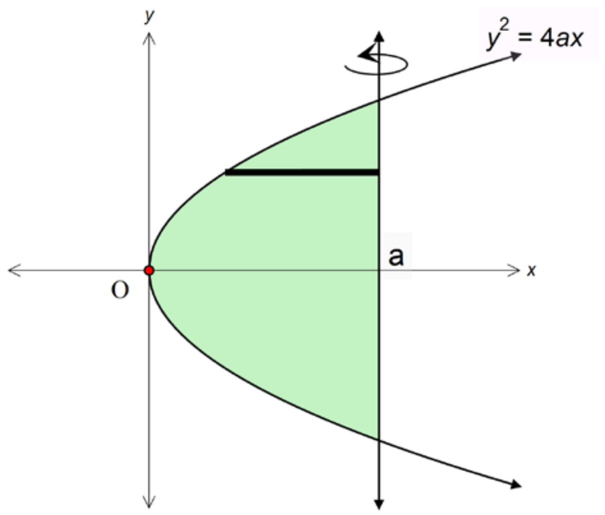
(ii) Deduce that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$ (2)

(iii) Hence, or otherwise, show that $\tan\dfrac{\pi}{12}$ is a root of the equation (3)

$x^3 - 3x^2 - 3x + 1 = 0$

(iv) Show that $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12} = 4$ (2)

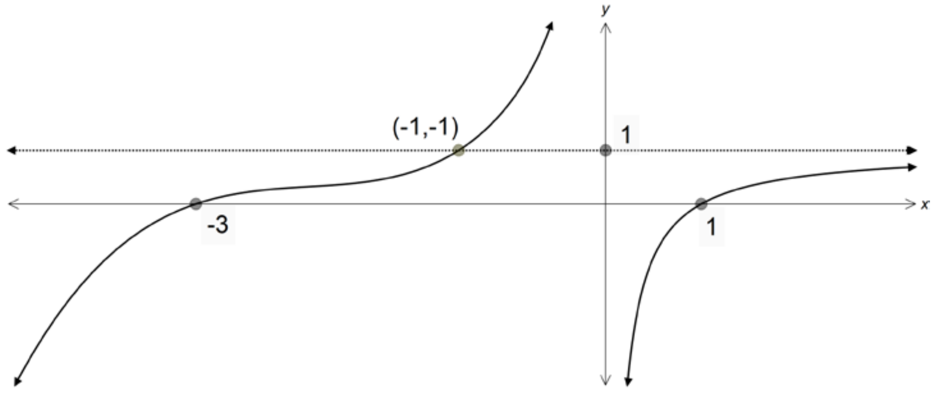
c) The figure drawn below shows the shaded area enclosed by the parabola $y^2 = 4ax$ and the line $x = a$.



The area is rotated around the line $x = a$. By considering slices perpendicular to the axis of rotation, find the volume of the solid of revolution. (4)

Question 15 START A NEW BOOKLET **15 marks**

a)



The diagram shows the graph $y = f(x)$. On separate axes, sketch graphs of each of the following functions. Show all details. [Your diagrams should be at least $\frac{1}{3}$ of a page in size, with axes drawn with a ruler and a clear scale on each axis.] (2)

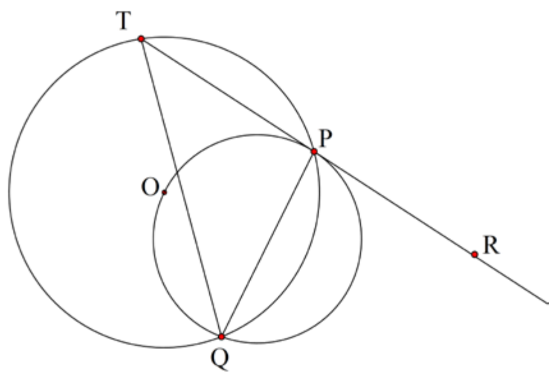
(i) $y = |f(x)|$ (1)

(ii) $y = f(-x)$ (1)

(iii) $y = \dfrac{1}{f(x)}$ (2)

(iv) $y = e^{f(x)}$ (2)

b) Two circles intersect at P and Q as shown. The smaller circle passes through the centre O of the larger circle. The tangent to the smaller circle RPT intersects the larger circle at T. PQ bisects $\angle RQO$



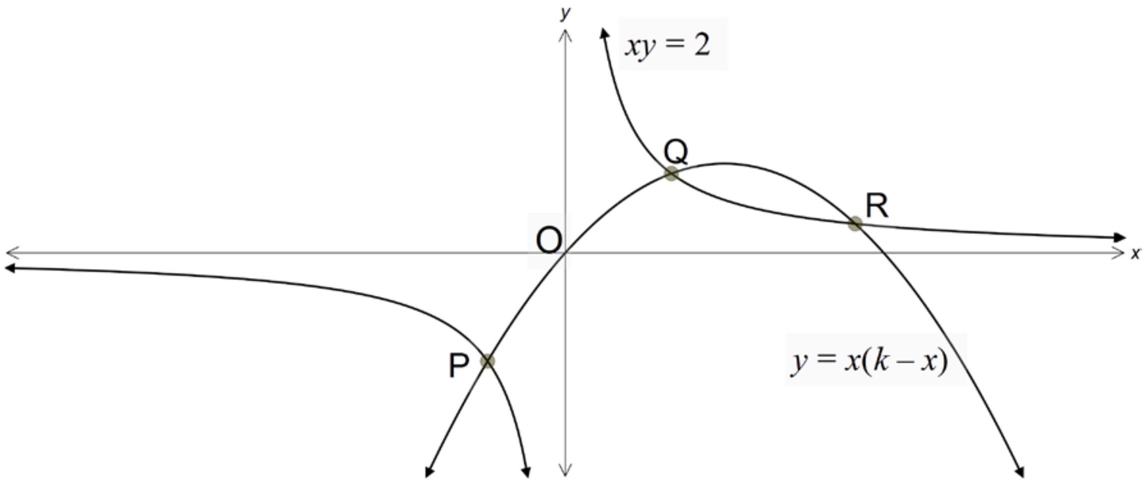
Let $\angle PTQ = \alpha$

(i) Show that ΔPQT is isosceles (3)

(ii) Show that P is the midpoint of RT (2)

Question 15 continued

c)



In the diagram, the curves $xy = 2$ and $y = x(k-x)$ intersect at the points P , Q and R with the x coordinates α, β and δ respectively.

(i) Show that α, β and δ satisfy the equation $x^3 - kx^2 + 2 = 0$ (1)

(ii) Find the value of k such that α, β and δ are consecutive terms in an arithmetic sequence. (3)

Question 16 START A NEW BOOKLET **15 marks**

a) A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the particle has velocity $v\ ms^{-1}$. The acceleration due to gravity is $10\ ms^{-2}$.

(i) The particle is projected vertically upwards with speed $U\ ms^{-2}$. Show that during its upward motion, its acceleration $a\ ms^2$ is given by
$a = -\frac{1}{10}\left(100 + v^2\right)$ (1)

(ii) Hence show that its maximum height, H metres, is given by
$H = 5\ \ln\left(\frac{U^2 + 100}{100}\right)$ (3)

(iii) The particle falls vertically from rest. Show that during its downward motion its acceleration $a\ ms^{-1}$ is given by $a = \frac{1}{10}\left(100 - v^2\right)$ (1)

(iv) Hence show that it returns to its point of projection with speed $V\ ms^{-1}$given by $V = \frac{10U}{\sqrt{U^2 + 100}}$ (3)

Question 16 continued

b) A particle P of mass m is attached to one end of a light inelastic string of length l, while the other end is fixed at point O.

The particle moves with velocity v in a horizontal circle of radius r so that the string describes a cone whose vertical axis passes through the centre C of the circle.

Let T be the tension in the string as the particle moves; let $OC = h$ and let Θ be the angle between the string and the vertical.

(i) Draw a diagram, clearly showing the forces acting on P.

(ii) Let ω be the constant angular velocity of P. Show that $h = \dfrac{g}{\omega^2}$ (1)

(iii) Hence show that

(α) $h = \dfrac{mgl}{T}$ (2)

(β) $r^2 = \dfrac{mv^2 l}{T}$ (1)

(iv) Show that $T = \dfrac{m}{2l}\left\{v^2 + \sqrt{v^4 + 4g^2l^2}\right\}$ (3)

End of Examination

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Answer sheet for multiple choice questions 1- 10

INSTRUCTIONS:

For Questions 1 to 10, place a cross in the box corresponding to your selected answer.

Student Number: ___________________________________

Question #	(A)	(B)	(C)	(D)
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$