NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2017

Mathematics Extension II

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Write your Student Number at the top of each page
- Answer Section I- Multiple Choice on Answer Sheet provided
- Answer Section II – Free Response in a separate booklet for each question.
- NESA approved calculators and templates may be used.

Section I Multiple Choice
- 10 marks
- Attempt all questions
- Allow about 15 minutes for this section

Section II – Free Response
- 90 marks
- Each question is of equal value
- All necessary working should be shown in every question.
- Allow about 2 hours 45 minutes for this section

Weighting: 40%

Section 1: Multiple Choice (10 marks)

Indicate your answer on answer sheet provided.

Allow approximately 15 minutes for this section.

Q1. What are the values of the real numbers p and q such that $(x+1)^2$ is a factor of the polynomial $px^5 + qx^4 + 1$?

A $p = -4$ and $q = -5$

B $p = 4$ and $q = -5$

C $p = -4$ and $q = 5$

D $p = 4$ and $q = 5$

Q2.

For a certain function $y=f(x)$, the function $y=f(|x|)$ is obtained by:

A A reflection of $y= f(x)$ in the y axis.

B A reflection of $y=f(x)$ in the x axis.

C A reflection of $y=f(x)$ in the x axis for $y \geq 0$

D A reflection of $y=f(x)$ in the y axis for $x \geq 0$

Q3. The eccentricity of a hyperbola with parametric equations $x = 4\sec\theta$ and $y = 3\tan\theta$ is:

A $\frac{4}{3}$

B $\frac{5}{3}$

C $\frac{5}{4}$

D $\frac{25}{16}$

Q4. The gradient of the curve $x^2y - xy^2 + 6 = 0$ at the point P(2,3) is equal to:

A -5

B $\frac{3}{8}$

C $\quad \frac{9}{8}$

D $\quad 1$

Q5. Which of the following is the locus of the point P representing the complex number z on the Argand plane such that $|z - 2i| = 2 + Imz$?

A circle

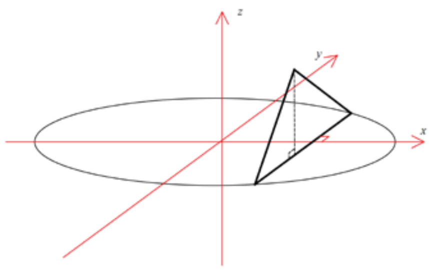
B hyperbola

C parabola

D straight line

Q6. The horizontal base of a solid is the circle $x^2 + y^2 = 1$.
Each cross section, taken perpendicular to the x axis, is a triangle with one side in the base of the solid. The length of the side of the triangle, in the base of the solid, is equal to the altitude of the triangle through the opposite vertex.

Which of the following is an expression for the volume of the solid?



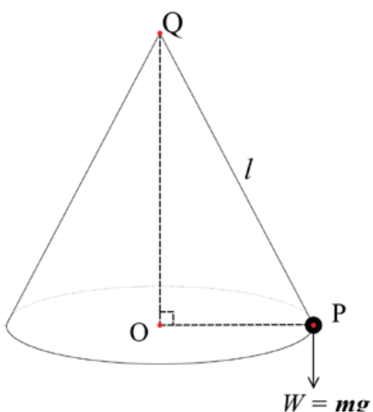
A $\quad \frac{1}{2}\int_{-1}^{1} (1 - x^2)\, dx$

B $\quad \int_{-1}^{1} (1 - x^2)\, dx$

C $\quad \frac{3}{2}\int_{-1}^{1} (1 - x^2)\, dx$

D $\quad 2\int_{-1}^{1} (1 - x^2)\, dx$

Q7.



A mass of m kg at point P is attached to a fixed point Q by a light, inextensible string QP of length l metres. The point P describes a horizontal circle with uniform angular velocity w rad/s under the force of its weight $W=mg$. Which of the following expressions represents the tension in the string?

A $\quad mlw$

B $\quad mlw^2$

C $\quad mglw$

D $\quad mgl$

Q8. The exact value of $\displaystyle\int_0^{\frac{\pi}{4}} x\sec^2 x \, dx$ is:

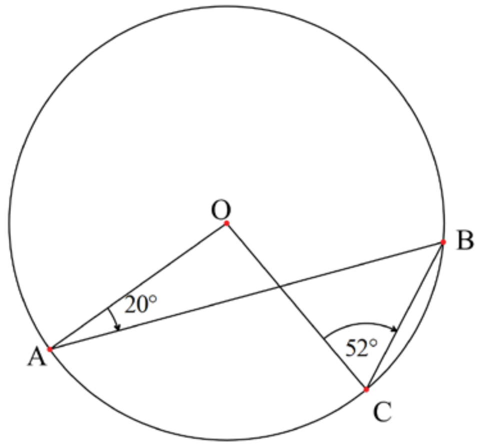
A $\quad \dfrac{\pi}{4}-\dfrac{1}{2}\ln 2$

B $\quad \dfrac{\pi}{4}+\dfrac{1}{2}\ln 2$

C $\quad \dfrac{\pi}{4}+2\ln\sqrt{2}$

D $\quad \dfrac{7}{3}$

Q9.



In the diagram above, O is the centre of the circle. Given that

$$\angle OAB = 20° \text{ and } \angle OCB = 52°$$

What is the size of $\angle ABC$?

A 32°

B 49°

C 56°

D 64°

Q10. An object of mass 5 kg is attached to a piece of rope 3 metres in length. The rope has a breaking strain of 240 N.The object and rope are then rotated in a horizontal circle.What is the angular velocity of the object at the moment the rope breaks?

A 2 rad s^{-1}

B 4 rad s^{-1}

C 8 rad s^{-1}

D 16 rad s^{-1}

<u>End of Multiple Choice</u>

Section II Total Marks is 90

Attempt Questions 11 – 16.

Allow approximately 2 hours & 45 minutes for this section.
Answer all questions, starting each new question in a new booklet with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.

Question 11. – Start New Booklet **15 marks**

a. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \; dx = \frac{2}{15}$ **2**

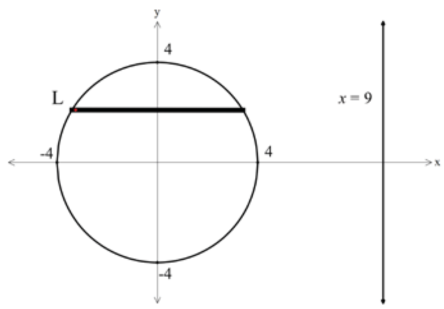
b. Let $\alpha = -\sqrt{3} + i$ and $\beta = 1 - i$.

(i) Express $\overline{\alpha}$ and β in modulus-argument form. **2**

(ii) Find $\overline{\alpha}\beta$ in modulus-argument form. **2**

(iii) Hence, or otherwise, find the exact value of $\tan\frac{\pi}{12}$. **2**

Express your answer in simplest form.



c. The circle $x^2 + y^2 = 16$ is rotated around the line $x = 9$ to form a torus. When the circle is rotated, the line segment L at height y sweeps out an annulus.

(i) Show that the area of the annulus is equal to $36\pi\sqrt{16 - y^2}$. **2**

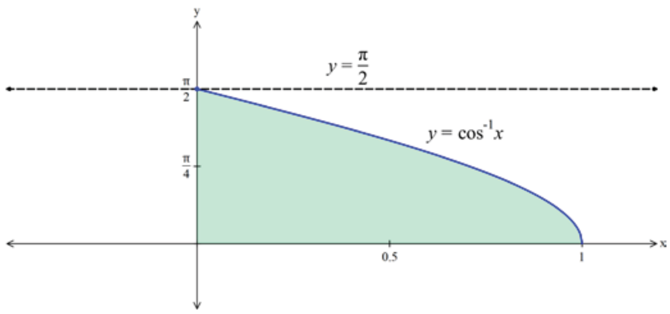
(ii) Find the volume of the torus. **2**

d. Sketch the region defined by $1 \le |z - 2 + 3i| \le 3$ **3**
and $\frac{\pi}{4} \le Arg(z - 2 + 3i) \le \frac{2\pi}{3}$.

End of Question 11

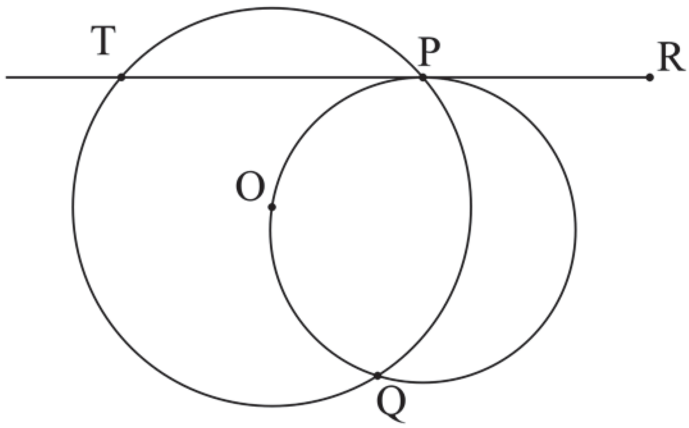
Question 12 – Start New Booklet **15 Marks**

a. The diagram shows the section of the curve $y = \cos^{-1} x$ from x=0 to x= 1. **3**



This shaded region shown is rotated around the line $y = \dfrac{\pi}{2}$ to form a solid.
Use the method of cylindrical shells to find the volume of the solid.

b.



Two circles intersect at P and Q as shown in the diagram above.
The smaller circle passes through the centre, O, of the larger circle.
The tangent RP to the smaller circle intersects the larger circle at T, and PQ bisects $\angle RQO$.

Let $\angle PTQ = \alpha$.

(i) Show that ΔPQT is isosceles. **3**

(ii) Show that P is the midpoint of RT. **2**

Question 12 continues next page.

Question 12 continued.

c. (i) If $\dfrac{16x}{x^4-16} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx}{x^2+4}$, find the values of A, *B* and *C*. **2**

(ii) Hence, show that $\displaystyle\int_4^6 \frac{16x}{x^4-16}\,dx = \log_e\left(\frac{4}{3}\right)$. **2**

d. Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$ in simplest exact form. **3**

End of Question 12

Question 13 – Start New Booklet **15 Marks**

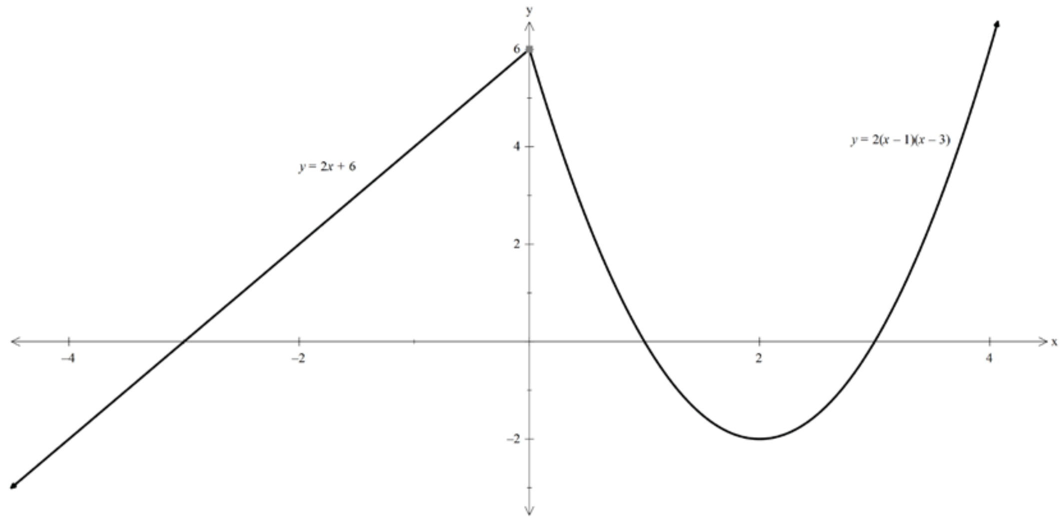
a. Consider the curve defined implicitly by $x^2 + y^2 - xy = 12$.

(i) Find $\frac{dy}{dx}$ in simplest exact form. **2**

(ii) Find the coordinates of the stationary points on the curve. **2**

b. The diagram below shows the graph of a function $y = f(x)$

.



Copy the diagram into your answer booklet.

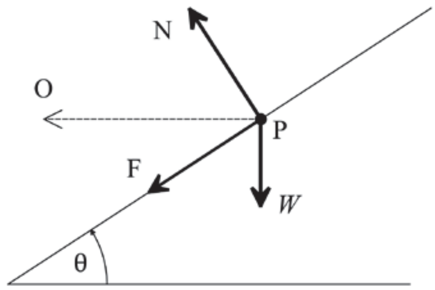
On separate diagrams, draw graphs of each of the following functions. Show clearly any asymptotes and intercepts on the coordinate axes. Your diagrams MUST be at least one quarter of a page.

(i) $|y| = f(x)$ **1**

(ii) $y = f\left(\frac{1}{x}\right)$ **3**

Question 13 continues next page.

Question 13 continued.



c. A car of mass m kg travels around a circular road of radius r metres. The centre of mass of the car is at point P in the diagram above. The forces acting on the car are its weight W , the normal reaction force N and the lateral force F.

(i) By resolving the forces F and N in the horizontal and vertical directions, show that when $F = 0$, $\tan\theta = \dfrac{v^2}{gr}$ where v is the linear speed of the car. **2**

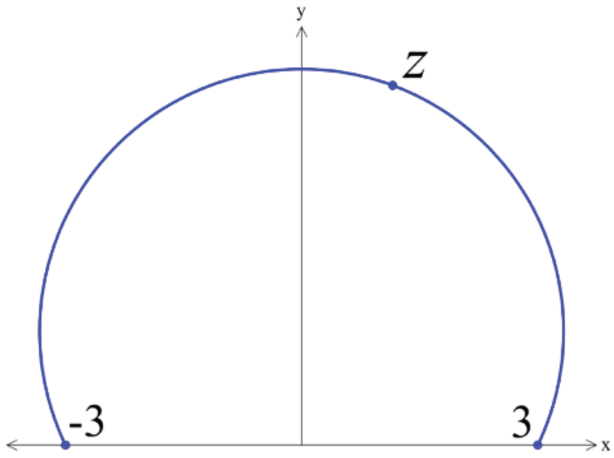
(ii) If the speed of the car is now reduced to $v/2$, prove that the lateral force F is given by **5**

$$F = -\frac{3mgv^2}{4\sqrt{v^4 + g^2r^2}}$$

End of Question 13

Question 14 – Start New Booklet **15 Marks**

a. The equation $Arg\left(\frac{x-3}{x+3}\right)=\frac{\pi}{3}$ defines a major arc of a circle as shown in the diagram below.



(i) Find the cartesian equation of z. **3**

(ii) Find the maximum value of $|z|$. **1**

b. The region bounded by the curves $y=6-x^2$ and $y=\frac{1}{2}x^2$ forms the base of a solid. Cross sections by planes perpendicular to the y axis are semi circles with their diameters in the base of the solid. Find the volume of the solid. **5**

c. Consider the polynomial $P(x)=ax^3+bx^2+cx+d$ where a,b,c and d real numbers. Given that two of the roots of $P(x)=0$ are -2 and $1-2i$ and that $P(-1)=-8$, find the values of a,b,c and d. **6**

End of Question 14

Question 15 – Start a new booklet **15 Marks**

a. Using the identity $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$, show that

$$\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x\ dx = \frac{\pi^3}{96}\ .$$ **3**

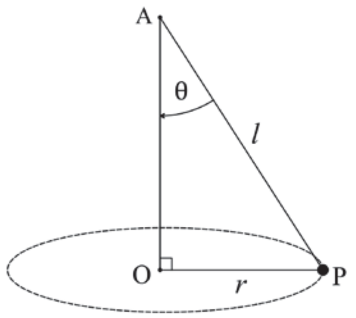
b. Consider the ellipse $E : 16x^2 + 25y^2 = 400$ with foci *S and S'*.

(i) Let *P* be any point on *E*. What is the value of $PS + PS'$? **1**

(ii) Find the eccentricity *e* of the ellipse. **1**

(iii) Find the coordinates of the foci and the equations of the directrices. **2**

(iv) Draw a neat, clearly labelled sketch of the ellipse *E*, showing the foci, directrices and the intercepts with the coordinate axes. **2**



c. In the diagram above, *A* and *P* are the endpoints of a light string of length *l* metres. *A* is a fixed point, while an object of mass *m* kg is attached to the string at *P*. Let the constant angular velocity of the object be ω rad s^{-1}, and let the acceleration due to gravity be $g\ m\ s^{-2}$. Let θ be the angle between the string and the vertical line *OA*, and let *T* Newtons be the tension in the string.

(i) Draw clear diagram showing all forces acting on the object. **1**

(ii) Deduce that $\cos\theta = \dfrac{g}{l\omega^2}$. **2**

(iii) Suppose the angular velocity of the object is increased to ω_1 rad s^{-1}, so that the angle θ is doubled.

Show that: $$\omega_1{}^2 = \frac{gl\omega^4}{2g^2 - l^2\omega^4}$$ **3**

End of Question 15

Question 16 – Start a new booklet **1 Marks**

a. Let $I_n = \int_0^1 x^n \ e^x \, dx$ where n is a positive integer.

(i) Show that $I_{n+1} = \ = e - (n+1)I_n$ **2**

(ii) Hence evaluate $\int_0^1 x^3 e^x \ dx$ in simplest exact form. **2**

b. Let $z = -\sqrt{3} + i$

(i) Express z in modulus argument form. **2**

(ii) Hence find $Im\left(\frac{z^{16}}{i}\right)$ **2**

c. $P\left(p, \frac{1}{p}\right)$ and $Q\left(q, \frac{1}{q}\right)$ are two points on the hyperbola $xy = 1$ where $p > 0$; $q > 0$.
M is the midpoint of the chord PQ.

(i) Show that the chord PQ has equation $x + pqy = p + q$. **2**

(ii) If P and Q move on the rectangular hyperbola in such a way that the perpendicular distance of the chord PQ from the origin $O(0,0)$ is always $\sqrt{2}$, show that $(p+q)^2 = 2(1 + p^2q^2)$. ***1***

(iii) Hence, find the equation of the locus of M. State any restrictions on its domain and range. **4**

End of Examination

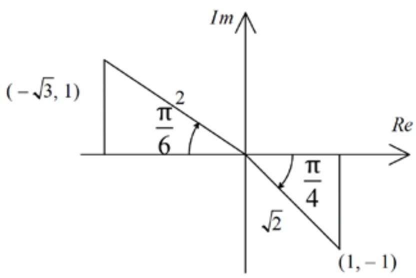
1:A 2:D 3:C 4:B 5:C 6:D 7:B 8:A 9:A 10B

Q1	$(x+1)^2$ is a double root so satisfies both P(x) and P'(x) $x = -1$ $P(x) = \; = p + q + 1 = 0$ $\therefore \quad p = q + 1$ $P'(x) = 5p - 4q = 0$ $5(q+1) - 4q = 0$ $q = -5$ $p = -4$	A
Q2	A reflection of y= $f(x)$ in y axis for $x \geq 0$	D
Q3	$x = 4\sec\theta \qquad y = 3\tan\theta$ $\therefore \; a = 4 \qquad b = 3$ $b^2 = a^2(e^2 - 1)$ $9 + 16(e^2 - 1)$ $e^2 = \frac{25}{16}$ $= \frac{5}{4}$	C
Q4	$x^2y - xy^2 + 6 = 0$ $2xy - y^2 + x^2\frac{dy}{dx} - 2xy\frac{dy}{dx} = 0$ $\frac{dy}{dx}(x^2 - 2xy) = y^2 - 2xy$ $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$ at (2,3) $\frac{dy}{dx} = \frac{9-12}{4-12}$ $= \frac{3}{8}$	B

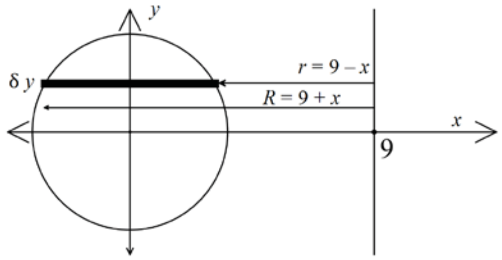
Q5	let $z = x + iy$ $\lvert z - 2i\rvert = 2 + Im\, z$ $\lvert x + iy - 2i\rvert = 2 + y$ $\lvert x + (y-2)i\rvert = 2 + y$ $\sqrt{x^2 + (y-2)^2} = 2 + y$ $x^2 + (y-2)^2 = (2+y)^2$ $x^2 + y^2 - 4y + 4 = 4 + 4y + y^2$ $x^2 = 8y$	C
Q6	$\delta V =$ Area of Trinagle $\times\ \delta x$ $= \frac{1}{2}(2y)(2y)\ \delta x$ $= 2y^2\ \delta x$ $y = \sqrt{1-x^2}$ $V = 2\int_{-1}^{1} (1 - x^2)\ dx$	D
Q7	$\sin\theta = \frac{r}{l}$ $r = l\sin\theta$ $T\sin\theta = mr\omega^2$ $T\sin\theta = ml\sin\theta\ \omega^2$ $T = ml\omega^2$	B

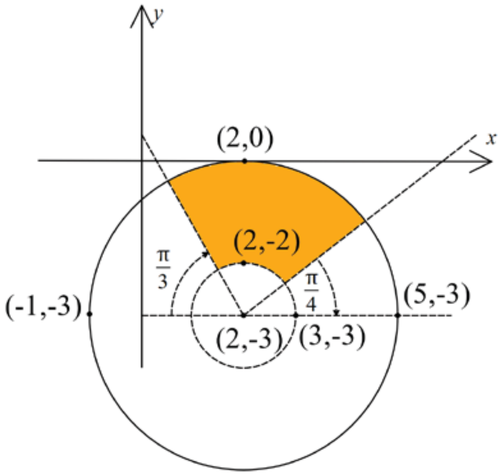
Q8	$\int_0^{\frac{\pi}{2}} x\sec^2 x \ dx$ $u = x \qquad v' = \sec^2 x$ $u' = 1 \qquad v = \tan x$ $\left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \ dx$ $= \left(\frac{\pi}{4} - 0\right) - \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{4} + \left(\ln\frac{1}{\sqrt{2}} - \ln 1\right)$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	A
Q9	Let $\angle AOC = 2\alpha$ $\therefore \qquad \angle ABC = \alpha$ $\therefore$ $20 + 2\alpha = 52 + \alpha$ (*same* exteranl angle of Δ) $\alpha = 32$	A
Q10	$T = 24\circ N$ $mr\omega^2 = 240N$ $5 \times 3 \times \omega^2 = 240$ $\omega^2 = 16$ $\omega = 4$	B

Q 11

a	$\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos^2 x \;\; dx$ $= \int_0^{\frac{\pi}{2}} \sin x \;\, (1-\cos^2 x)\cos^2 x \, dx$ $= \int_0^{\frac{\pi}{2}} \sin x\cos^2 x - \sin x\cos^4 x \;\; dx$ $= \left[-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x \right]_0^{\frac{\pi}{2}}$ $= \left[\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x \right]_0^{\frac{\pi}{2}}$ $= 0 - \left(\frac{1}{5} - \frac{1}{3} \right)$ $= \frac{2}{15}$	2 marks – correct solution 1 mark – correct expanded integral using trig identity.
b-i	 $\alpha = 2\text{cis}\frac{5\pi}{6}$ $\therefore \bar{\alpha} = 2\text{cis}\left(\frac{-5\pi}{6} \right)$ $\beta = \sqrt{2}\text{cis}\left(\frac{-\pi}{4} \right)$	2 marks – 2 correct solutions 1 mark – one correct solution

b-ii	$\overline{\alpha}\beta = 2\sqrt{2}\text{cis}\left(\left(-\frac{5}{6}-\frac{1}{4}\right)\pi\right)$ $= 2\sqrt{2}\text{cis}\left(-\frac{13}{12}\pi\right)$ $= 2\sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$	2 marks – correct solution 1 mark – correct modulus, incorrect argument
b-iii	$\overline{\alpha}\beta = 2\sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$ $\overline{\alpha}\beta = (-\sqrt{3}-i)(1-i)$ $= (-\sqrt{3}-1)+(\sqrt{3}-1)i$ $\therefore 2\sqrt{2}\cos\left(\frac{11\pi}{12}\right) = -(\sqrt{3}+1)$ $-2\sqrt{2}\cos\left(\frac{\pi}{12}\right) = -(\sqrt{3}+1)$ $2\sqrt{2}\cos\left(\frac{\pi}{12}\right) = (\sqrt{3}+1)$ $2\sqrt{2}\sin\left(\frac{11\pi}{12}\right) = (\sqrt{3}-1)$ $2\sqrt{2}\sin\left(\frac{\pi}{12}\right) = (\sqrt{3}-1)$ $\tan\frac{\pi}{12} = \dfrac{2\sqrt{2}\sin\frac{\pi}{12}}{2\sqrt{2}\cos\frac{\pi}{12}}$ $= \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ $= \dfrac{(\sqrt{3}-1)^2}{3-1}$ $= \dfrac{4-2\sqrt{3}}{2}$ $= 2-\sqrt{3}$	2 marks – correct solution in simplest form 1 mark – expression for either $2\sqrt{2}\sin\left(\frac{11\pi}{12}\right)$ or $2\sqrt{2}\cos\left(\frac{11\pi}{12}\right)$

c-i	 Annulus Area $= \pi(R^2 - r^2)$ $R = 9 + x \qquad r = 9 - x$ $x = \sqrt{16 - y^2}$ $= \pi[(R + r)(R - r)]$ $= \pi[(9 + \sqrt{\ }) + (9 - \sqrt{\ })][(9 + \sqrt{\ }) - (9 - \sqrt{\ })$ $= \pi\left(18 \times 2\sqrt{16 - y^2}\right)$ $= 36\pi\sqrt{16 - y^2}$	2 marks – correctly and fully demonstrated 1 incomplete demonstration
c-ii	Disk volume $= \pi(R^2 - r^2)\ \delta y$ $R = 9 + x \qquad r = 9 - x$ $x = \sqrt{16 - y^2}$ $\delta V = \pi[(R + r)(R - r)]\ \delta y$ $= \pi[(9 + \sqrt{\ }) + (9 - \sqrt{\ })][(9 + \sqrt{\ }) - (9 - \sqrt{\ })]$ $= \pi\left(18 \times 2\sqrt{16 - y^2}\right)\delta y$ $= 36\pi\sqrt{16 - y^2}\ \delta y$ $V = \int_{-4}^{4} 36(16 - y^2)\ dy$ $= 2\int_{0}^{4} 36\ \sqrt{16 - y^2}\ dy$ (even function) $= 72 \times \frac{1}{4}\pi \times 4^2$ (1/4 of a circle) $= 288\pi^2\ units^3$	2 marks – correct solution 1 mark – lacks δV

d	$\lvert z-2+3i\rvert$ $=\lvert z-(2-3i)\rvert$ Circle – centre $(2,\ -3)$ radius 3 	3 marks – all features correct 2 features – shading correct for arguments but incorrect circles. 1 mark – correct circles

Question 12

a

$$\delta V = 2\pi r.h.\delta y$$

$$V \cong \lim_{\delta y \to 0} \sum_{y=\frac{\pi}{2}}^{0} 2\pi r.h.\delta y$$

$$r = \frac{\pi}{2} - y \qquad h = x \qquad \delta V = \text{circum} \times \ \text{heigt} \times \ \text{thickness}$$

$$y = \cos^{-1} x$$

$$\therefore \ x = \cos y$$

$$V = 2\pi\left(\frac{\pi}{2} - y\right)x\,dy$$

$$= 2\pi\left(\frac{\pi}{2} - y\right)\cos y\,dy$$

$$V = 2\pi\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - y\right)\cos y\,dy$$

$$= \int_0^{\pi} \left(\frac{\pi}{2}\cos y - y\cos y\right)dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{\pi}{2}\cos y\,dy - \int_0^{\frac{\pi}{2}} y\cos y\,dy$$

$$= \frac{\pi}{2}\Big[\sin y\Big]_0^{\frac{\pi}{2}} - \int_0^{\pi} y\cos y\,dy$$

$$u = y \qquad v' = \cos y$$

$$u' = 1 \qquad v = \sin y$$

$$= 2\pi\left\{\frac{\pi}{2}\Big[\sin y\Big]_0^{\frac{\pi}{2}} - \left\{\Big[y\sin y\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin y\,dy\right\}\right\}$$

$$= 2\pi\left\{\frac{\pi}{2}(0-1) - \left\{\left(\frac{\pi}{2} - 0\right) - \Big[-\cos y\Big]_{\frac{\pi}{2}}^{0}\right\}\right\}$$

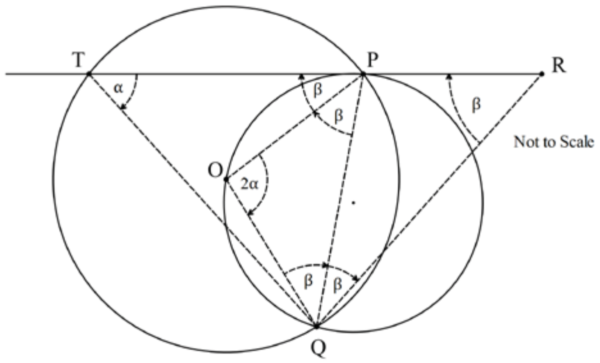
$$= 2\pi\left\{\frac{\pi}{2} \times \frac{\pi}{2} - \frac{\pi}{2} - \Big[\cos y\Big]_0^{\frac{\pi}{2}}\right\}$$

$$= 2\pi\{0 - 0 + 1\} = 2\pi$$

3 marks – correct solution

2 marks – forms correct volume elements and finds correct primitive functions
or
correctly applies IBP to obtain a volume from an incorrect volume element

1 mark – correctly applies IBP to a volume element containing the expression $y\cos y$

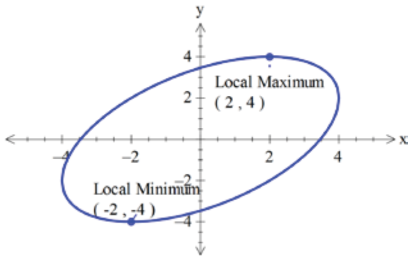
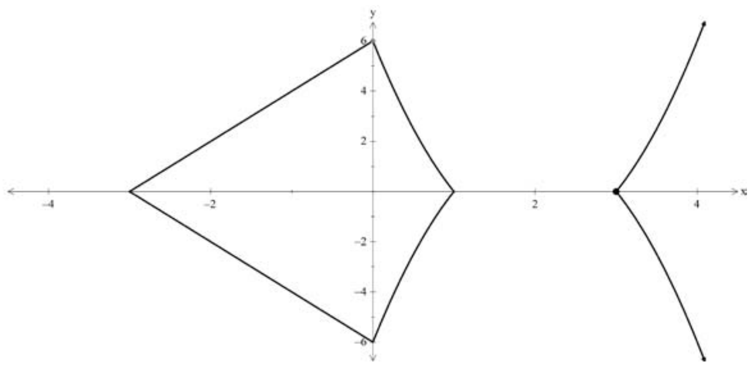
		
	Let $PTQ = \alpha$ $\therefore \quad \angle POQ = 2\alpha$ ($\angle$ at centre twice $\angle$ at circum $\angle RQP = \angle PQO = \beta$ (given) $\angle TPO = \angle PQO = \beta$ (alternate $\angle$ theorem) OP = OQ radii $\therefore \quad \Delta\, POQ$ isosceles $\therefore \quad \angle OPQ = \angle OQP = \beta$ $\therefore \quad 2\alpha + 2\beta = 180$ $\angle PTQ + \angle TPQ + \angle PQT = 180$ $\alpha + 2\beta + \angle PQT = 180$ $\angle PQT = \alpha$ $\therefore \quad \angle PQT = \angle PTQ$ $\therefore \quad \Delta\, TPQ$ isosceles $\therefore \quad$ PT = PQ	3 marks – correct solution 2 marks – Identifies $\angle POQ = 2\alpha$ and $\angle QPR = 2\alpha$ owith reasons r equivalent progress 1 mark - Identifies $\angle POQ = 2\alpha$ or equivalent progress

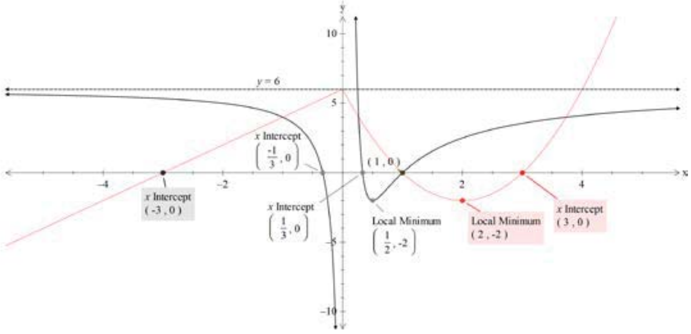
	$\angle TPQ = \angle PRQ + \angle PQR$ $(ext\ \angle\ =$ sum of opp interior $\angle)$ $2\beta = \angle PRQ = \beta$ $\angle PRQ = \beta$ $\therefore$ ΔPRQ isosceles $\overrightarrow{PQ} = \overrightarrow{PR}$ as $\overrightarrow{TP} = \overrightarrow{PQ}$ then $\overrightarrow{TP} = \overrightarrow{PR}$ $\therefore$ P is midpoint of $\overrightarrow{TR}$	
c-i	$\dfrac{16x}{x^4 - 16} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx}{x^2+4}$ $\therefore$ $16x = A(x+2)(x^2+4) + B(x-2)(x^2+4) + Cx(x-2)(x+2)$ Let $x = 2$ $32 = A \times 4 \times 8 + 0 + 0$ $A = 1$ Let $x = -2$ $-32 = 0 + B \times -4 \times 8 + 0$ $B = 1$ Let $x = 1$ $16 = 1 \times 3 \times 5 + 1 \times -1 \times 5 + C \times 1 \times -1 \times 3$ $6 = -3C$ $C = -2$	

c-ii	$\int_4^6 \frac{16x}{x^4-16}\,dx$ $= \int_4^6 \frac{1}{x+2}+\frac{1}{x-2}-\frac{2x}{x^2+4}\,dx$ $= \Big[\ln(x+2)\Big]_4^6 + \Big[\ln(x-2)\Big]_4^6 - \Big[\ln(x^2+4)\Big]_4^6$ $= (\ln 8 - \ln 6) + (\ln 4 - \ln 2) - (\ln 40 - \ln 20)$ $= \ln\left(\frac{8}{6}\right) + \ln\left(\frac{4}{2}\right) - \ln\left(\frac{40}{20}\right)$ $= \ln\left(\frac{4}{3}\times\frac{4}{2} \div \frac{4}{2}\right)$ $= \ln\left(\frac{4}{3}\right)$	
	$\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$ $\tan\frac{x}{2} = t \Rightarrow x = 2\tan^{-1}t$ $dx = \frac{2}{1+t^2}$ $x = 0 \Rightarrow t = 0$ $x = \frac{\pi}{3} \Rightarrow t = \frac{1}{\sqrt{3}}$ $\tan x = \frac{2t}{1-t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$	

	$\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$ $=\int \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{dt}$ $=2\int \frac{\frac{2t}{1-t^2}}{1+t^2+1-t^2}\,\mathrm{dt}$ $=2\int \frac{\frac{2t}{1-t^2}}{2}\,\mathrm{dt}$ $=\int \frac{2t}{1-t^2}\,\mathrm{dt}$ $= -\left[\ln(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}}$ $= -\left\{\ln\frac{2}{3} - \ln 1\right\}$ $= -\left\{\ln\left(\frac{2}{3}\right) - 0\right\}$ $= \ln\left(\frac{3}{2}\right)$	

Question 13

ia-	$x^2+y^2-xy=12$ $2x+2y\dfrac{dy}{dx}-y-x\dfrac{dy}{dx}=0$ $(2x-y)+(2y-x)\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}=-\dfrac{2x-y}{2y-x}$ $=\dfrac{y-2x}{2y-x}$	2 marks – correct solution 1mark -
a-ii	Stat Points at $\dfrac{dy}{dx}=0$  $\therefore$ $2-y=0$ $y=2x$ $x^2+y^2-xy=12$ $x^2+4x^2-2x^2=123x^2=12$ $x^2=4$ $x=\pm 2$ $x=2 \Rightarrow y=4$ $x=\text{-}2 \Rightarrow y=\text{-}4$ nb $2y-x \neq 0$	2 marks – correct solution 1 mark -
b-i		1 mark – correct answer

<table>
<tr>
<td>b-ii</td>
<td>
</td>
<td>2 marks – graphs as shown with all detail including coordinates

2 mark – lacks horizontal asymptote or minimum position – must have all other features.

1 mark – one feature correct</td>
</tr>
<tr>
<td>c-i</td>
<td>Horizontally

$N\sin\theta + F\cos\theta = \dfrac{mv^2}{r} \qquad (1)$

Vertically

$N\cos\theta - F\sin\theta = \text{mg} \qquad (2)$

$F = 0$

$\dfrac{N\sin\theta}{N\cos\theta} = \left(\dfrac{mv^2}{r}\right) \div \text{mg}$

$\tan\theta = \dfrac{v^2}{rg}$</td>
<td>2 marks – correct solution

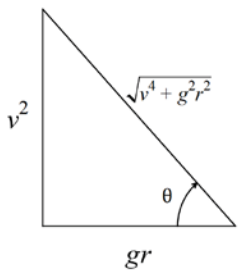
1 mark – correct resolution horizontally or vertically</td>
</tr>
<tr>
<td>c-ii</td>
<td>$N\sin\theta + F\cos\theta = \dfrac{mrv^2}{r} \qquad (1)$

$N\cos\theta - F\sin\theta = \text{mg} \qquad (2)$

$\tan\theta = \dfrac{v^2}{rg}$

$\Rightarrow \qquad \cos\theta = \dfrac{rg}{\sqrt{v^4 + g^2r^2}}$

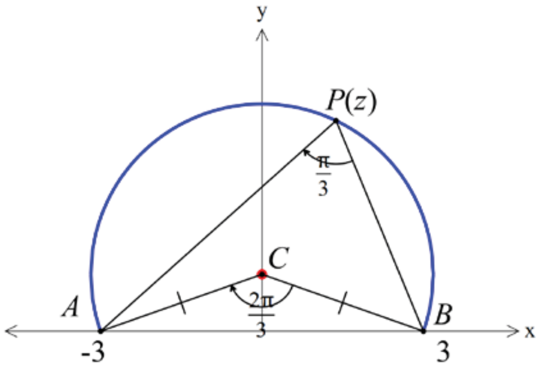
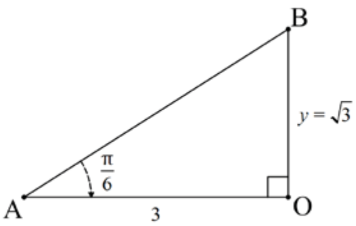
$\Rightarrow \qquad \sin\theta = \dfrac{v^2}{\sqrt{v^2 + g^2r^2}}$

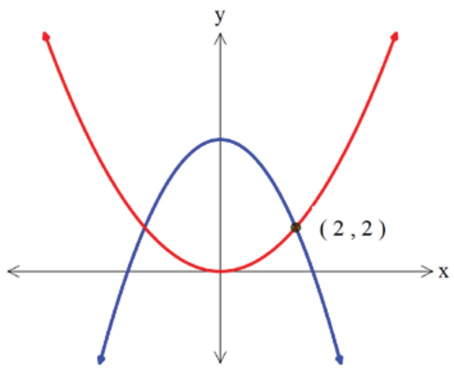

</td>
<td>5 marks

- Derivation of $\cos\theta$ or $\sin\theta$
- Correct replacement of v with $\dfrac{v}{2}$
- Expression for F</td>
</tr>
</table>

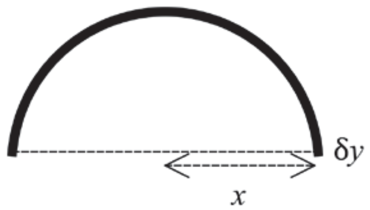
	eqn (1) $\times$ cosθ – *eqn* (2) $\times$ sinθ $F(\cos^2\theta + \sin^2\theta) = \frac{mrv^2}{r} s\cos\theta - mg\sin\theta$ $v \Rightarrow \frac{v}{2}$ $F = \frac{mrv^2}{4r} \times \frac{gr}{\sqrt{v^4 + g^2r^2}} - \frac{mgv^2}{\sqrt{v^2 + g^2r^2}}$ $= \frac{-3mgv^2}{2\sqrt{v^4 + g^2\,r^2}}$	

Q14

		
a	$$Arg\left(\frac{z-3}{z+3}\right)=\frac{\pi}{3}$$ $$Arg(z-3)-Arg(z=3)=\frac{\pi}{3}$$ $\therefore$ $$\angle APB=\frac{\pi}{3}$$ $$\angle ACB=\frac{2\pi}{3}$$ $\Delta\ ACB$ is isosceles $\angle CAB=\angle CBA=\frac{1}{2}\left\{\pi-\frac{2\pi}{3}\right\}=\frac{\pi}{6}$ therefore C is $(0,\sqrt{3})$ $r=\text{AC}$ $r^2=\text{AC}^2=3^2+(\sqrt{3})^2$ $=12$ $x^2+(y-\sqrt{3})^2=12; y\geq 0$	3 marks – correct solution 2 marks – correct centre and radius but incorrect equation or correct equation formed from one of centre or radius being incorrect 1 mark correct centre or radius
	Maximum value occurs at y – intercept therefore $max\|z\|=\sqrt{3}+2\sqrt{3}=3\sqrt{3}$	1makr correct answer

<table>
<tr>
<td>b</td>
<td>


$$y = 6 - x^2$$
$$y = \frac{x^2}{2}$$
$$6 - x^2 = \frac{x^2}{2}$$
$$3x^2 = 12$$
$$x^2 = 4 = \pm 2 \quad \Rightarrow \quad y = 2$$



$$A = \frac{1}{2}\pi r^2 = \pi x^2 \frac{1}{2}$$
$$\delta V_1 = \frac{\pi x^2}{2}\delta y$$

for $y = 6 - x^2$

for $y = \frac{x^2}{2}$

$$\delta V_2 = \frac{\pi}{2} 2y\delta y$$
$$= \pi y\, \delta y$$

Total Vol = $V_1 + V_2$
</td>
<td>
5 marks – correct solution

4 marks – solution contains one error in working

3 marks – solution contains two errors in working

2 marks – 3errors

1 mark - one correct volume element
</td>
</tr>
</table>

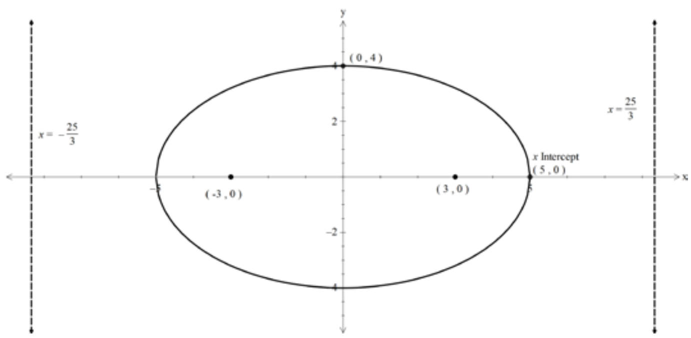
	$V_1 = \frac{\pi}{2}\int_2^6 (6-y)\ dy$ $= \frac{\pi}{2}\left[6y - \frac{y^2}{2}\right]_2^6$ $= \frac{\pi}{2}[(36-18)-(12-2)]$ $= 4\pi$ $V_2 = \frac{\pi}{2}\int_0^2 y dy$ $= \frac{\pi}{2}\left[y^2\right]_0^2$ $= \frac{\pi}{2}(4-0)$ $= 2\pi$ $V = 6\pi$	
c	Consider the polynomial $P(x) = ax^3 + bx^2 + cx + d$ where *a,b,c* and *d* real numbers. Given that two of the roots of *P(x)*= 0 are -2 and $1-2i$ and that *P*(-1) = -8, find the values of *a,b,c* and *d.* As *a, b, c* and *d* are real, complex roots are in conjugate pairs. $P(x) = 0$ for $x = -2$, $1-2i$, $1+2i$ $P(-1) = -8$ $-8 = -a + b - c + d$ $8 = a - b + c - d \quad (1)$ $\alpha + \beta + \gamma = -\frac{b}{a}$ $-2 + 1 - 2i + 1 + 2i = -\frac{b}{a}$ $0 = \frac{b}{a}$ $b = 0 \quad (2)$ $\alpha\beta\gamma = -\frac{d}{a}$ $(1+2i)(1-2i)(-2) = -\frac{d}{a}$ $(1-4i^2)(-2) = -\frac{d}{a}$ $(5)(-2) = -\frac{d}{a}$ $\frac{d}{a} = 10$ $d = 10a \quad (3)$	6 marks – correct solution 5 marks – obtains four correct numerical values with one error on signs 4 marks – obtains 3 correct numerical values or equivalent progress 3 marks – obtains 2 correct values or equivalent progress 2 marks – significant progress towards finding umeral values for a, b, c and d. 1 mark relevant progress towards finding numerical values for a, b ,c and d.

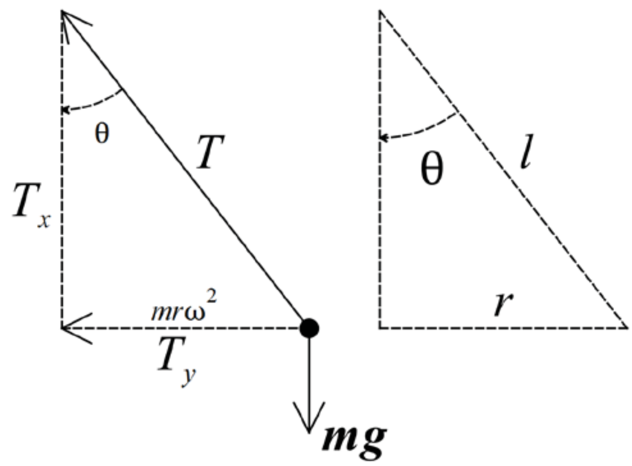
	$\frac{c}{a} = \alpha\beta + \alpha\gamma + \beta\gamma$ $\frac{c}{a} = (1 + 2i)(1 - 2i) + (-2)(1 + 2i) + (-2)(1 - 2i)$ $\frac{c}{a} = 5 - 2 - 2i - 2 + 2i$ $\frac{c}{a} = 1$ $c = a \qquad\qquad (4)$ Substitute (2) (3) and (4) into (1) $a - 0 + a - 10a = 8$ $-8a = 8$ $a = -1$ $\therefore$ $c = -1$ $d = -10$ $b = 0$ $a = -1 \quad b = 0 \quad c = -1 \quad d = -10$	

Q15

a	$f(x) = x\left(\frac{\pi}{2}-x\right)\sin^2 x$ $f\left(\frac{\pi}{2}-x\right) = \left(\frac{\pi}{2}-x\right)\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right]\sin^2\left(\frac{\pi}{2}-x\right)$ $= x\left(\frac{\pi}{2}-x\right)\cos^2 x$ $\int f(x)\ dx = \int \left\{\frac{1}{2}f(x) + \frac{1}{2f}(x)\right\}dx$ therefore	3 marks – correct solution 2 marks – production of 2I expression 1 mark – demonstration of use of $\int_0^a f(a-x)\ dx$

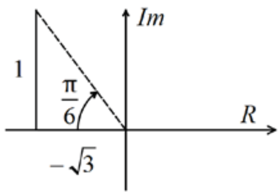
	$I = \int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x \ dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x \ dx$ as $\int_0^a f(x) \ dx = \int_0^a f(a-x) \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x \ dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)(x)\sin^2\left(\frac{\pi}{2}-x\right) \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\sin^2 x \ dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\cos^2 x \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{x\left(\frac{\pi}{2}-x\right)\sin^2 x + x\left(\frac{\pi}{2}-x\right)\cos^2 x\right\} \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{x\left(\frac{\pi}{2}-x\right)(\sin^2 x + \cos^2 x\right\} \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right) \ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}x-x^2\right) \ dx$ $= \frac{1}{2}\left[\frac{\pi}{4}x^2-\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{2}\left\{\left(\frac{\pi}{4}.\frac{\pi^2}{4}-\frac{\pi^3}{24}\right)-0\right\}$ $= \frac{1}{2}\left\{\frac{\pi^3}{48}\right\}$ $= \frac{\pi^3}{96}$	

b-i	By definition $PS + PS' = 2a$ $16x^2 + 25y^2 = 400$ $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ $a = 5 \quad y = 4$ $\therefore \quad 2a = 10$	1 mark.
b-ii	$e^2 = 1 - \dfrac{b^2}{a^2}$ $= 1 - \dfrac{16}{25} = \dfrac{9}{25}$ $e = \dfrac{3}{5}$	1 mark
b-iii	Foci $(\pm ae, 0)$ Directrices $x = \pm\dfrac{a}{e}$ Foci $\left(\pm 5 \times \dfrac{3}{5}, 0\right)$ Directrices $x = \dfrac{\pm 5}{\frac{3}{5}} = \pm\dfrac{25}{3}$	2 marks – both correct 1 mark – one correct.
b-iv		2 marks – all features correctly shown. 1 mark – correct shape but not all features labelled.

c		1 mark
	$\sin\theta = \frac{r}{l}$ (1) $r = l\sin\theta$ $T\cos\theta = \text{mg}$ $T\sin\theta = mr\omega^2$ $\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{\text{mg}}$ (2) Subst (1) into (2) $\frac{\sin\theta}{\cos\theta} = \frac{r\omega^2}{g}$ $\frac{r}{l\cos\theta} = \frac{r\omega^2}{g}$ $\cos\theta = \frac{g}{l\omega^2}$	2 marks

	$T\cos(2\theta) = \text{mg}$ $T\sin(2\theta) = mr_1(\omega_1)^2$ $\therefore$ from part a $\cos(2\theta) = \dfrac{g}{l(\omega_1)^2}$ $2\cos^2\theta - 1 = \dfrac{g}{l(\omega_1)^2}$ $2\left[\dfrac{g}{l\omega^2}\right]^2 - 1 = \dfrac{g}{l(\omega_1)^2}$ $\dfrac{2g^2}{l^2\omega^4} - 1 = \dfrac{g}{l(\omega_1)^2}$ $\dfrac{2g^2 - l^2\omega^4}{l^2\omega^4} = \dfrac{g}{l(\omega_1)^2}$ $\dfrac{(\omega_1)^2}{g} = \dfrac{l\omega^4}{2g^2 - l^2\omega^4}$ $(\omega_1)^2 = \dfrac{gl\omega^4}{2g^2 - l^2\omega^4}$	3 marks 2 marks - correct substitution for $\cos^2\theta$ and some simplification. 1 mark – Line 2 – use of Trig Identity.

Q16

a-i	$I_n = \int_0^1 x^n \ e^x \, dx$ $u = x^n \qquad dv = e^x$ $du = nx^{n-1} \qquad v = e^x$ $I_n = \left[x^n e^x\right]_0^1 - \int_0^1 nx^{n-1} e^x \, dx$ $= e - n\int_0^1 x^{n-1} e^x \, dx$ $I_n = e - nI_{n-1}$ $n \Rightarrow n+1$ $I_{n+1} = e - (n+1)I_n$	2 marks – correct solution 1 mark – obtains $I_n = e - nI_{n-1}$ or equivalent expression.
a-ii	$I_3 = e - 3I_2$ $I_2 = e - 2I_1$ $I_1 = \int_0^1 xe^x \, dx : n > 0$ $= \left[xe^x\right]_0^1 - \int_0^1 e^x \ dx$ (by IBP) $= e - \left[e^x\right]_0^1$ $= e - (e-1)$ $= 1$ $I_2 = e - 2I_1 = e - 2$ $I_3 = e - 3I_2 = e - 3(e-2)$ $= 6 - 2e$	2 marks – correct solution 1 mark – obtains $I_3 = 6 - 2e$ using $I_0 = \int_0^1 e^x \, dx$ or equivalent progress.
	$z = -\sqrt{3} + i$ $z = 2\text{cis}\left(\dfrac{5\pi}{6}\right)$ 	2 marks - correct solution 1 mark – correct modulus or argument
		2 marks – correct solution 1 mark – correct gradient

	$\frac{z^{16}}{i}$ $= 2^{16}\frac{\text{cis}\left(16 \times \frac{5\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{2}\right)}$ $= 2^{16}\,\text{cis}\left(\frac{40\pi}{3} - \frac{\pi}{2}\right)$ $= 2^{16}\text{cis}\left[\frac{(80-3)\pi}{6}\right]$ $= 2\text{cis}\,\frac{77\pi}{6}$ $\therefore Im\left(\frac{z^{16}}{i}\right) = 2^{16}\sin\left(\frac{77\pi}{6}\right)$ $= 2^{16}\sin\left(\frac{77\pi}{6}\right)$ $= 2^{16}\sin\left(6 \times 2\pi + \frac{5\pi}{6}\right)$ $= 2^{16}\sin\left(\frac{5\pi}{6}\right)$ $= 2^{16} \times \frac{1}{2}$ $= 2^{15} = 32768$	

c-i	$$m = \frac{\frac{1}{p} - \frac{1}{q}}{p-q}$$ $$= \frac{q-p}{pq} \times \frac{1}{p-q}$$ $$= -\frac{1}{pq}$$ $$y - \frac{1}{p} = -\frac{1}{pq}(x-p)$$ $$-pqy + q = x - p$$ $$p + q = x + pqy$$ $$x + pqy = p + q$$	2 marks – correct solution 1 mark – correct gradient
c-ii	$$\sqrt{2} = \frac{\lvert x + pqy - (p+q)\rvert}{\sqrt{1 + p^2q^2}}$$ $$x = 0 \qquad y = 0$$ $$\sqrt{2} = \frac{\lvert p+q\rvert \times \lvert -1\rvert}{\sqrt{1+p^2q^2}}$$ $$\sqrt{2}\sqrt{1+p^2q^2} = \sqrt{(p+q)^2}$$ $$2(1+p^2q^2) = (p+q)^2$$	1 mark – correct solution

c-iii	$M=\left(\dfrac{p+q}{2},\dfrac{1}{2}\left(\dfrac{1}{p}+\dfrac{1}{q}\right)\right)$ $=\left(\dfrac{p+q}{2},\dfrac{p+q}{2pq}\right)$ $x=\dfrac{p+q}{2}\qquad y=\dfrac{x}{pq}$ From part (ii) $\dfrac{(p+q)^2}{2}=1+p^2q^2$ $2\left(\dfrac{p+q}{2}\right)^2=1+p^2q^2$ $2x^2-1=p^2q^2$ $y^2=\dfrac{x^2}{p^2q^2}$ $y^2=\dfrac{x^2}{2x^2-1}$ $y=\dfrac{\pm x}{\sqrt{2x^2-1}}$ $2x^2-1>0$ $\therefore\qquad x<-\dfrac{1}{\sqrt{2}}$ or $x>\dfrac{1}{\sqrt{2}}$ $y^2=\dfrac{x^2}{2x^2-1}$ $2y^2x^2-y^2=x^2$ $x^2(2y^2-1)-y^2=0$ $x^2=\dfrac{y^2}{2y^2-1}$ $\therefore\qquad y<-\dfrac{1}{\sqrt{2}}$ or $y>\dfrac{1}{\sqrt{2}}$ But p,q > 0 $\Rightarrow\ x>0,\ y>0$ $\therefore\qquad y^2=\dfrac{x^2}{2x^2-1}$ for $x>\dfrac{1}{\sqrt{2}},\ y>\dfrac{1}{\sqrt{2}}$	4 marks – correct solution 3 marks – find $2x^2y^2=x^2+y^2$ or equivalent AND determines $\lvert x\rvert>\dfrac{1}{2}$ and $\lvert y\rvert>\dfrac{1}{2}$ 2 marks - find $2x^2y^2=x^2+y^2$ or equivalent AND determines $\lvert x\rvert>\dfrac{1}{2}$ or $\lvert y\rvert>\dfrac{1}{2}$ 1 mark – finds $2x^2y^2=x^2+y^2$ or equivalent expression.