Student Number: ____________________

NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2016

Mathematics Extension II

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Write your Student Number at the top of each page
- Answer Section I- Multiple Choice on Answer Sheet provided
- Answer Section II – Free Response in a separate booklet for each question.
- Board approved calculators and templates may be used.

Section I Multiple Choice
- 10 marks
- Attempt all questions
- Allow about 15 minutes for this section

Section II – Free Response
- 90 marks
- Each question is of equal value
- All necessary working should be shown in every question.
- Allow about 2 hours 45 minutes for this section

Weighting: 40%

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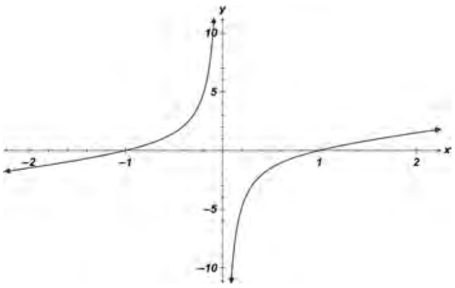
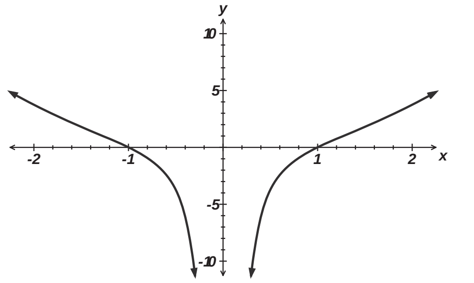
Section 1: Multiple Choice (10 marks)
Indicate your answer on answer sheet provided.

Q1. Which of the graphs below shows $y = x - \frac{1}{x^2}$?

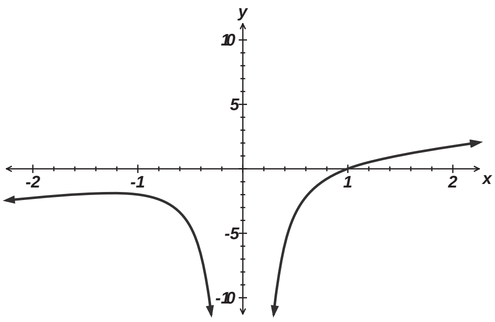
(A)



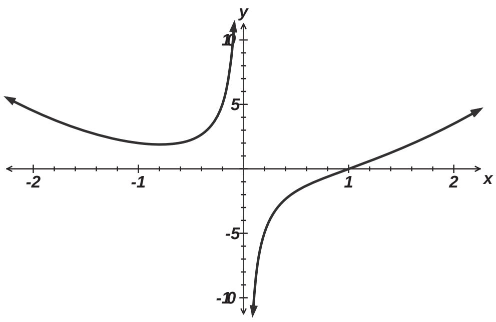
(B)



C)



(D)



Q2. What is the eccentricity of the ellipse $16x^2 + 25y^2 = 400$

A 0.25

B 0.36

C 0.6

D 0.75

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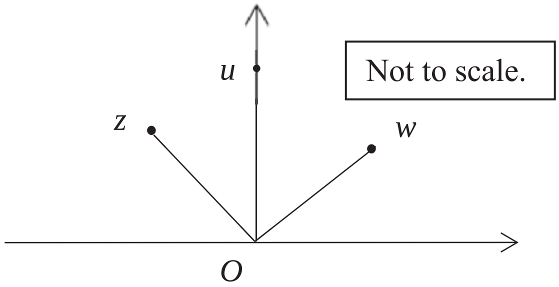
Q3. The Argand diagram shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the positive imaginary axis.



A $u = zw$ and $u = z + w$

B $u = zw$ and $u = z - w$

C $z = uw$ and $u = z + w$

D $z = uw$ and $u = z - w$

Q4. The value of $\displaystyle\int_0^2 \frac{2x+2}{x^2+4}\,dx$ is?

A $\ln 2 + \dfrac{\pi}{4}$

B $\ln 2 + \dfrac{\pi}{2}$

C $\ln 2$

D $2\ln 2$

Q5. A curve is defined implicitly by the equation $x + x\log_e y = y$.

The correct expression for $\dfrac{dy}{dx}$ is?

A $\dfrac{dy}{dx} = \dfrac{xy}{1-x}$

B $\dfrac{dy}{dx} = \dfrac{y\left(y - 1 - \log_e y\right)}{x}$

C $\dfrac{dy}{dx} = -\dfrac{y\log_e y}{x}$

D $\dfrac{dy}{dx} = \dfrac{y\left(1 + \log_e y\right)}{y - x}$

Q6. The horizontal base of a solid is the circle $x^2 + y^2 = 1$. Each cross section is a triangle with one side in the base of the solid. The length of this side of the triangle is equal to the altitude of the triangle through the opposite vertex.

Which of the following is an expression for the volume of the solid?

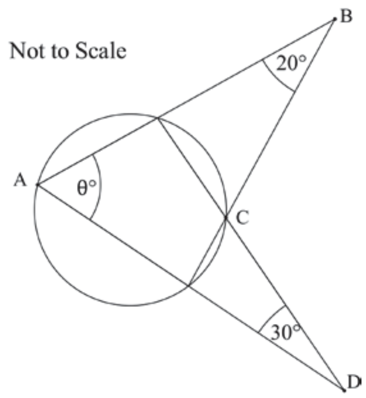
A $\quad \frac{1}{2}\int_{-1}^{1} (1 - x^2)\ dx$

B $\quad \int_{-1}^{1} (1 - x^2)\ dx$

C $\quad \frac{3}{2}\int_{-1}^{1} (1 - x^2)\ dx$

D $\quad 2\int_{-1}^{1} (1 - x^2)\ dx$

Q7.



The size of $\angle BAD$ in the diagram above is:

A $\quad 50^0$

B $\quad 55^0$

C $\quad 60^0$

D $\quad 65^0$

Q8. The equation $2x^3+4x-3=0$ has roots $x=\alpha, x=\beta$ and $x=\gamma$.
The equation with roots $x=0, x=2\alpha,\ x=2\beta$ and $x=2\gamma$ is?

A $x^3+8x-12=0$

B $x^4+8x^2-12x=0$

C $4x^3+8x-6=0$

D $4x^4+8x^2-6x=0$

Q9. A particle of mass *m* kg is suspended from a fixed point O. The particle rotates in a horizontal circle on the end of a string of length *l* with constant angular velocity ϖ. The tension in the string is *T* Newtons.

If the length *l* of the string and the angular velocity ϖ are doubled, the tension in the string is now:

A T

B $\frac{T}{2}$

C $4T$

D $8T$

Q10. A particle *P* of mass *m* kg is attached to a string of length *l* and is suspended from a point O above a smooth horizontal table. The particle moves on the flat surface of the table with linear velocity *v* in a circle of radius *r* metres. The forces acting on the particle are its weight *mg*, the normal reaction force *N* and the tension *T* in the string. The string forms an angle α at the point *O*.

Which of the following is the correct resolution of the forces on *P* in the vertical and horizontal directions.

A $T\sin\alpha - N = \text{mg}$ and $T\cos\alpha = \frac{mv^2}{r}$

B $T\sin\alpha + N = \text{mg}$ and $T\cos\alpha = \frac{mv^2}{r}$

C $T\cos\alpha - N = \text{mg}$ and $T\sin\alpha = \frac{mv^2}{r}$

D $T\cos\alpha + N = \text{mg}$ and $T\sin\alpha = \frac{mv^2}{r}$

End of Multiple Choice

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Section II Total Marks is 90

Attempt Questions 11 – 16.

Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question in a new booklet with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.

Question 11. – Start New Booklet **15 marks**

a) Write $2x^3 - 15x^2 + 38x - 30$ as a product of real factors given that $x = 3 + i$ is a zero. **2**

b) Shade the region on the Argand diagram defined by the two inequalities **3**

$$|z| \leq 2 \text{ and } |z-2| \leq |z|$$

c) (i) Find all the solutions to the equation $z^6 = 1$ in $x + iy$ form. **2**

(ii) If ω is a non-real solution to $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$ **2**

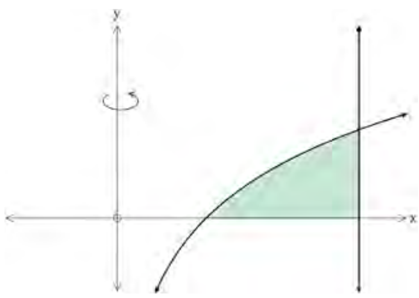
d) Consider the locus of z such that $|z - \sqrt{2} - i| = 1$.

(i) Sketch the locus of z in the complex plane. **1**

(ii) Find the minimum value of $|z|$. **1**

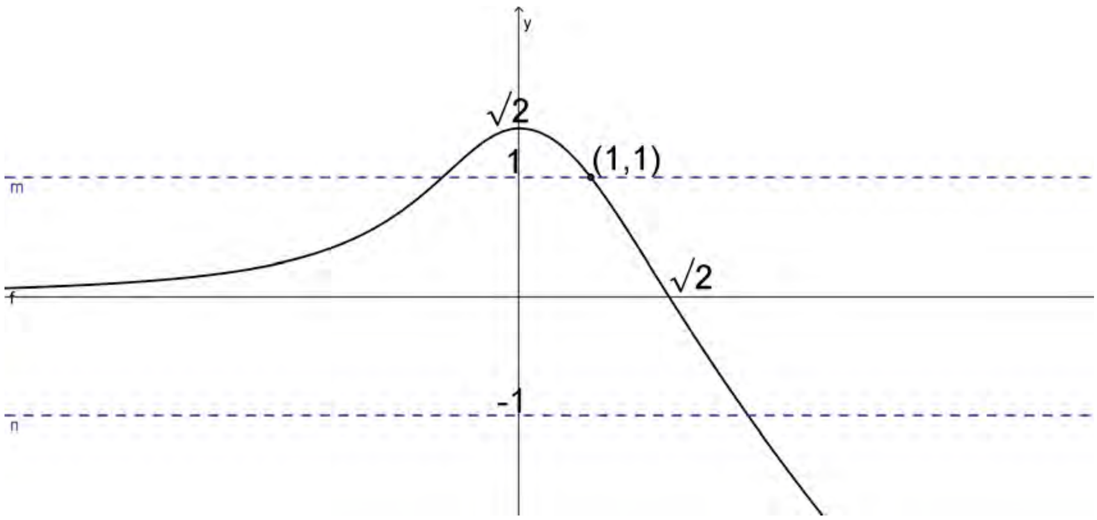
(iii) Find the maximum value of the Arg(z) for $0° < Arg(z) < 90°$. **1**

e) The region bounded by $y = \ln x$, $x = e$ and the x-axis is rotated about the y-axis. Find the volume so formed using the cylindrical shells method. **3**



End of Question 11

Question 12 – Start New Booklet **15 Marks**



a) The curve $y = f(x)$, sketched above, has an asymptote at $y = 0$. Copy or trace the above graph onto three separate number planes. Use your diagrams to show sketches of the following graphs, showing all essential features.

(i) $y = (f(x))^2$ **1**

(ii) $|y| = f(x)$ **1**

(iii) $y = \ln(f(x))$ **2**

b) Given the function $x^2 - 2xy = 4$

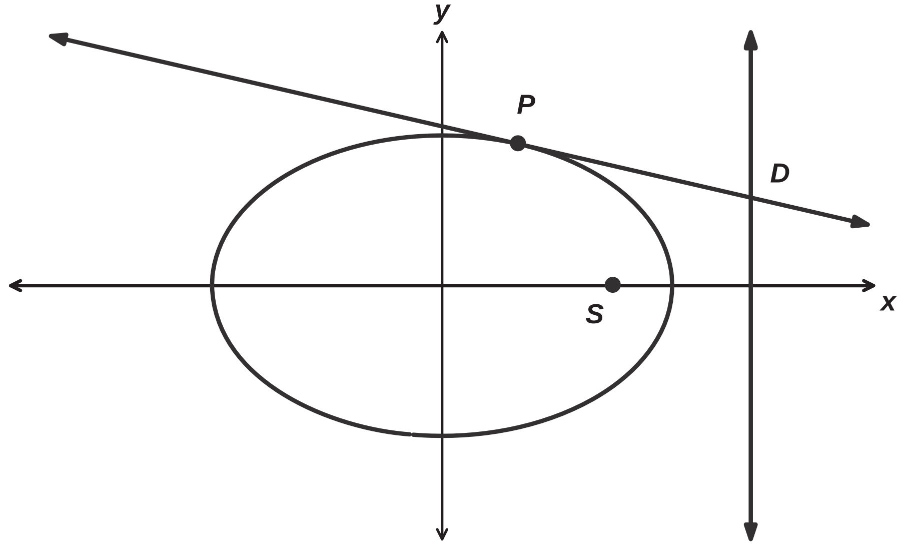
(i) Show that is increasing for all x, $x \neq 0$ **2**

(ii) Sketch the curve showing all asymptotes and intercepts on the coordinate axes. **2**

c) A hyperbola has eccentricity $\frac{3}{2}$ and directrices $x = -4$ and $x = 4$. Find the equation of this hyperbola. **2**

Question 12 continues next page.

Question 12 continued.



d) $P\left(a\cos\theta, b\sin\theta\right)$ is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The tangent at *P* meets a directrix of the ellipse at *D*. *S* is the corresponding focus.

(i) Show that the tangent at *P* has equation **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) Show that $\angle PSD = 90^\circ$. **3**

End of Question 12

Student Number: ____________________

Question 13 – Start New Booklet **15 Marks**

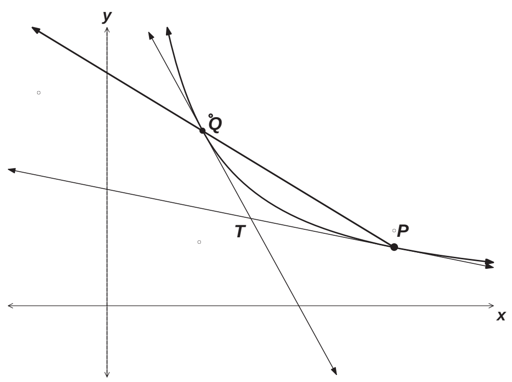
a) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx$ **2**

b) Use the method of partial fractions to find a primitive of $\dfrac{4x}{(x^2+1)(x-1)}$. **3**

c) The equation $x^3 + px + q = 0$ has roots α, β and γ.
Show that $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$. **3**

d) The variable points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ lie on the hyperbola $xy = c^2$ as shown in the diagram below.



The tangents to the hyperbola at P and Q intersect at the point T.
M is the midpoint of PQ.

The equation of the tangent at P is $x + p^2 y = 2cp$.

(i) Show that the coordinates of T are $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$. **2**

(ii) If O is the origin, show that O, T and M are collinear . **2**

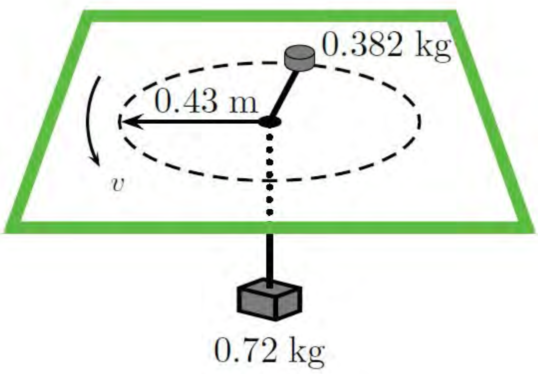
(iii) Find an expression for q in terms of p if T, M and S are collinear, where S is a focus of the hyperbola. **2**

End of Question 13

Student Number: ____

Question 14 – Start New Booklet **15 Marks**

a) An air puck of mass 0.382 kg is tied to a light inelastic string and allowed to revolve in a circle of radius 0.43m on a horizontal, frictionless table. The other end of the string passes through a hole in the centre of the table and a mass of 0.72 kg is tied to this end. The suspended mass remains at equilibrium while the puck moves. (g = 9.8 ms^{-2})



What is the speed of the puck which keeps the lower weight at an equilibrium. **2**

b) (i) If α and β are roots of $x^2 - 2x + 4 = 0$ prove $\alpha^n - \beta^n = 2^{n+1} i \sin(\frac{n\pi}{3})$. **3**

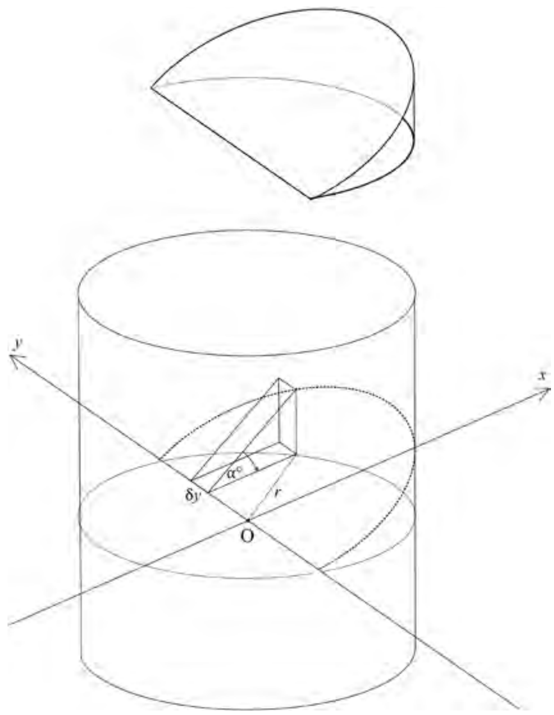
(ii) Hence find an exact expression for the value of $\dfrac{\alpha^8 - \beta^8}{2^8}$. **1**

c) Evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$ using the method of integration by parts. **4**

Question 14 continues on next page.

Question 14 continued.

d) A wedge is cut from a right cylinder of radius r by two planes, one perpendicular to the axis of the cylinder while the second makes an angle α with the first and intersects it at the centre O of the circle.



A is the area of the triangle that forms on face of the slice.

(i) Show that $A = \dfrac{1}{2}(r^2 - y^2)\tan\alpha$ **2**

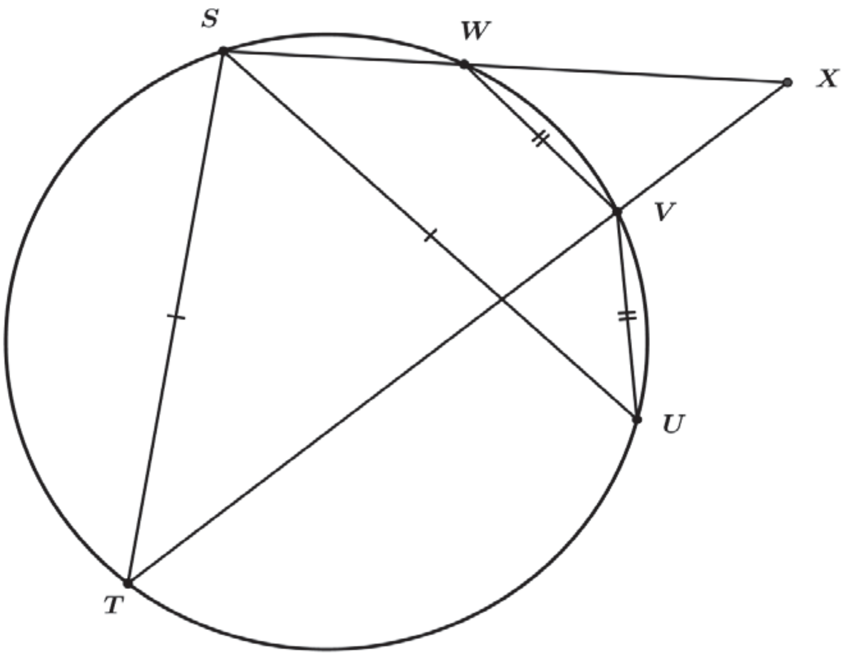
(ii) Hence show that the volume of the wedge is $\dfrac{2}{3}r^3\tan\alpha$. **3**

End of Question 14

Question 15 – Start a new booklet **15 Marks**

a) Show that $(x-1)^3$ is a factor of $P(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1$ for integers $n \geq 2$ **2**

b) The diagram below shows points S, T, U, V and W lying on a circle.
Chords ST and SU are equal and chords UV and VW are equal.
SW produced and TV produced meet at X.



(i) Copy the diagram into your booklet.

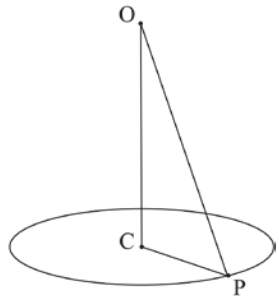
(ii) Show that $\angle SUT = \angle SXV + \angle XSV$. **2**

(iii) Show that $ST = SX$. **2**

Question 15 continues on the next page

Question 15 continued.

c) A light inelastic string OP is fixed at O. A particle, of mass m, attached to the string at point P is moving uniformly with an angular velocity ω, in a horizontal circle whose centre C is vertically below O and is distance h from O.



Prove that the period of motion is $2\pi\sqrt{\frac{h}{g}}$ **4**

d) (i) If $I_n = \int tan^{\,n} x\, dx$ show that $I_n = \frac{1}{n-1} tan^{n-1}x - I_{n-2}$. **3**

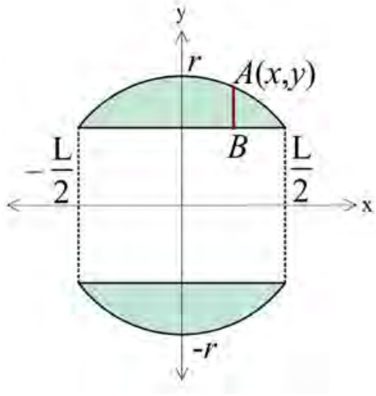
(ii) Evaluate $\int \tan^5 x\, dx$ between the limits of $x = 0$ and $x = \frac{\pi}{4}$. **2**

End of Question 15

Student Number: ____

Question 16 – Start a new booklet **13 Marks**

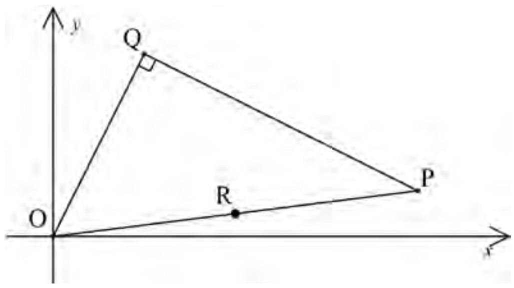
a) The diagram shows the cross section of a sphere through which a cylindrical hole of length L has been drilled.



(i) Show that the annulus formed by rotating the interval AB about the x-axis has area $\frac{\pi}{4}(L^2 - 4x^2)$ square units. **2**

(ii) Hence show that the volume of the solid remaining is the same as the volume of a sphere of diameter L. **2**

b) In the Argand diagram below, OPQ is a triangle which is right-angled at Q. The point R is the midpoint of OP.



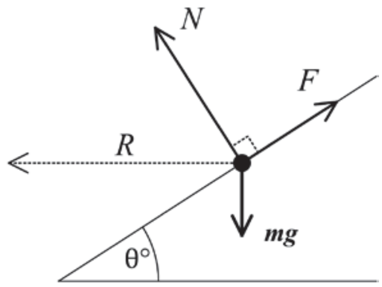
(i) If $\overrightarrow{OP} = z$ and $\overrightarrow{OQ} = w$ show that $\overrightarrow{OR} = \frac{1}{2}(1 - ki)w$ where k is a constant, $k > 0$. **2**

(ii) Express $\overrightarrow{RQ}$ in terms of w and hence show $|\overrightarrow{OR}| = |\overrightarrow{RQ}|$. **2**

Question 16 continues on next page.

Question 16 continued

c) The question refers to the diagram below.



A particle of mass m travels at constant speed v around a circular track of radius R, centre C. The track is banked inwards at angle of θ, and the particle does not move up or down the bank.

The reaction exerted by the track on the particle has a normal component N, and a component F due to friction, directed up or down the bank.

The force F lies in the range from $-\mu N$ to μN where μ is a positive constant and N is the normal component; the sign of F is positive when the force is directed up the bank. The acceleration due to gravity is g. The acceleration related to the circular motion is of magnitude $\frac{v^2}{R}$ and is directed towards the centre of the track.

(i) Show that $mg = N\cos\theta + F\sin\theta$ **1**

(ii) Show that $\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$ **2**

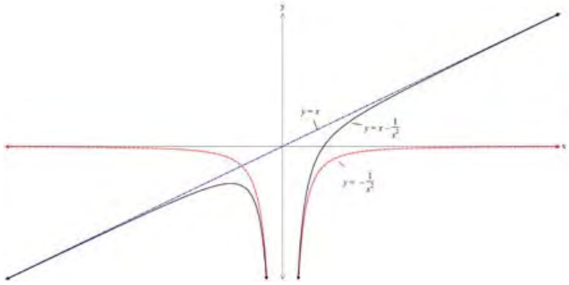
(iii) The particle will begin to move up the bank when the lateral force F first exceeds the maximum force acting down the track. Show that the maximum speed V at which the particle can travel without slipping up the track is given by **2**

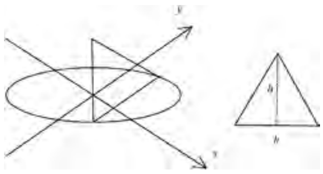
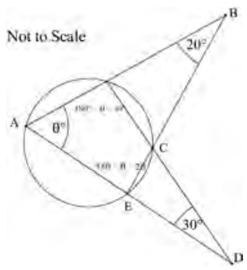
$$V^2 = Rg\left\{\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right\}$$

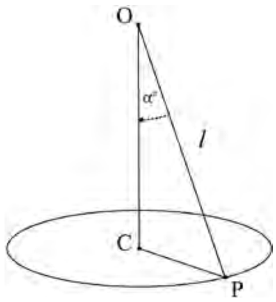
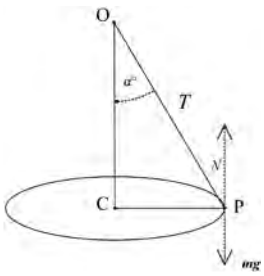
(iv) Let U be the minimum speed that the particle can travel around the track without sliding down the bank. Find an expression for U^2 in terms of R, g, μ and $\tan\theta$. **1**

(v) Show that if $\mu \geq \tan\theta$, then the particle will not slide down the bank regardless of its speed. **1**

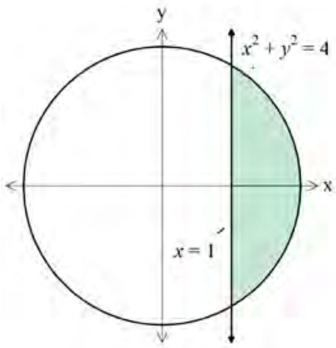
End of Examination

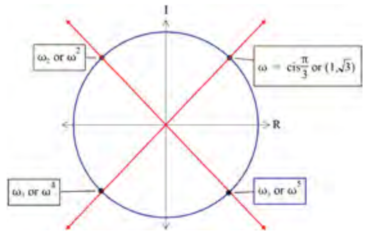
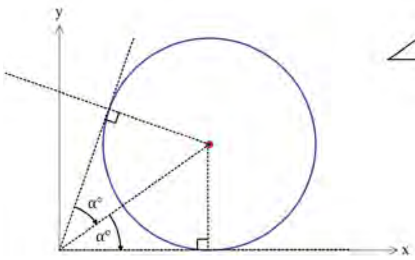
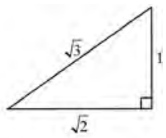
Q1	Method 1 – combine the graphs 	C
Q2	$16x^2 + 25y^2 = 400$ $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ $\Rightarrow \quad a = 5 \;\; b \Rightarrow 4$ $b^2 = a^2(1 - e^2)$ $e = \sqrt{1 - \dfrac{b^2}{a^2}}$ $e = \sqrt{1 - \dfrac{16}{25}}$ $= 0{\cdot}6$	C
Q3	$z = u.w$ as $Arg(z) = Arg(u) + Arg(w)$ $u = z + w$ vector addition	C
Q4	$\displaystyle\int_0^2 \frac{2x + 2}{x^2 + 4}\, dx$ $\displaystyle = \int_0^2 \frac{2x}{x^2 + 4}\, dx + 2\int_0^2 \frac{1}{x^2 + 4}\, dx$ $\displaystyle = \Big[\ln(x^2 + 4)\Big]_0^2 + 2\left[\tan^{-1}\frac{x}{2}\right]_0^2$ $= \ln 8 - \ln 4 + 2(\tan^{-1} 1 - \tan^{-1} 0)$ $\displaystyle = \ln\frac{8}{4} + 2\left(\frac{\pi}{4} - 0\right)$ $\displaystyle = \ln 2 + \frac{\pi}{2}$	B

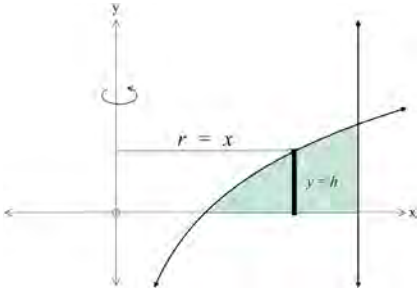
Q5	$x + x\ln y = y$ $1 + \ln y + \frac{x}{y}\frac{dy}{dx} = \frac{dy}{dx}$ $1 + \ln y = \left(1 - \frac{x}{y}\right)\frac{dy}{dx}$ $\frac{dy}{dx} = (1 + \ln y) \div \frac{y-x}{y}$ $\frac{dy}{dx} = \frac{y(1 + \ln y)}{y - x}$	D
Q6	 Area of triangle $\frac{1}{2}bh = \frac{h^2}{2}$ $h = 2y$ $h^2 = 4y^2$ $h^2 = 4(1 - x^2)$ $\delta V = \frac{1}{2} \times 4(1 - x^2)\delta x$ $V = \int_{-1}^{1} 2(1 - x^2)\ dx$ $= 2\int_{-1}^{1} (1 - x^2)\, dx$	D
Q7	 Opp angles of cyclic quad. $180° = 180° - θ° - 30° + 180° - θ° - 20°$ $2θ° = 130°$ $θ = 65°$	D

Q8	$x = 2\alpha$ $\frac{x}{2} = \alpha$ $2\left(\frac{x}{2}\right)^3 + 4\left(\frac{x}{2}\right) - 3$ $= \frac{x^3}{4} + 2x - 3$ $\therefore$ if $x = 0$ is a factor as well $4x\left(\frac{x^3}{4} + 2x - 3\right) = x^4 + 8x^2 - 12x = 0$	B
Q9	$T\sin\alpha = mr\omega^2$ $T_1\frac{r}{l} = mr\omega^2$ $T = ml\omega^2$ if $l \Rightarrow 2l$ and $\omega \Rightarrow 2\omega$ $T_2 = 8ml\omega^2 = 8T_1$ 	D
Q10	 Horizontally $T\sin\alpha = \frac{mv^2}{r}$ Vertically $T\cos\alpha + N = \text{mg}$	C

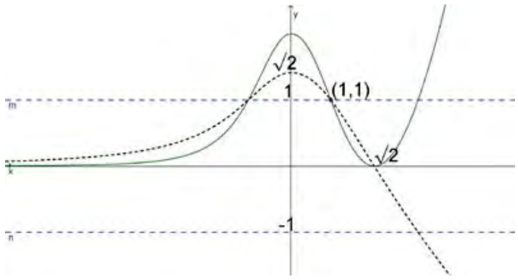
Q11

a	$2x^3 - 15x^2 + 38x - 30$ Real coefficients therefore roots in conjugate pairs. Factors $\{x-(3+i)\}\{x-(3-i)\}(ax+b)$ $= \{(x-3)-i\}\{(x-3)+i\}(ax+b)$ $= \{(x-3)^2+1\}(ax+b)$ $= (x^2-6x+10)(ax+b)$ $\therefore\ a = 2$ coeff of x^2 $c = {}^-3$ constant $(x^2-6x+10)(2x-3)$ $= 2x^3 - 3x^2 - 12x^2 + 18x + 20x - 30$ $= 2x^3 - 15x^2 + 38x - 30$	2 marks – correct solution 1 mark – obtains $x^2 - 6x + 10$ as a quadratic factor
b	$\|z\| = 2$ $\left(\sqrt{x^2+y^2}\right)^2 = 2^2$ $x^2 + y^2 = 4$ $\sqrt{(x-2)^2+y^2} = \sqrt{x^2+y^2}$ $x^2 - 4x + 4 = x^2 + y^2$ ${}^-4x = {}^-4$ $x = 1$ 	3 marks – correct solution 2 marks – correct representation of |z|=2 and |z-2|=|z| 1 mark - correct representation of |z|=2 or |z-2|=|z|

c-i	 Or Roots are $1\pm\sqrt{3}\,i$; $^-1\pm\sqrt{3}\,i$	2 marks – correct solution 1 mark correct values in trig form only.
c-ii	$z^6 - 1 = 0$ $(z^2)^3 - 1 = 0$ $(z^2 - 1)[(z^2)^2 + z^2 + 1] = 0$ $(z^2 - 1)(z^4 + z^2 + 1) = 0$ If ω is a non real root $\omega^4 + \omega^2 + 1 = 0$ $\omega^4 + \omega^2 = {}^-1$	2 marks – correct solution 1 mark – factoris $z^6 - 1$ and attempts to obtain required result or equivalent trig method
d-i	$\lvert z - \sqrt{2} - 1\rvert = 1$ $\lvert z - (\sqrt{2} + 1)\rvert = 1$ 	1 mark – correct solution
d-ii	Minimum $\lvert z\rvert = \sqrt{3} - 1$	1 mark – correct solution
d-iii	Max $Arg\,(z) = 2\tan^{-1}\frac{1}{\sqrt{2}} = 70°31'$	1 mark – correct solution

e	 $\delta V = 2\pi r \times h\,\delta x$ $= 2\pi xy\,\delta x$ $= 2\pi x\ln x\,\delta x$ $\Delta\text{ Volume} = \lim_{\delta x \to 0} \sum_{1}^{e} 2\pi x\ln x\ \ \delta x$ $V = 2\pi\int_{1}^{e} x\ln x dx$ If $y = x\ln x$ then $\frac{dy}{dx} = x\ln x + x \times \frac{1}{x}$ $\therefore \int \frac{dy}{dx}\,dx = \int (x\ln x + 1)\ dx$ $x\ln x - \int 1\,dx = \int x\ln x\ \ dx$ $\therefore 2\pi\int_{1}^{e} x\ln x\ \ dx = 2\pi\left[x\ln x - x\right]_{1}^{e}$ $= 2\pi\{(e\ln(e) - e) - (1\ln(1) - 1)\}$ $= 2\pi\{e - e - \circ + 1)$ $= 2\pi$	3 marks – correct solution 2 marks – forms a correct expression for V as an integral and finds the correct primitive function. 1 mark – forms a correct expression for $\delta\ v$ or for V as an integral

Q12

<table>
<tr>
<td>a-i</td>
<td>
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<td>1 mark – shape – students should attempt to show “above/below” w.r.t to $y = 1$</td>
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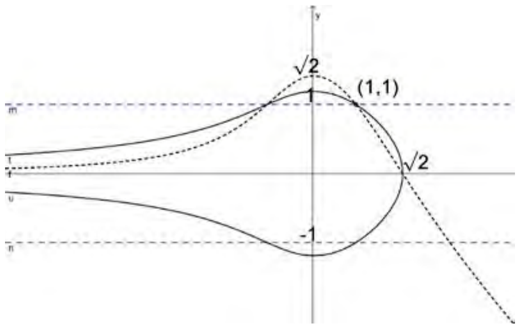
<td>a-ii</td>
<td>
</td>
<td>1 mark – shape – students should attempt to show “above/below” w.r.t to $y = 1$</td>
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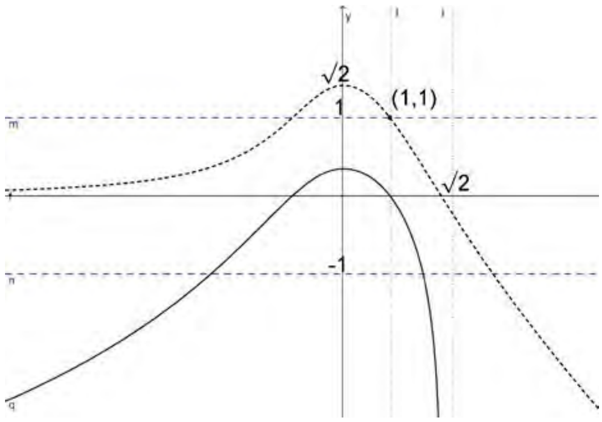
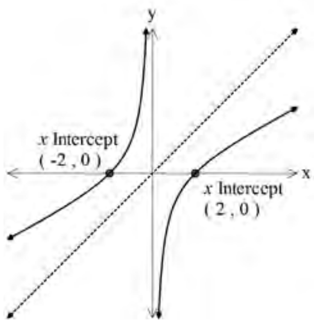
<td>a-ii</td>
<td>
</td>
<td>2 marks – correct asymmetric shape with asymptote

1 mark – symmetric shape and asymptote</td>
</tr>
<tr>
<td>b-i</td>
<td>$$x^2 - 2xy = 4$$
$$2x - 2y - 2x\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{y - x}{x}$$
$$y = \frac{x^2 - 4}{x}$$
$$\frac{dy}{dx} = \frac{x \times 2x - (x^2 - 4)}{x^2}$$
$$= \frac{x^2 + 4}{x^2}$$
which is > 0 for all $x \neq 0$</td>
<td>2 marks – correct solution

1 mark
- Correct dy/dx
- Correct explanantion for incorrect dy/dx</td>
</tr>
</table>

c-ii		2 marks – correct graph including oblique asymptote and intercepts 1 mark – graph with intercepts or asymptote
c	$b^2 = a^2(e^2 - 1) x = \pm\frac{a}{e}$ $4 = \frac{a}{e}$ $a = 4 \times \frac{3}{2} = 6$ $b = \sqrt{36\left(\frac{9}{4} - 1\right)} = 3\sqrt{5}$ $b^2 = \frac{80}{9}$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\frac{x^2}{36} - \frac{y^2}{45} = 1$	2 marks – correct solution 1 mark – correct solution from incorrect eccentricity

d-i	$x = a\cos\theta \qquad \dfrac{dx}{d\theta} = -a\sin\theta$ $y = b\sin\theta \qquad \dfrac{dy}{d\theta} = b\cos\theta$ $\therefore \quad \dfrac{dy}{dx} = \dfrac{b\cos\theta}{-a\sin\theta}$ $y - y_1 = m(x - x_1)$ $y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ $ay\sin\theta - absin^2\theta = -bx\cos\theta + ab\cos^2\theta$ $ay\sin\theta + bx\cos\theta = ab(\sin^2\theta + \cos^2\theta)$ $ay\sin\theta + bx\cos\theta = ab$ $\dfrac{y\sin\theta}{b} + \dfrac{x\cos\theta}{a} = 1$ $\therefore \quad \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$	2 marks – correctly demonstrated 1 mark – correctly formed *dy/dx* - Cartesian equation for tangent quoted

d-ii	Point D $y = \left(1 - \frac{x\cos\theta}{a}\right) \times \frac{b}{\sin\theta}$ $x = \frac{a}{e}$ $\therefore y = \left(1 - \frac{a}{e}\frac{\cos\theta}{a}\right)\frac{b}{\sin\theta}$ $= \left(1 - \frac{\cos\theta}{e}\right)\frac{b}{\sin\theta}$ $P = (a\cos\theta, b\sin\theta) \quad S = (ae, 0)$ $D = \left(\frac{a}{e}, \frac{b(e - \cos\theta)}{e\sin\theta}\right)$ $m_{SP} = \frac{b\sin\theta}{a\cos\theta - ae} = \frac{b\sin\theta}{a(\cos\theta - e)}$ $m_{SD} = \frac{\frac{b(e-\cos\theta)}{e\sin\theta}}{\left(\frac{a}{e} - ae\right)} = \frac{b(e - \cos\theta)}{a\sin\theta(1 - e^2)}$ $m_{SP} \times m_{SD} = \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$ $= \frac{-b^2}{a^2(1-e^2)}$ $= \frac{-b^2}{b^2} = -1$ $\therefore$ perpendicular	3 marks – correct solution 2 marks – correct gradients but incomplete demonstration of $m_{SP} \times m_{SD} = -1$ 1 mark – one correct gradient

Q13

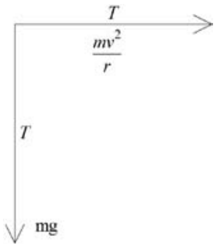
	$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. $\int_0^a f(x)\,dx$ $x \Rightarrow x - a$ $x = 0 \Rightarrow a - x = a$ $x = a \Rightarrow a - x = 0$ $\frac{d(a-x)}{dx} = -1$ $d(x-a) = -dx$ $\int_a^0 f(a-x)(-dx)$ $= \int_0^a f(a-x)\,dx$	1 mark – correct solution
	$I_n = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\cos^3\left(\frac{\pi}{2} - x\right) + \sin^3\left(\frac{\pi}{2} - x\right)}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}$ $\therefore\ 2I_n = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}$ $= \int_0^{\frac{\pi}{2}} \frac{\cos^3 x + \sin^3 x}{\cos^3 x + \sin^3 x}\,dx$ $= \int_0^{\frac{\pi}{2}} 1\,dx$ $= \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$	2 marks – correct solution 1 mark – forms correct alternative form for I_n

b	$$\frac{4x}{(x^2+1)(x-1)} = \frac{ax+b}{x^2+1} + \frac{c}{x-1}$$ $$(ax+b)(x-1) + c(x^2+1) = 4x$$ $$(a+c)x^2 + (b-a)x + (c-a) = 4x$$ $\therefore$ $$a+c=0 \quad (1)$$ $$b-a=4 \quad (2)$$ $$c-b=0 \quad (3)$$ From $(1)+(2)$ $b+c=4$ From (3) $c=b$ $\therefore$ $$2b=4$$ $$b=2$$ $$c=2$$ $$a={}^{-}2$$ $$\int \frac{4x}{(x^2+1)(x-1)}\,dx$$ $$= \int \frac{{}^{-}2x+2}{x^2+1} + \frac{2}{x-1}\,dx$$ $$= -\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx + 2\int \frac{1}{x-1}\,dx$$ $$= -\ln(x^2+1) + 2\tan^{-1}(x) + 2\ln(x-1) + C$$ $$= 2\tan^{-1}(x) + \ln\frac{(x-1)^2}{x^2+1} + C$$	3 marks – correct solution 2 marks – obtains two correct primitive functions 1 mark – obtains the correct primitive function or correctly evaluates coefficients.

c	$$x^3 + px + q = 0$$ $$\alpha + \beta + \gamma = -\frac{b}{a} = 0$$ $$\alpha\gamma + \alpha\beta + \beta\gamma = \frac{c}{a} = p$$ $$\alpha\gamma\beta = -\frac{d}{a} = -q$$ As α γ and β are roots $$\alpha^3 + p\alpha + q = 0$$ $$\beta^3 + p\beta + q = 0$$ $$\gamma^3 + p\gamma + q = 0$$ $$(\alpha^3 + \beta^3 + \gamma^3) + p(\alpha + \beta + \gamma) + 3q = 0$$ $$(\alpha^3 + \beta^3 + \gamma^3) + p \times 0 + 3q = 0$$ $$(\alpha^3 + \beta^3 + \gamma^3) = -3q$$ $$(\alpha^3 + \beta^3 + \gamma^3) = 3\alpha\gamma\beta$$	3 marks – correct solution 2 marks – obtains expression for $\alpha + \beta + \gamma =$ $\alpha\beta + \alpha\gamma + \beta\gamma =$ $\alpha\beta\gamma =$ $\alpha^3, \beta^3, \gamma^3$ 1 mark – obtains correct expressions for $\alpha + \beta + \gamma =$ $\alpha\beta + \alpha\gamma + \beta\gamma =$ $\alpha\beta\gamma =$ and makes progress towards required result.
d-i	T – intersection of 2 tangents $$x + p^2 y = 2cp$$ $$x + q^2 y = 2cq$$ $$y(p^2 - q^2) = 2c(p - q)$$ $$y(p + q) = 2c \qquad p \neq q$$ $$y = \frac{2c}{p + q}$$ $$x + p^2 \times \frac{2c}{p + q} = 2cp$$ $$x = 2cp - \frac{2cp^2}{p + q}$$ $$= \frac{2cp^2 + 2cpq - 2cp^2}{p + q}$$ $$= \frac{2cpq}{p + q}$$	2 marks – correct solution 1 mark – obtains one correct coordinate of T

d-ii	If collinear then $m_{OT} = m_{MO}$ $M = \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq} \right)$ $m_{\Omega} = \frac{c(p+q)}{2pq} \times \frac{2}{c(p+q)} = \frac{1}{pq}$ $m_{OT} = \frac{2c}{p+q} \times \frac{p+q}{2cpq} = \frac{1}{pq}$	2 marks – correct solution 1 mark – obtains coordinates of M or correct expression for either OM or OT.
d-iii	Focus of hyperbola in first quadrant: $(\sqrt{2}\ c, \sqrt{2}\ c)$ If all points are collinear then $m_{OS} = \frac{1}{pq}$ $\frac{c\sqrt{2}}{c\sqrt{2}} = \frac{1}{pq}$ $pq = 1$ $q = \frac{1}{p}$	2 marks – correct solution. 1 mark – significant progress in applying gradients OT, OM and OS.

Q14

a	 $T = \frac{mv^2}{r}$ $= \frac{0{\cdot}382 \times v^2}{0{\cdot}43}$ $T = {\cdot}72 \times 9{\cdot}8 = 70{\cdot}56$ $70{\cdot}56 = \frac{0{\cdot}382 \times v^2}{0{\cdot}43}$ $v^2 = \frac{70{\cdot}56 \times 0{\cdot}43}{0{\cdot}382}$ $speed = 2{\cdot}82\ ms^{-1}$	2 marks – correct solution 1 mark – equating two expressions for *T*.

b-i	$x^2 - 2x + 4 = 0$ $(x-1)^2 + 3 = 0$ $(x-1)^2 = {}^-3$ $x = 1 \pm \sqrt{3}\, i$ cis form $\alpha = 2\text{cis}\frac{\pi}{3}$ $\beta = 2\text{cis}\left(-\frac{\pi}{3}\right)$ $\alpha^n = 2^n \text{ cis}(n\pi^{\cdot 3})$ $\beta^n = 2^n \text{ cis}(n\pi^{\cdot 3})$ $a^n - \beta^n = 2^n \text{ cis}(n\pi^{\cdot 3}) - 2^n \text{ cis}(n\pi^{\cdot 3})$ $= 2^n\left\{ \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} - \left[\cos\left(-\frac{n\pi}{3}\right) - \sin\left(-\frac{n\pi}{3}\right)\right]\right\}$ $= 2^n\left\{ \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} - \cos\left(\frac{n\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)\right]$ $= 2^n \times 2i\sin\frac{n\pi}{3}$ $= 2^{n+1} i\sin\frac{n\pi}{3}$	3 marks – correct solution 2 marks – incomplete solution ie. Important explanation not included eg. $\sin(-x) = -\sin x$ 1 mark – some progress towards solution
	$\frac{\alpha^8 - \beta^8}{2^8} = \frac{2^9\, i\sin\left(\frac{8\pi}{3}\right)}{2^8}$ $= 2i\sin\left(\frac{2\pi}{3}\right)$ $= 2i\frac{-\sqrt{3}}{2}$ $= -\sqrt{3}\, i$	1 mark – correct solution by correct process

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$$I_n = \int_0^{\frac{\pi}{2}} e^x \cos x \; dx$$

$u = e^x \quad v' = \cos x$
$u' = e^x \quad v = \sin x$

$$I_n = \Big[e^x \sin x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$u = e^x \quad v' = \sin x$
$u' = e^x \quad v = -\cos x$

$$I_n = = \Big[e^x \sin x \Big]_0^{\frac{\pi}{2}} - \left\{ \Big[-e^x \cos x \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \cos x \, dx \right\}$$

$$2I_n = \Big[e^x \sin x + e^x \cos x \Big]_0^{\frac{\pi}{2}}$$

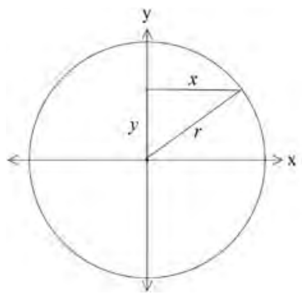
$$I_n = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

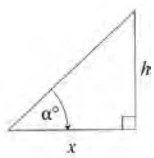
</td>
<td>4 marks –correct solution

3 – producing $2I_n$

2 marks – correct first integration

1 mark – correct process in identifying v and u' to use in integration by parts</td>
</tr>
<tr>
<td>d-i</td>
<td>





From diagram

$x^2 = r^2 - y^2$

$\tan\alpha = \frac{h}{x} \Rightarrow h = x\tan\alpha$

$\text{Area} = \frac{1}{2}\text{bh}$

$= \frac{1}{2}x^2\tan\alpha$

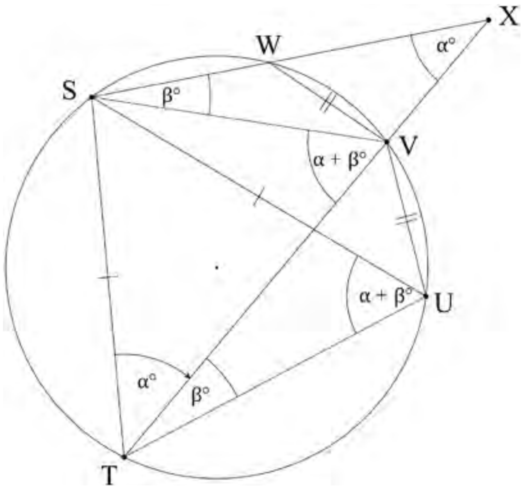
$= \frac{1}{2}(r^2 - y^2)\tan\alpha$

</td>
<td>2 marks – correct solution

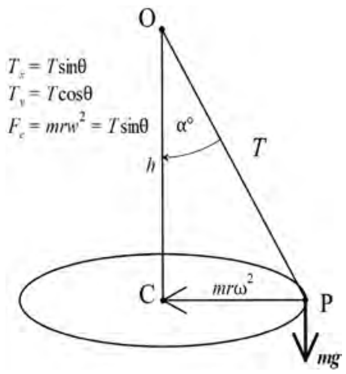
1 mark – expression for h</td>
</tr>
</table>

d-ii	$\delta V = \frac{1}{2}(r^2 - y^2)\tan\alpha \ \delta y$ $V = \lim_{\delta y \to 0} \sum_{-r}^{r} \frac{1}{2}(r^2 - y^2)\tan\alpha \ \delta y$ $V = \int_{-r}^{r} \frac{1}{2}(r^2 - y^2)\tan\alpha \ dy$ $= \frac{\tan\alpha}{2}\int_{-r}^{r} (r^2 - y^2)\, dy$ $= \frac{\tan\alpha}{2}\left[r^2 y - \frac{y^3}{3}\right]_{-r}^{r}$ $= \frac{\tan\alpha}{2}\left\{\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right\}$ $= \frac{\tan\alpha}{2} \times \frac{4r^3}{3}$ $= \frac{2}{3r^3}\tan\alpha$	3 marks – final solution 2 marks – integral 1 mark – formation of Σ statement

Q15

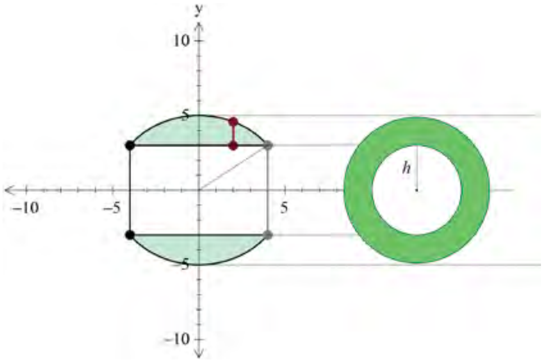
a	$P(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1$ $P(1) = 1 - n + n - 1 = 0$ $P'(x) = 2nx^{2n-1} - n(n+1)x^{n} + n(n-1)x^{n-2}$ $P'(1) = 2n - n^2 - n + n^2 - n = 0$ $P''(x) = 2n(2n-1)x^{2n-2} - n^2(n+1)x^{n-1} + n(n-1)(n-2)x^{n-3}$ $P''(1) = 4n^2 - 2n - n^3 - n^2 + n^3 - 2n^2 - n^2 + 2n = 0$ $P(x) = x^4 - 2x^3 + 2x^1 - 1$ Hence $(x-1)^3$ is a factor as $P(1) = P'(1) = P''(1) = 0$	2 marks – correct solution 1 mark – obtains *P'(x)* and *P"(x)* and either *P'*
	 	
b	let $\angle SXV = \alpha$ let $\angle XSV = \beta$ $\angle SVT = \angle SXV + \angle XSV$ $= \alpha + \beta$ ($ext\ \angle$ = sum of opp interior $\angle$) $\angle AVT = \angle SUT$ ($\angle$'s standing is same segment are equal) ∴ $\angle SUT = \angle SXV + \angle XSV$ $= \alpha + \beta$	2 marks – correct solution 1 mark – shows $\angle SVT = \angle SXV + \angle XSV$ With reason or $\angle SVT = \angle SUT$ with reason or equivalent progress

	$\angle STU = \angle SUT = \alpha + \beta$ ($\angle$'s opp = sides in isoc Δ) $\angle XSV = \angle VTU = \beta$ ($\angle$'s on =chords are $equ\alpha$) $\angle STV = \angle STU - \angle VTU$ $= \alpha + \beta - \beta = \alpha$ $= \angle SXV$ $\therefore$ ΔSXT is isosceles $\therefore$ ST = SX (sides opp = $\angle s$ in isoc Δ)	2 marks – correct solution 1 mark – establishes at least two relevant facts relating to the solution.

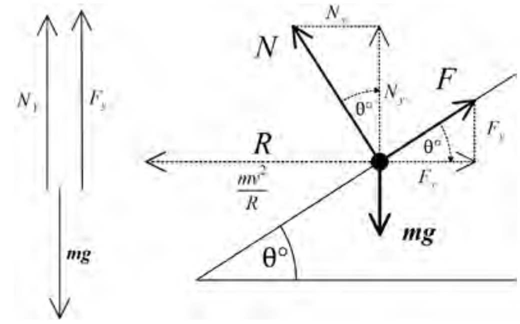
	$T\cos\alpha = mg$ $T\sin\alpha = m\ r\omega^2$ $\tan\alpha = \dfrac{r\omega^2}{g}$ $\tan\alpha = \dfrac{r}{h}$ $\dfrac{r}{h} = \dfrac{r\omega^2}{g}$ $\omega^2 = \dfrac{g}{h}$ $\omega = \sqrt{\dfrac{g}{h}}$ $\omega = \dfrac{2\pi}{n}$ $\text{Period} = 2\pi \times \dfrac{1}{\omega}$ $\text{Period} = 2\pi\sqrt{\dfrac{h}{g}}$  	4 marks – correct solution 3 marks – correct derivation of $\omega = \sqrt{\dfrac{g}{h}}$ 2 marks – correct expressions for Tsinɵ or Tcosɵ
d-i	$I_n = \int \tan^n x\ dx$ $= \int (\sec^2 x - 1)\ \tan^{n-2} x\, dx$ $= \int \sec^2 x \tan^{n-2} x\, dx - \int \tan^{n-2} x\ dx$ $= \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}$	3 marks – correct solution 2 marks – obtains tow correct integral expressions 1 mark applies identity for $\tan^2$x

d-ii	$\int_0^{\frac{\pi}{4}} \tan^5 x \; dx = \frac{1}{4}\tan^5 x - I_3$ $\int_0^{\frac{\pi}{4}} \tan^3 x \; dx = \frac{1}{2}\tan^3 x - I_1$ $I_1 = \int_0^{\frac{\pi}{4}} \tan x \; dx$ $= \left[-\ln(\cos x) \right]_0^{\frac{\pi}{4}}$ $= \left[\ln(\sec x) \right]_0^{\frac{\pi}{4}}$ $= \ln(\sqrt{2}) - \ln 1 = \ln(\sqrt{2})$ $\therefore$ $I_5 = \frac{1}{4} \times 1^5 - \left(\frac{1}{2} \times 1^3 - \ln(\sqrt{2}) \right)$ $= \ln(\sqrt{2}) - \frac{1}{4}$	2 marks – correct solution 1 mark – correct expression for I_3 and I_5 but incorrect evaluation.

Q16

a-i	 $\text{Area} = \pi(R^2 - h^2)$ $r^2 = h^2 + \left(\frac{L}{2}\right)^2$ $\therefore h^2 = r^2 - \frac{L^2}{4}$ $r^2 = R^2 + x^2$ $R^2 = r^2 - x^2$ $\text{Area} = \pi\left\{(r^2 - x^2) - \left(r^2 - \frac{L^2}{4}\right)\right\}$ $= \pi\left(\frac{L^2}{4} - x^2\right)$ $= \frac{\pi}{4}(L^2 - 4x^2)$	2 marks – correct solution 1 mark – determination of either R^2 or h^2 Note $\frac{L}{2}$ is NOT indicated as inner radius of annulus in diagram. – it is half length of the cylinder.

a-ii	$\delta V = \frac{\pi}{4}(L^2 - 4x^2)\delta x$ $V \approx \lim_{\delta x \to 0} \sum_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\pi}{4}(L^2 - 4x^2)\delta x$ $V = \frac{\pi}{4}\int_{-\frac{L}{2}}^{\frac{L}{2}} (L^2 - 4x^2)\, dx$ $V = \frac{\pi}{2}\int_{0}^{\frac{L}{2}} (L^2 - 4x^2)\, dx$ (even function) $V = \frac{\pi}{2}\left[L^2x - \frac{4x^3}{3} \right]_0^{\frac{L}{2}}$ $= \frac{\pi}{2}\left(\frac{L^3}{2} - \frac{4L^3}{24} \right)$ $= \frac{\pi}{2}\frac{L^3}{3} = \frac{\pi L^3}{6} u^3$	2 marks – correct solution ; note question asked to compare with sphere of diameter $\frac{L}{2}$ so this comparison should be represented, 1 mark – correctly formed integral
b-i	$\overrightarrow{QP}$ is equivalent to 90° clockwise rotation of $\overrightarrow{OQ}$ multiplied by some scalar factor, ie $\overrightarrow{QP} = -ki\overrightarrow{OQ}$ $\therefore \overrightarrow{QP} = -kiw$ $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$ *vector addition* $\overrightarrow{OP} = w + (-ikw)$ $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OP}$ $\overrightarrow{OR} = \frac{1}{2}\{w - ikw\}$ $= \frac{1}{2w}(1 - ki)$	2 marks – correctly demonstrated 1 mark - OP= *ki*OQ

b-ii	$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$ $= w - \frac{1}{2}(1 - ik)w$ $= \frac{1}{2}w + \frac{1}{2}ikw$ $\overrightarrow{OR} = \frac{1}{2}w - \frac{1}{2}ikw$ $\therefore$ *$\overrightarrow{OR}$ and $\overrightarrow{RQ}$ are conjugates* *and hence have the same modulus.*	2 marks – correct solution 1 mark – conjugates formed but no comment made on equal modulus
	$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$ $= w - \frac{1}{2}(1 - ik)w$ $= \frac{1}{2}w + \frac{1}{2}ikw$ $\overrightarrow{OR} = \frac{1}{2}w - \frac{1}{2}ikw$ $\therefore$ *$\overrightarrow{OR}$ and $\overrightarrow{RQ}$ are conjugates* *and hence have the same modulus.*	1 mark for showing OR = RQ Note: Question said “hence” therefore required connection with RQ.
c-i	 Resolving forces vertically. $N_y + F_y - \text{mg} = 0$ $N\cos\theta + F\sin\theta - \text{mg} = 0$ $\text{mg} = N\cos\theta + F\sin\theta$ (1)	1 mark – correct solution Answer must clearly demonstrate from where the vertical components are derived – not just restating the equation in question.

c-ii	Resolving forces horizontally. $N\sin\theta - F\cos\theta = \frac{mv^2}{r}$ ② ② $\div$ ① $\frac{mv^2}{R} \times \frac{1}{\text{mg}} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$ $\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$	2 mark – correct solution from clearly demonstrated method. 1 mark – resolving forces horizontally.
c-iii	For sliding upwards to occur – then must overcome maximum downwards friction ie. F= –μN. $\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$ becomes $\frac{V^2}{Rg} = \frac{N\sin\theta - (-\mu N)\cos\theta}{N\cos\theta + (-\mu)N\sin\theta}$ $\frac{V^2}{Rg} = \frac{N\sin\theta + \mu N\cos\theta}{N\cos\theta - \mu N\sin\theta} \div \frac{\cos\theta}{\cos\theta}$ $\frac{V^2}{Rg} = \frac{N(\tan\theta + \mu)}{N(1 - \mu\tan\theta)}$ $V^2 = Rg\left\{\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right\}$	2 marks – includes explanation of why –μN used. 1 mark – correct division process

c-iv	Similarly to slide down – need to overcome upwards friction F = μN. $$\frac{v^2}{Rg}=\frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$$ becomes $$\frac{U^2}{Rg}=\frac{N\sin\theta-(\mu N)\cos\theta}{N\cos\theta+(\mu)N\sin\theta}$$ $$\frac{U^2}{Rg}=\frac{N\sin\theta-\mu N\cos\theta}{N\cos\theta+\mu N\sin\theta}\div\frac{\cos\theta}{\cos\theta}$$ $$\frac{U^2}{Rg}=\frac{N(\tan\theta-\mu)}{N(1+\mu\tan\theta)}$$ $$U^2=Rg\left\{\frac{\tan\theta-\mu}{1+\mu\tan\theta}\right\}$$	1 mark – correct solution
c-v	$$U^2=Rg\left\{\frac{\tan\theta-\mu}{1+\tan\theta}\right\} \quad ③$$ If $\mu \geq \tan\theta$ then $\mu-\tan\theta \geq 0$ therefore $U^2 \leq 0$ from ③ but $U^2 \geq 0$ therefore $U=0$ ∴ for any speed $v>0$ the particle will *NOT* slide down the track.	1 mark –c orrect expression derived