NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2018

Mathematics Extension II

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Write your Student Number at the top of each page
- Answer Section I- Multiple Choice on Answer Sheet provided
- Answer Section II – Free Response in a separate booklet for each question.
- Answer Section II - Question 13d in the supplied additional sheet
- NESA approved calculators and templates may be used.

Section I Multiple Choice

- 10 marks
- Attempt all questions
- Allow about 15 minutes for this section

Section II – Free Response

- 90 marks
- Each question is of equal value
- All necessary working should be shown in every question.
- Allow about 2 hours 45 minutes for this section

Weighting: 40%

Section 1: Multiple Choice (10 marks)

Indicate your answer on answer sheet provided.

Allow approximately 15 minutes for this section.

Q1. A hyperbola has the equation $x^2 - 4y^2 = 4$. The distance between its two directrices is:

A $\sqrt{5}$

B $\frac{4\sqrt{5}}{5}$

C $2\sqrt{5}$

D $\frac{8\sqrt{5}}{5}$

Q2. The equation $x^3 - 4x^2 + 7x + 3 = 0$ has roots $x = \alpha$, $x = \beta$ and $x = \gamma$. Which of the equations below has roots $x = -\alpha$, $x = -\beta$ and $x = -\gamma$?

A $x^3 - 4x^2 + 7x + 3 = 0$

B $x^3 + 4x^2 + 7x - 3 = 0$

C $x^3 - 4x^2 + 7x - 3 = 0$

D $x^3 + 4x^2 - 7x - 3 = 0$

Q3. Which of the following is $\int_0^{\frac{\pi}{2}} 2x\cos x \; dx$?

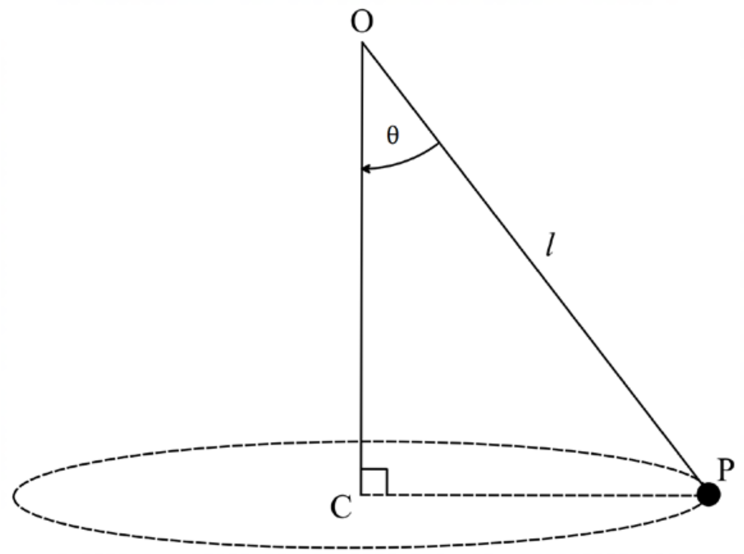
A $\pi - 2$

B $\pi + 2$

C $\frac{\pi^2}{4}$

D $\frac{\pi^2}{8}$

Q4.



The diagram shows a particle *P* of mass *m* kilograms suspended from a fixed point *O* by a light inextensible string of length l metres.

P moves in a circle with centre *C* directly below *O* and has uniform angular speed ω rads^{-1}.

The string makes an angle θ with the vertical line *CO* and the acceleration due to gravity is g ms^{-2}.

Which of the following is the tension *T* in the string?

A $m\,l\,\omega$ newtons

B $m\,l\,\omega^2$ newtons

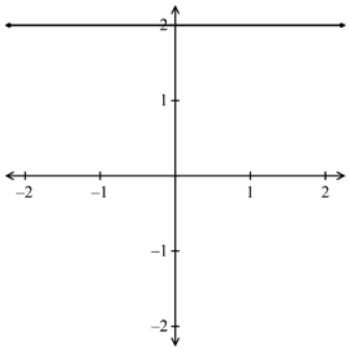
C $m\,g\,l\,\omega$ newtons

D $m\,g\,l\,\omega^2$ newtons

Q5. The complex number z is such that $Im\left(\frac{1}{z}\right) = -\frac{1}{2}$.

Which of the diagrams below represents the locus of z ?

A



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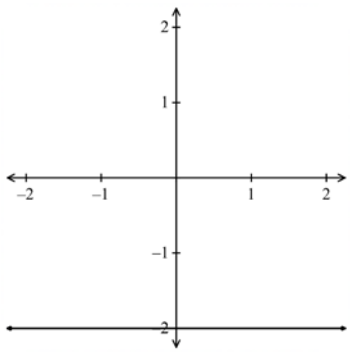
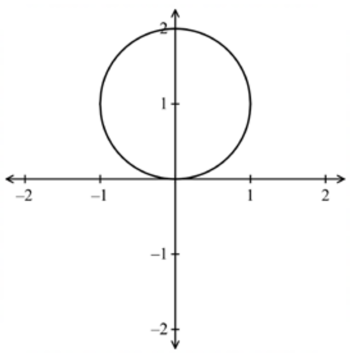


C



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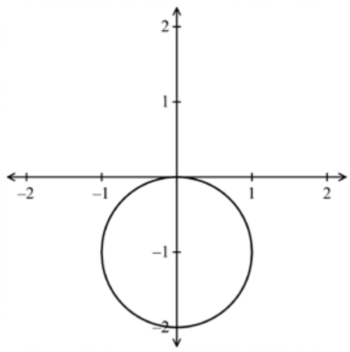


Q6. The equation $|z - 3| + |z + 3| = 10$ defines an ellipse.

The length of the semi minor axis is:

A 4

B 5

C 8

D 10

Q7. Consider the following statements:

$$\text{I} \quad \int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}}$$

$$\text{II} \quad \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\ dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\ dx$$

Which of these statements is correct?

A Both statements I and II are correct

B Only statement I is true

C Only statement II is true

D Both statements are false

Q8. The hyperbola with equation $xy = 8$ is the hyperbola $x^2 - y^2 = a^2$ referred to different axes.

What is the value of a?

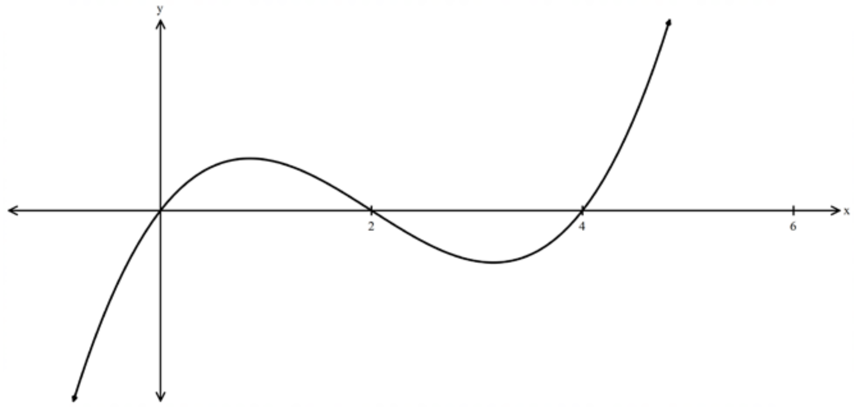
A 2

B 4

C 8

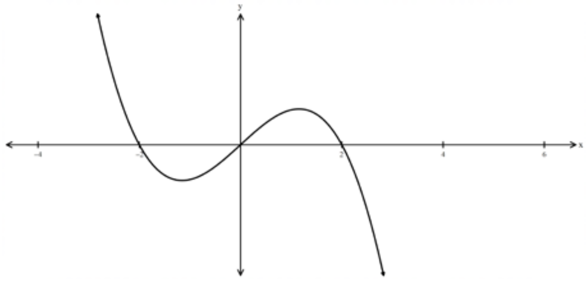
D 16

Q9.



The graph of $y = f(x)$ is shown above. The graph of $y = f(2-x)$ is:

A



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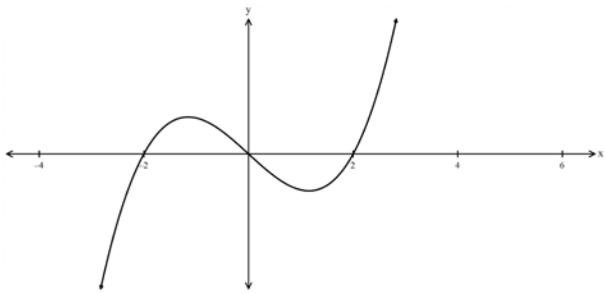

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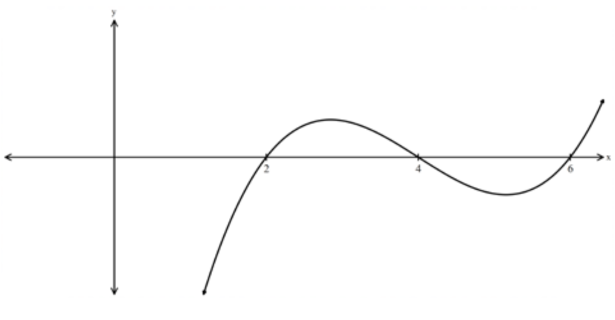

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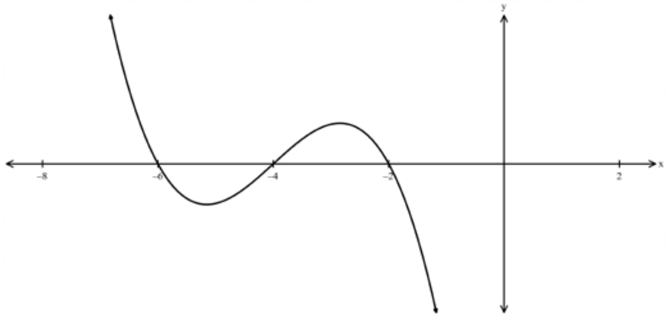
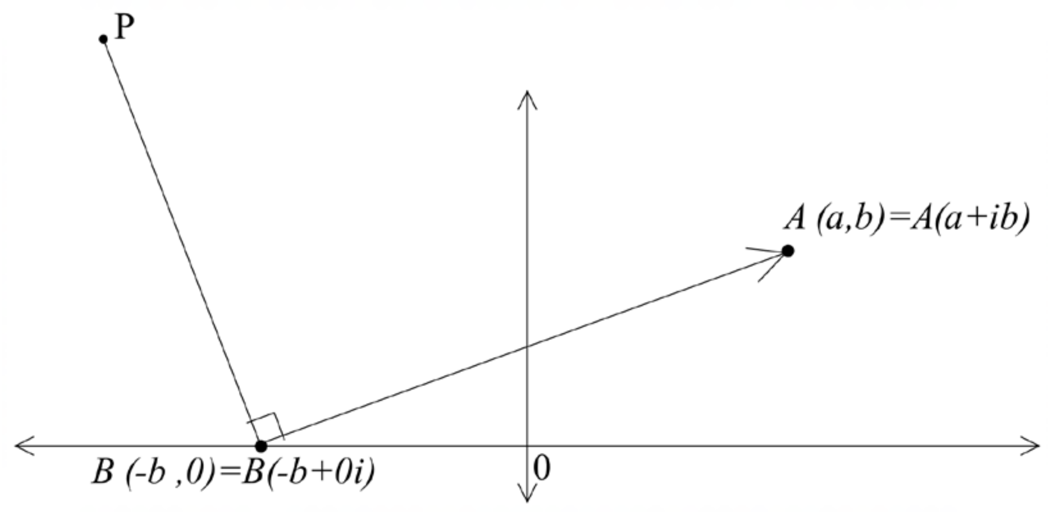


Q10.



The Argand diagram above shows the point $A(a\ ,b)$ representing the complex number $z = a + ib$, where a and b are real. B is the point $(-b\ ,0)$.

P is a point such that $PB=2AB$ and $\angle ABP = 90°$.

Which of the following complex numbers does P represent?

A $\quad -2b + 2ai$

B $\quad -b + ai$

C $\quad -2b + (2a + 2b)i$

D $\quad -3b + (2a + 2b)i$

<u>End of Multiple Choice</u>

Section II Total Marks is 90

Attempt Questions 11 – 16.

Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question in a new booklet with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper.

All necessary working must be shown in each and every question.

Question 11. – Start New Booklet **15 marks**

a) Given $z = 1 + i$, represent on an Argand diagram each of the following: 3

i. z

ii. $\frac{1}{z}$

iii. z^2

b) Prove that for any two complex numbers z_1 and z_2 ,

i. $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ 2

ii. Give a geometric interpretation of this result 1

c) Find the equation of the normal to the curve $x^3 - 6xy + y^3 = 5$ at the point (1 , -2). 3

d) Show that $\int \frac{\sin^3 x}{\cos^2 x}\, dx = \sec x + \cos x$ 3

e) The hyperbola H: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where $a > 0$, $b > 0$) 3

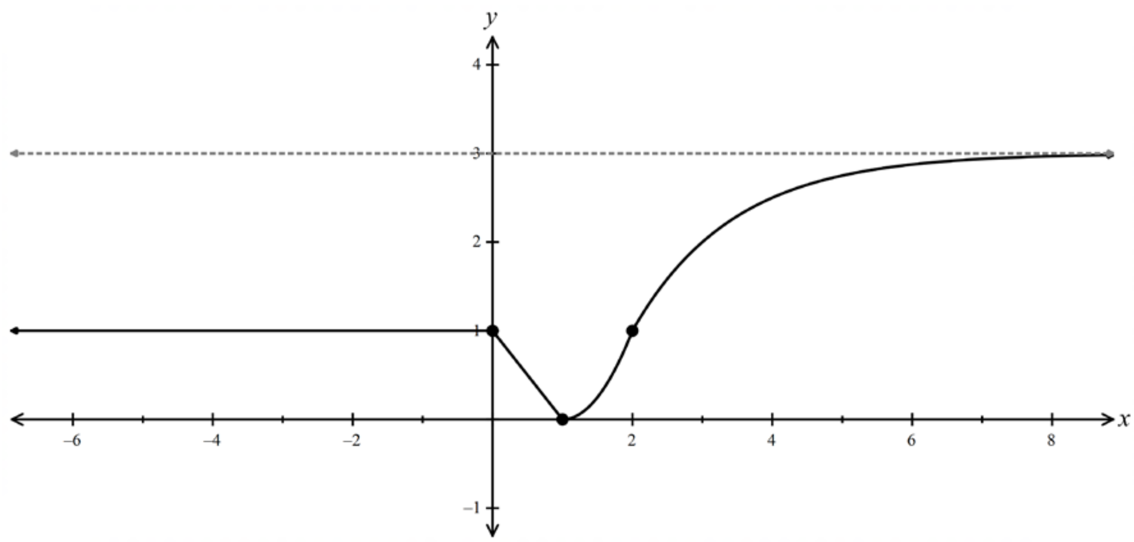
has a focus at the point $(2\sqrt{13}\ , 0\ )$. The line $y = \frac{2}{3}x$ is an asymptote.

Find the values of a and b.

End of Question 11

Question 12. – Start New Booklet **15 marks**

a)



The diagram shows the graph $y = f(x)$.

On separate diagrams, draw $\frac{1}{3}$ page sketches of the following graphs:

(i) $y = f(|2x|)$ 2

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y = \sqrt{f(x)}$ 2

(iv) $y = e^{f(x)}$ 2

(v) $y = x^{f(x)}$ 3

b)

i. Find $\sqrt{-8 + 6i}$ in cartesian form 2

ii. Hence, solve the equation 2

$$2z^2 - (3 + i)z + 2 = 0$$

Express the result in the form $x + iy$

End of Question 12

Question 13. – Start New Booklet **15 marks**

a)

i. On the same diagram, sketch the graphs of $y = |x^3 - 1|$ and $y = 1 - x$ 1

ii. Hence, or otherwise, solve $|x^3 - 1| < 1 - x$ 2

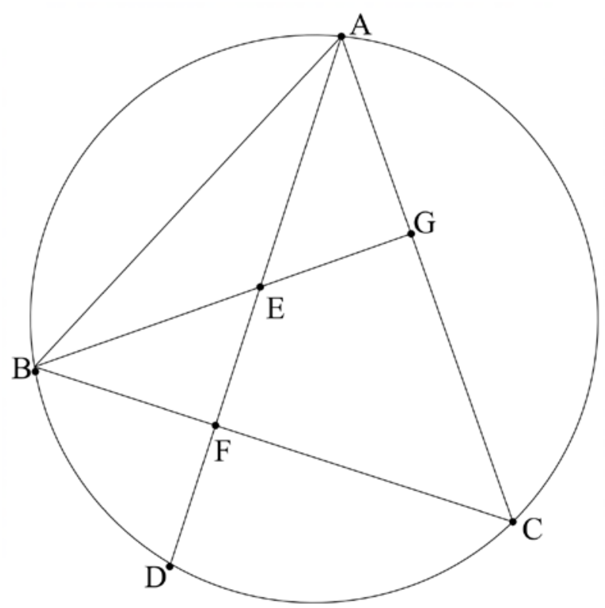
b) Draw neat, labelled sketches to indicate the regions of the Argand plane defined by:

i. $|z| \leq 2$ and $0 \leq \arg z \leq \dfrac{\pi}{4}$ 2

ii. $|z - \bar{z}| \leq 6$ and $0 \leq Re(2z) \leq 4$ 2

c) The equation $8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$ has a triple root. Solve the equation completely. 4

d)



The diagram shows ΔABC inscribed in a circle.
G is the point on AC such that $BG \perp AC$. F is the point on BC such that $AF \perp BC$.
AF and BG intersect at E.
AF produced meets the circle at D.

USE THE SUPPLIED SHEET TO ANSWER THE FOLLOWING:

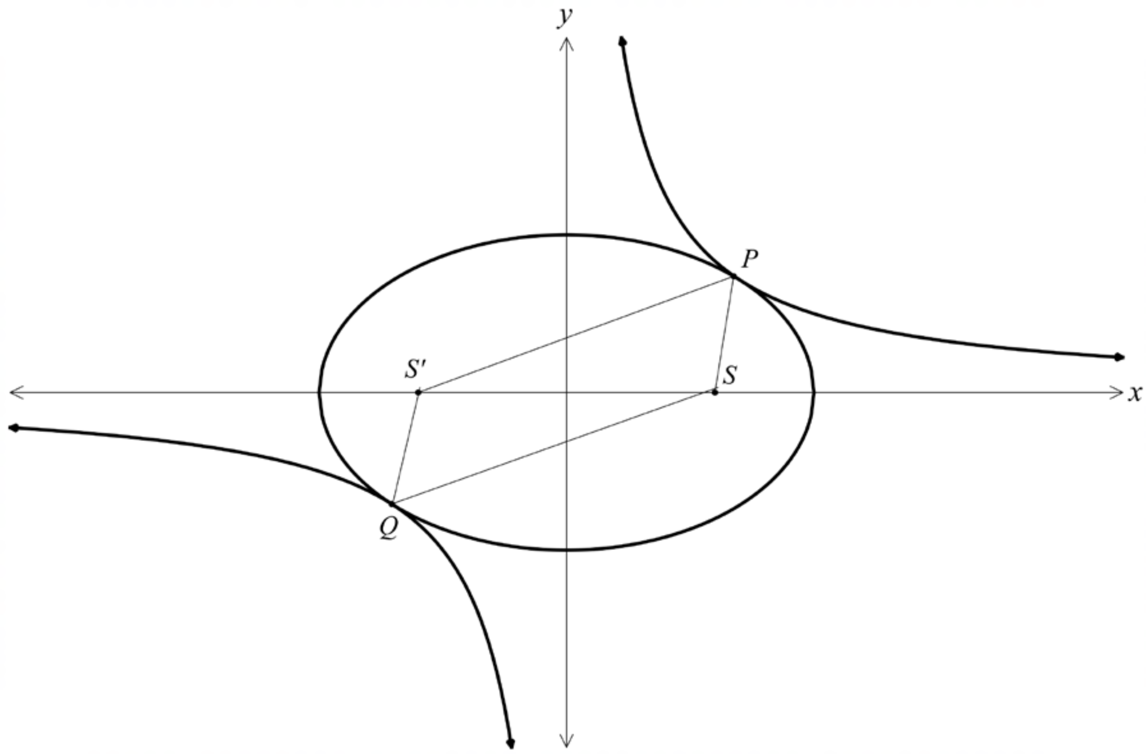
i. Explain why $ABFG$ is a cyclic quadrilateral. 1

ii. Show that $DF=EF$. 3

End of Question 13

Question 14. – Start New Booklet **15 marks**

a) The ellipse E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a > b > 0$) has area $A = \pi ab$, eccentricity e and foci S and S'. The ellipse E touches the hyperbola H: $xy = \frac{1}{2}ab$ at the points P and Q.



i. Find the coordinates of P in terms of a and b. 2

ii. Show that the ratio of the area of the quadrilateral $PSQS'$ to the area of the ellipse E is $e\sqrt{2} : \pi$ 2

b) Find $\int \frac{x^2+6}{x^2+x-6}\,dx$ 3

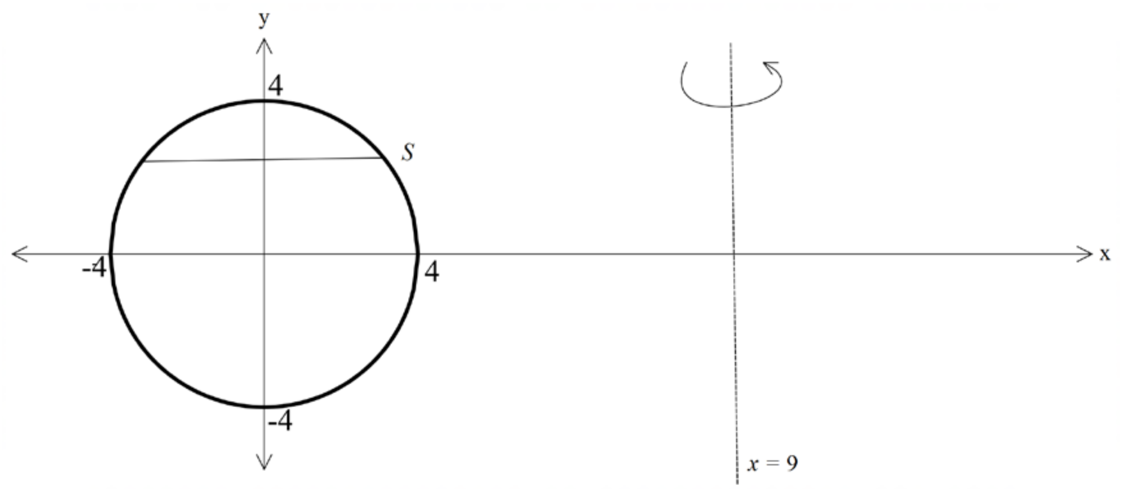
c) Using the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, or otherwise, evaluate 3

$$\int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\cos x-\sin x}\,dx$$

Question 14 continues next page.

Question 14 continued

d) The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a torus.
When the circle is rotated, the line segment S at height y sweeps out an annulus.



i. Show that the area A of the annulus is given by $A = 36\pi\sqrt{16 - y^2}$ **3**

ii. Find the volume of the torus in exact form. **2**

End of Question 14

Question 15. – Start New Booklet **15 marks**

a)

i. Let ω be a complex root of the equation $x^3 = 1$. Show that the other complex root is ω^2 1

ii. Show that $1 + \omega + \omega^2 = 0$ 1

iii. Find the monic cubic equation for which the roots are $\alpha + \beta$, $\alpha\omega + \beta w^2$ and $\alpha w^2 + \beta\omega$ where α, β are real numbers. 3

b) Using the substitution $t = \tan\left(\frac{x}{4}\right)$, or otherwise, evaluate 4

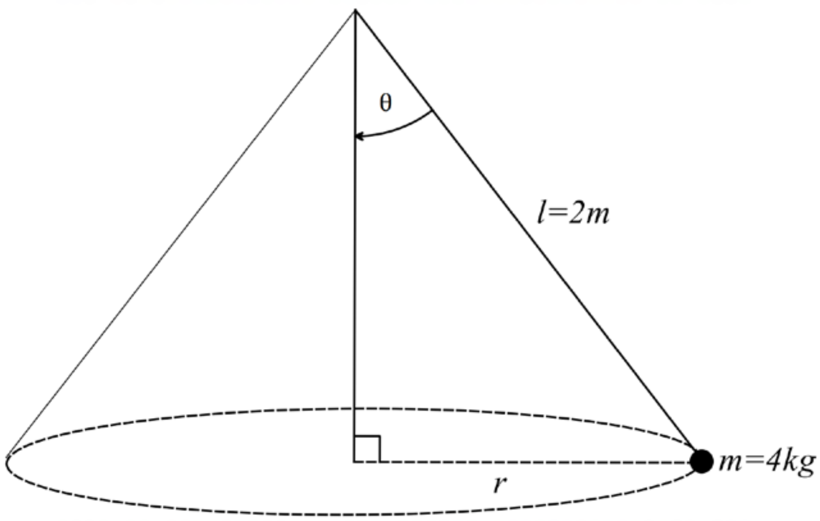
$$\int_0^{\pi} \frac{1}{1 + \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}\, dx$$

State your answer in simplest exact form.

Question 15 continues next page.

Question 15 continued.

c)



A particle of mass 4kg is attached to a string 2 metres in length.
The particle and string revolve as a conical pendulum.
The constant speed of the particle is $v = \sqrt{g}\ ms^{-1}$, where $g\ ms^{-2}$ is the acceleration due to gravity.

Let θ be the angle of inclination of the string to the vertical, and let r metres be the radius of the horizontal circle in which the particle is revolving, and let T newtons be the tension in the string.

i. Show that $\tan\theta = \dfrac{1}{r}$ 2

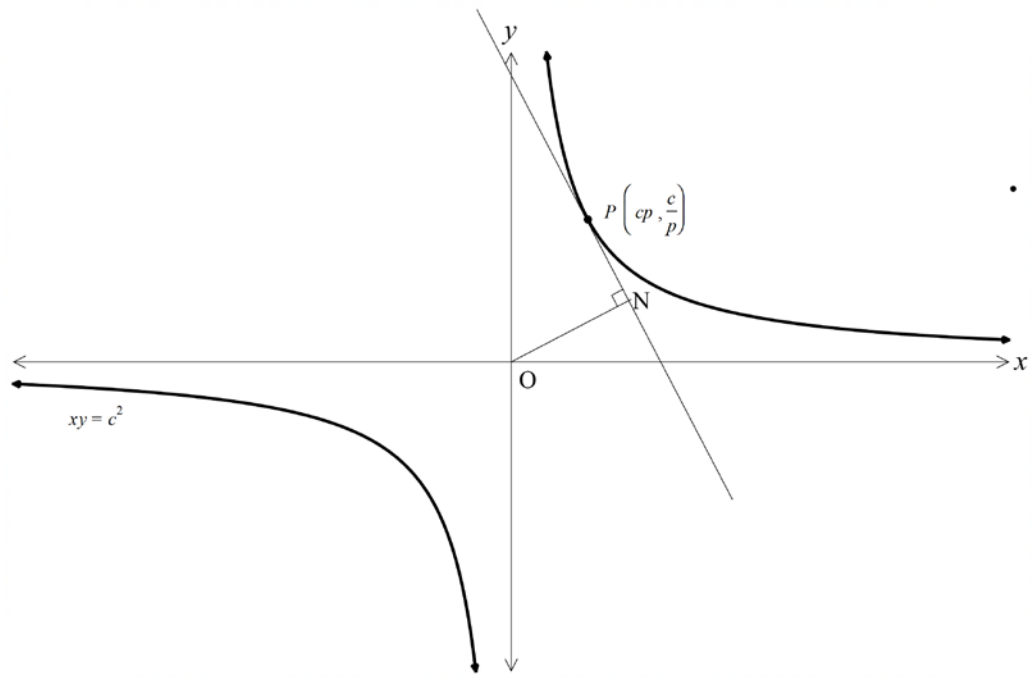
ii. Hence show that $\cos\theta = \dfrac{\sqrt{17}-1}{4}$ 3

iii. Find the value of T, correct to one decimal place, given $g = 9.8ms^{-2}$. 1

End of Question 15

Question 16. – Start New Booklet **15 marks**

a)



The equation of the tangent to the hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is

$$x + p^2 y = 2cp$$

The point N is the foot of the perpendicular line ON drawn from the origin O to the tangent at P.

(i) Show that the coordinates of N are $\left(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right)$ 2

(ii) Show that the Cartesian equation of the locus of N is $(x^2 + y^2)^2 = 4c^2 xy$ 2

Question 16 continues next page.

Question 16 continued.

b)

i. Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \; dx$ for $n = 0, 1, 2, \ldots.$ 3

Show that $I_n = \dfrac{n-1}{n} I_{n-2}$

ii. Show that $\dfrac{I_{2n}}{I_0} = \dfrac{(2n)!}{2^{2n} \times (n!)^2}$ for $n = 0, 1, 2, \ldots.$ 3

iii. Let $y = f(x)$ be a continuous function over $0 \leq x \leq a$. 2

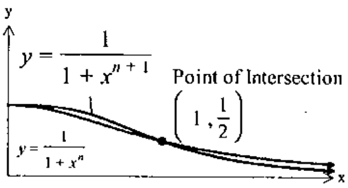
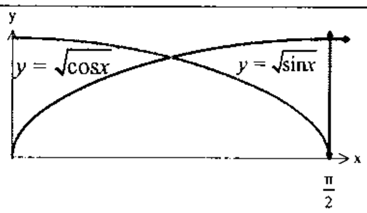
Show that $\displaystyle\int_0^a f(x) \; dx = \int_0^{\frac{a}{2}} \{f(x) + f(a-x)\} \; dx$

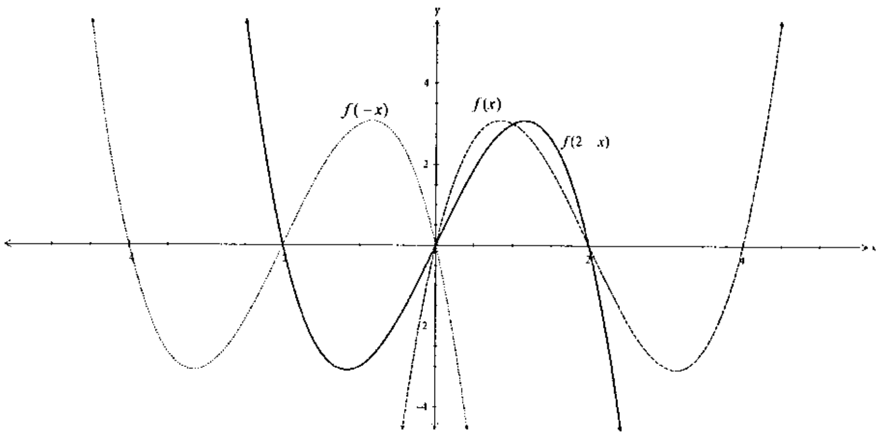
iv. Using parts (i) and (iii) , show that 3

$$\int_0^{\pi} x \cos^6 x \, dx = \frac{5\pi^2}{32}$$

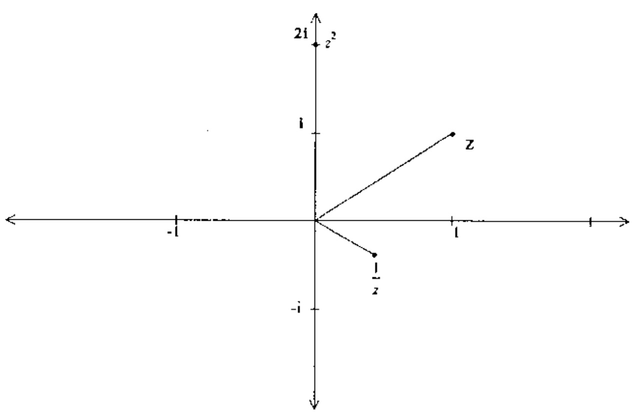
End of Examination

MULTIPLE CHOICE

1	$x^2 - 4y^2 = 4$ $\frac{x^2}{4} - y^2 = 1$ Distance $= 2\frac{a}{e}$ $a = 2\ b = 1$ $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{5}{4}}$ $= \frac{\sqrt{5}}{2}$ $D = 2 \times \frac{2}{\left(\frac{\sqrt{5}}{2}\right)}$ $= \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}$	D Check.
2		B
3		A
4		B
5		C
6		A
7	  Both are true.	A
8	$xy = c^2 = 8$ $x^2 - y^2 = a^2$ $c^2 = \frac{1}{2}a^2 = 8$ $\therefore\ a^2 = 16$ $a = 4$	B

9		A
10	$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$ $= -b + i(2\overrightarrow{AB})$ $= -b + 2i(a + ib - (-b + 0i))$ $= -b + 2i(a + b) + 2i^2 b$ $= -3b + 2i(a + b)$	D

QUESTION 11

a	$z = 1 + i$ $z^2 = 1 + 2i + i^2$ $= 2\mathbf{i}$ $\frac{1}{z} = \frac{1}{1+i} \times \frac{1-i}{1-i}$ $= \frac{1-\mathbf{i}}{2}$ 	3 marks- correct solution for each
bi	$\|z_1 - z_2\|^2 + \|z_1 + z_2\|^2$ $= (z_1 - z_2)\overline{(z_1 - z_2)} + (z_1 + z_2)\overline{(z_1 + z_2)}$ $= (z_1 - z_2)\left(\overline{(z_1)} - \overline{(z_2)}\right) + (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right)$ $= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$ $= \|z_1\|^2 + \|z_2\|^2 + \|z_1\|^2 + \|z_2\|^2$ $= 2\|z_1\|^2 + 2\|z_2\|^2$	2 marks- correct solution 1 mark- partial correct expansion and simplification of RHS/LHS

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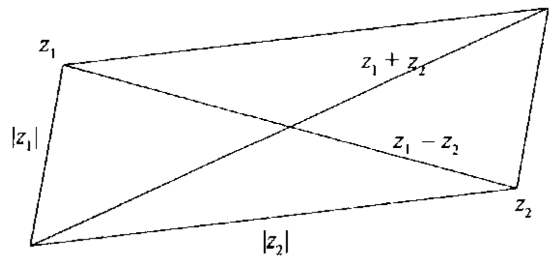
<td>


2 x magnitudes of sides = sum squares of magnitudes of diagonals
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<td>1 mark- correct explanation</td>
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<tr>
<td>c</td>
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$$x^3 - 6xy + y^3 = 5$$

$$\frac{d}{dx}\{x^3 - 6xy + y^3\} = \frac{d}{dx}\{5\}$$

$$3x^2 + 3y^2\frac{dy}{dx} - 6\left\{\frac{xdy}{dx} + y\right\} = 0$$

$$x^2 + y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y = 0$$

$$\frac{dy}{dx}\{y^2 - 2x\} = 2y - x^2$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2} - 2x$$

at $(1. -2)$

$$m_1 = \frac{-4 - 1}{(-2)^2 - 2(1)}$$

$$= -\frac{5}{2}$$

$$m_2 = \frac{2}{5}$$

$$y + 2 + \frac{2}{5}(x - 1)$$

$$2x - 5y - 12 = 0$$

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<td>3 marks- correct solution

2 marks- partial correct with only one error

1 mark- correct application to eqn of line but incorrect diff and gradient</td>
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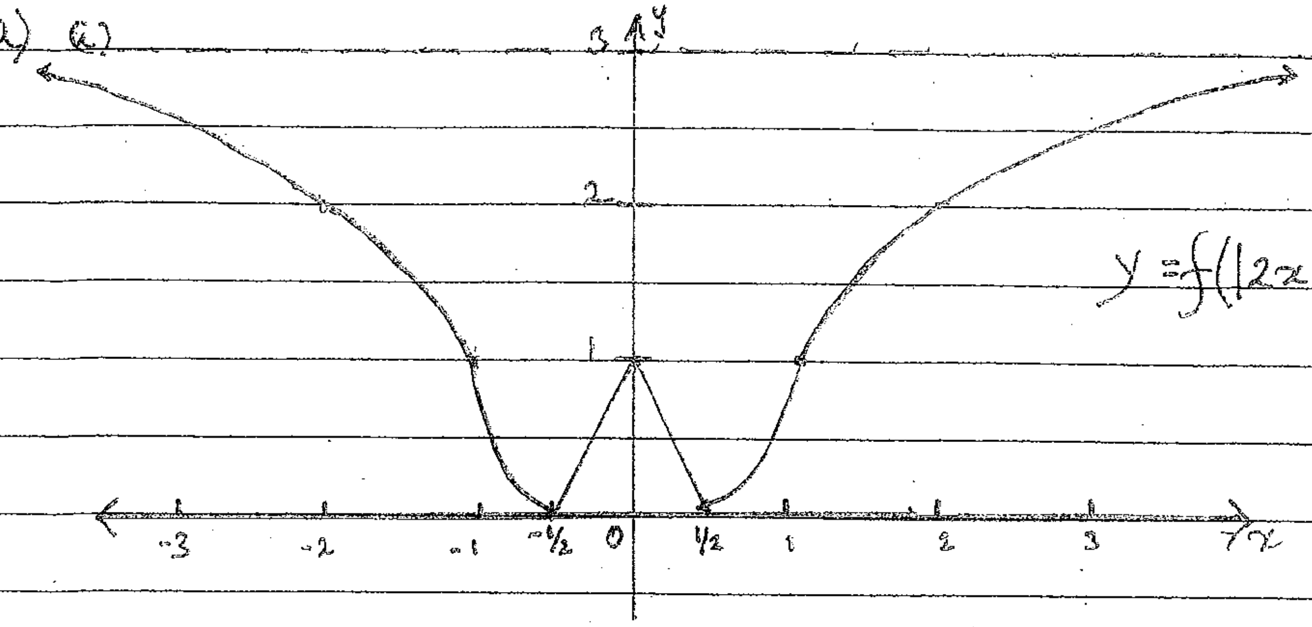
d	$\int \frac{\sin^3 x}{\cos^2 x}$ $= \int \sin x \left\{ \frac{1-\cos^2 x}{\cos^2 x} \right\} dx$ $= \int \sin x \cos^{-2} x \; dx - \int \frac{\sin x \cos^2 x}{\cos^2 x} dx$ $= \sin x (\cos x)^{-2} dx - \int \sin x \; dx$ $= -\int (-\sin x)(\cos x)^{-2} \; dx - \int \sin x \; dx$ $= -\frac{(\cos x)^{-1}}{-1} - (-\cos x)$ $= \frac{1}{\cos x} + \cos x$ $= \sec x + \cos x$	3 marks- correct solution 2 marks- partial correct with only one error 1 mark- one correct technique in relevant progress to int
e	$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $S(ae, 0) \Rightarrow S(2\sqrt{13}, 0) \Rightarrow ae = 2\sqrt{13}$ $y = \frac{b}{a}x \Rightarrow y = \frac{2}{3}x \Rightarrow \frac{b}{a} = \frac{2}{3} \Rightarrow \frac{b^2}{a^2} = \frac{4}{9}$ $a^2 e^2 = 4 \times 13$ $a^2 \left[1 + \frac{b^2}{a^2} \right] = 52$ $a^2 \left[1 + \frac{4}{9} \right] = 52$ $a^2 = 36$ $a = 6$ $\frac{b}{6} = \frac{2}{3}$ $b = 4$	3 marks- correct solution 2 marks- partial correct only one error in correct progress 1 mark- correct solution for values of a, b and e

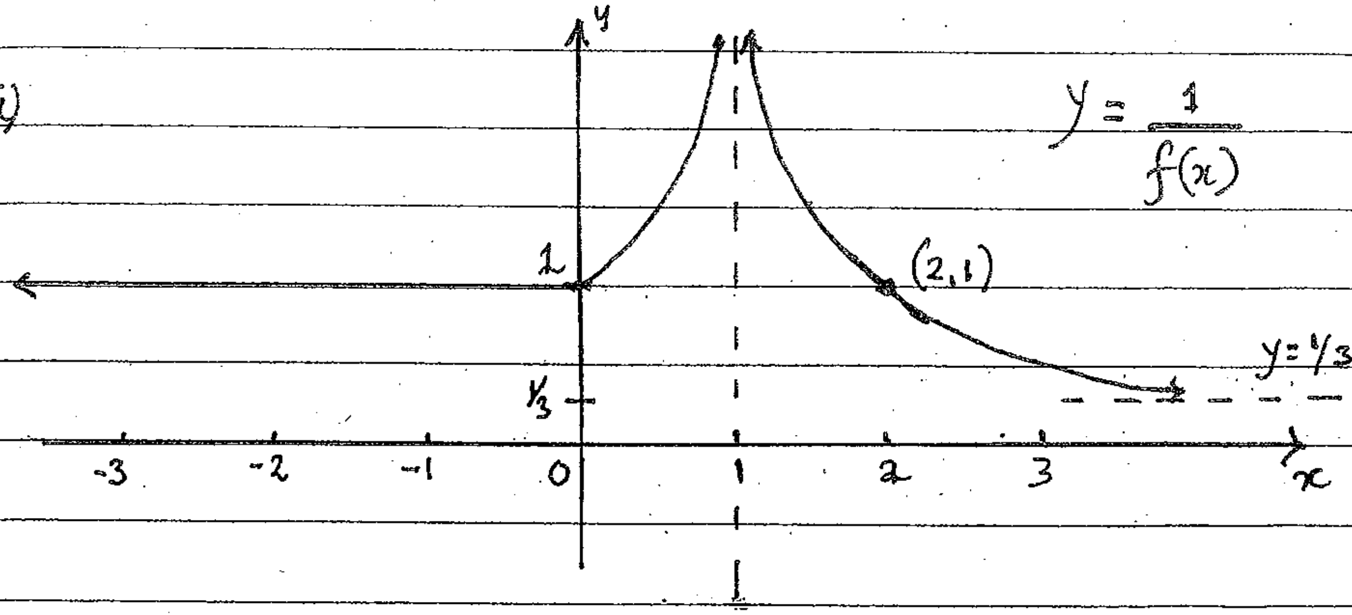
Question 12

a) (i)



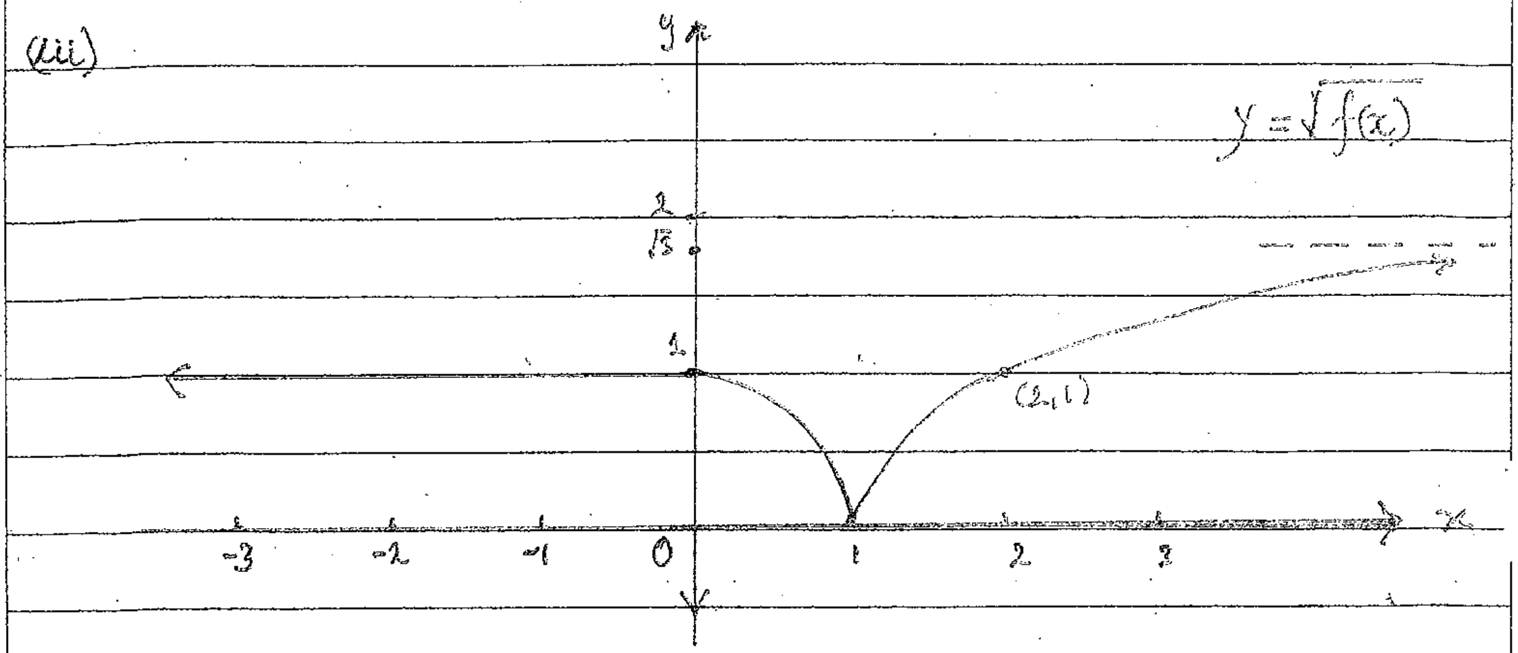
2 Marks: correct answer
1 Mark: correct shape

(ii)



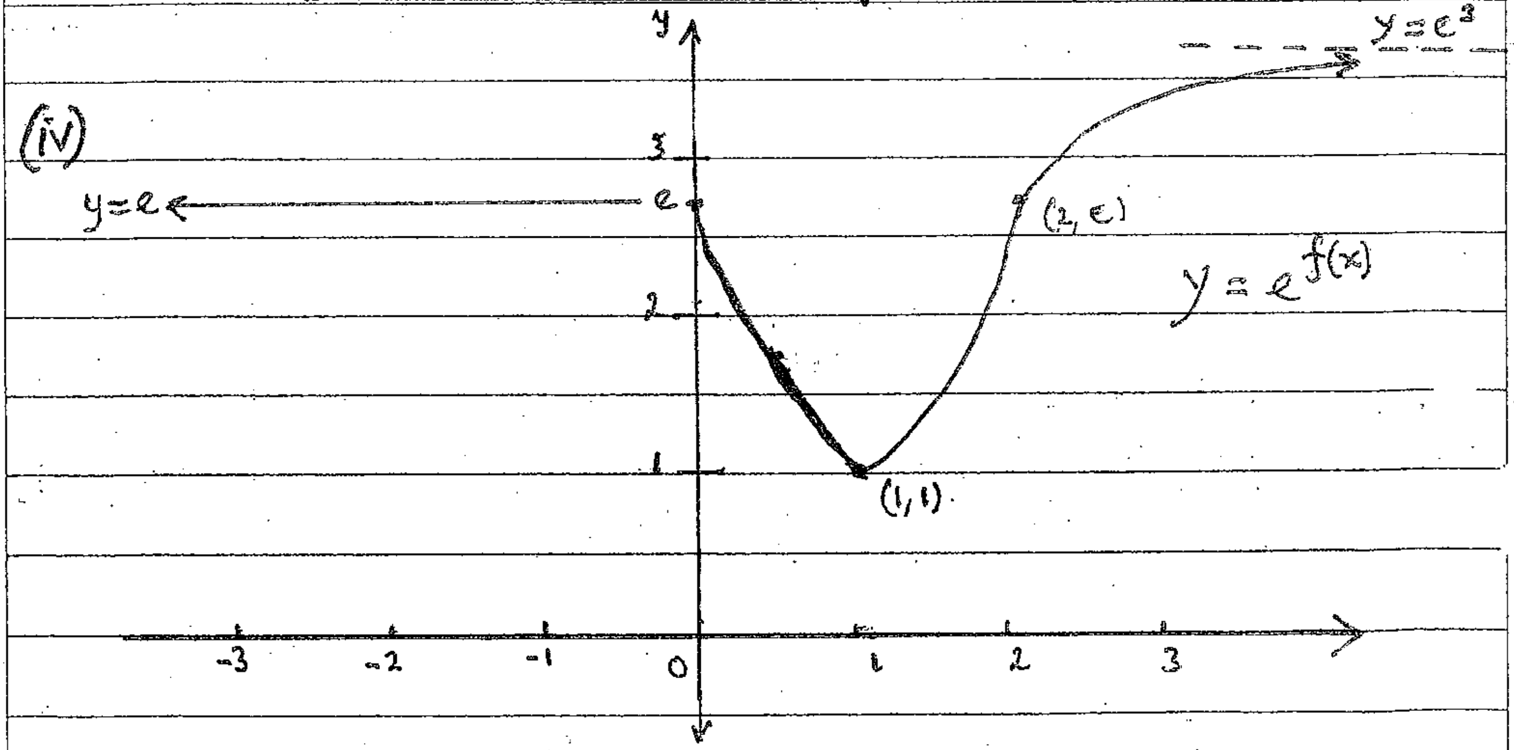
2 Marks: correct answer
1 Mark: correct shape

(iii)



2 Marks: correct answer
1 Mark: correct shape

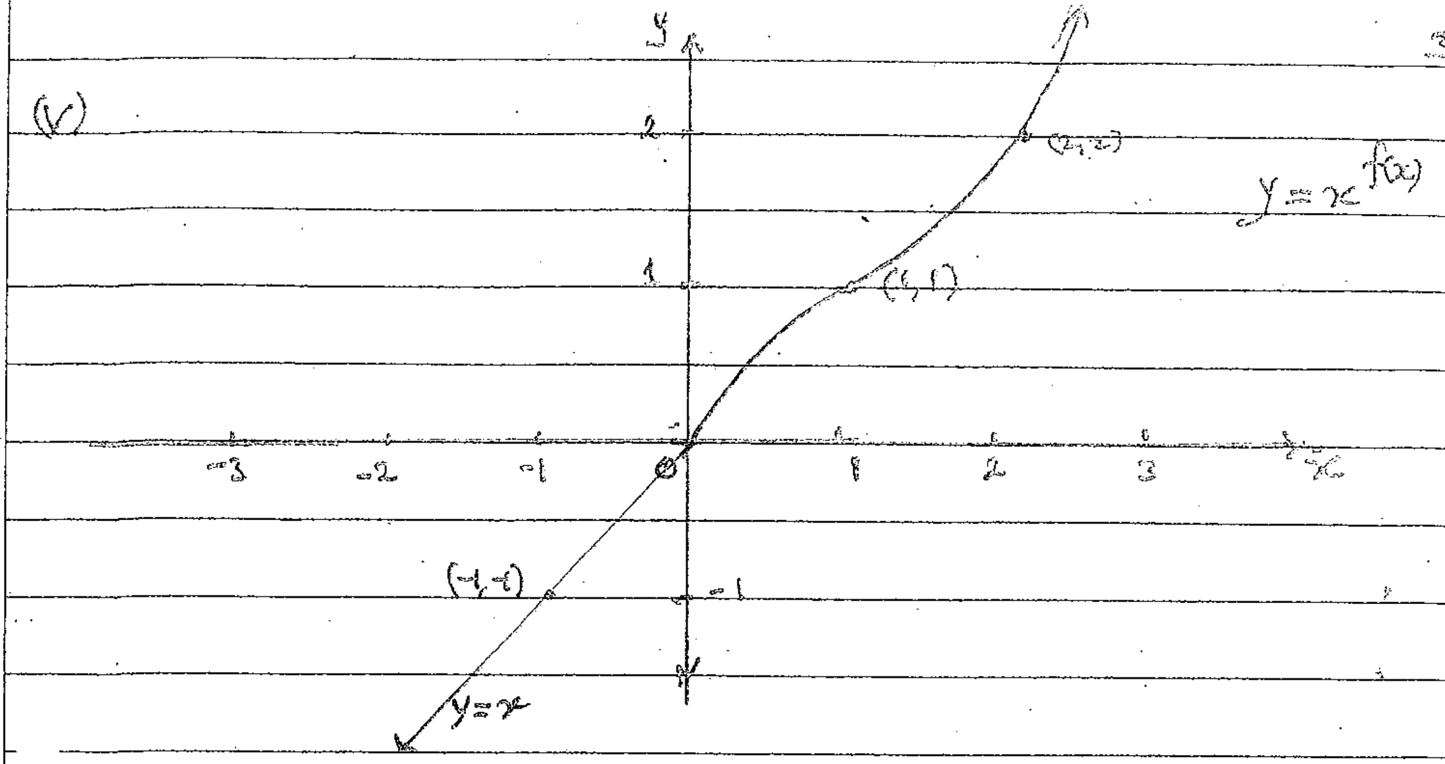
(iv)



2 Marks: correct answer
1 Mark: correct shape

3

(v)



3 Marks: correct answer
2 Marks: two correct parts
1 Mark: one correct part

(b) (i) $z = x + iy = \sqrt{-8+6i}$

$x^2 - y^2 + i(2xy) = -8 + 6i$

$x^2 - y^2 = -8 \quad xy = 3 \Rightarrow y = 3/x$

$x^2 - (3/x)^2 = -8$

$\Rightarrow \quad x^4 + 8x^2 - 9 = 0$

$\cancel{(x^2+9)}(x^2-1) = 0$

$x = \pm 1 \Rightarrow y = \pm 3$

$z = \pm 1 \pm 3i$

$\therefore$

$\sqrt{-8+6i} = 1 + 3i$

2 Marks: correct solution (other methods possible)
1 Mark: significant progress depending on method

(ii) $$2z^2 - (3+i)z + 2 = 0$$

$$z = \frac{(3+i) \pm \sqrt{(3+i)^2 - 4 \times 2 \times 2}}{2 \times 2}$$

$$z = \frac{(3+i) \pm \sqrt{-8+6i}}{4}$$

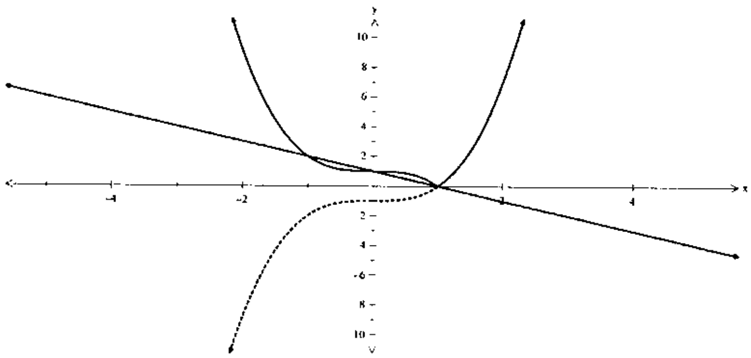
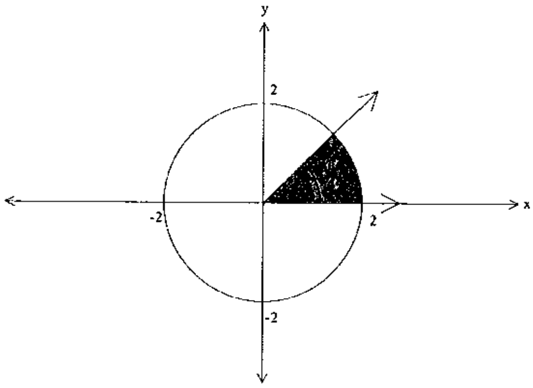
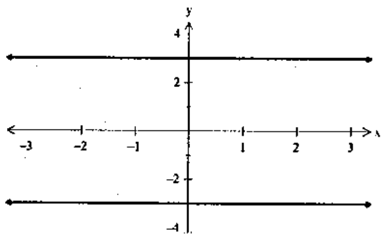
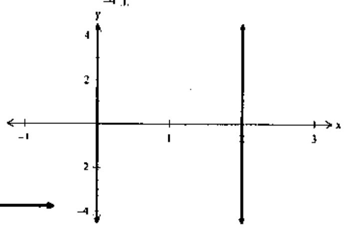
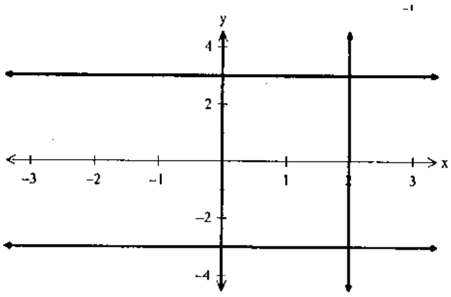
$$z_1 = \frac{3+i-(1+3i)}{4} \qquad z_2 = \frac{3+i+1+3i}{4}$$

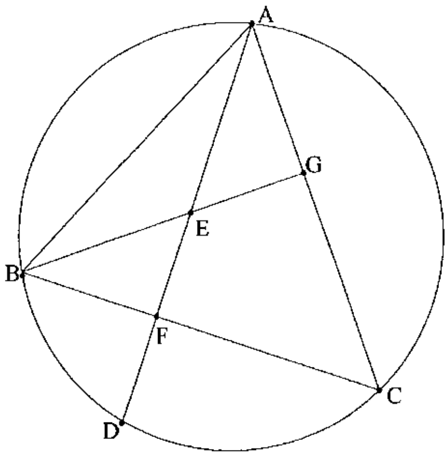
$$= \frac{2-2i}{4} \qquad = \frac{4+4i}{4}$$

$$z_1 = \frac{1-i}{2} \qquad z_2 = 1+i$$

2 Marks: correct answer
1 Mark: one correct value, z_1 or z_2

QUESTION 13

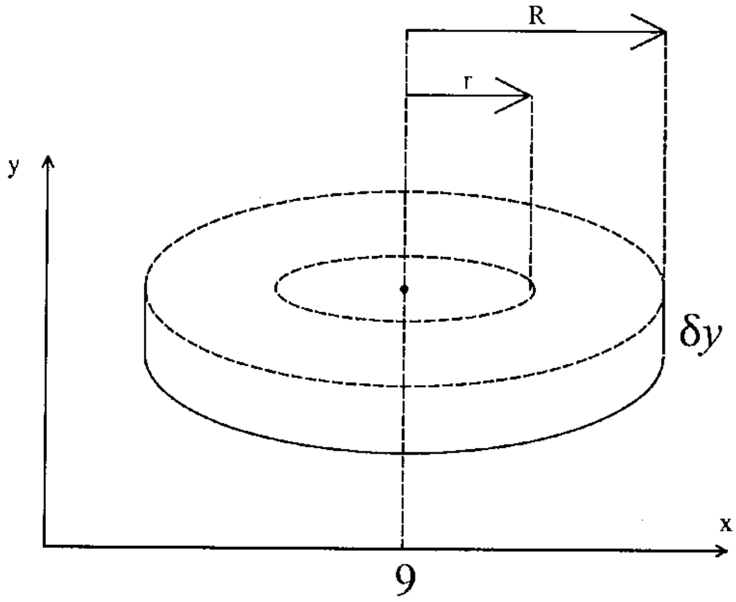
ai		1 mark- correct solution
aii	$-(x^3-1)=1-x$ $1-x^3=1-x$ $(1-x)(1+x+x^2)=1-x$ $1+x+x^2=1 \quad ,x \neq 1$ $x(x+1)=0$ $x=0\ ,\ -1$ $\therefore \quad \lvert x^3-1\rvert < 1-x$ $-1<x<0$	2 marks- correct solution 1 mark- correct solution from incorrect graph or only partial correct region (only one error)
bi		2 marks- correct solution 1 mark- partial correct region (only one error)
bii	$\lvert z-\bar{z}\rvert \leq 6$ $\lvert x+iy-(x-iy)\rvert \leq 6$ $\lvert 2iy\rvert \leq 6$ $\lvert 2y\rvert \leq 6$ $\lvert y\rvert \leq 3$ $0 \leq Re(2z) \leq 4$ $0 \leq 2x \leq 4$ $0 \leq x \leq 2$   	2 marks- correct solution 1 mark- partial correct region (only one error)

c	$P(x) = 8x^4 + 44x^3 + 54x^2 + 25x + 4$ $P'(x) = 32x^3 + 132x^2 + 108x + 25$ $P''(x) = 96x^2 + 264x + 108$ $\therefore 24x^2 + 66x + 27 = 0$ $x = \dfrac{-66\left(\pm\sqrt{66^2 - (4 \times 24 \times 27)}\right)}{48}$ $= -\dfrac{9}{4}, -\dfrac{1}{2}$ $P'\left(-\dfrac{9}{4}\right) = \dfrac{343}{4}$ $P'\left(-\dfrac{1}{2}\right) = 0$ $\therefore \quad x = -\dfrac{1}{2} \Rightarrow 2x + 1 = 0$ $P(x) = (2x + 1)^3(\alpha x + \beta)$ $\alpha = 1, \beta = 4$ $P(x) = (2x = 1)^3(x + 4)$ $x = -\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}, -4$ $P(-4) = 0$	4 marks- correct solution 3 marks- correct solution from incorrect triple root 2 marks- partial correct with solution to triple root and check 1 mark- only correctly solves for values of x on $P''(x)$
di	 $\angle AGB = 90°$ (given) $\angle AFB = 90°(given)$ $\therefore \angle AFB = \angle AGB$ $\therefore ABFG$ is cyclic (= $\angle$ on chord AB)	1 mark- correct solution with correct reasons
dii	join AB $\angle DAC = \angle DBC = x$ (= $\angle$ on "arc CD circle ABDC) ang FAG=ang FBG = x (= ang on "arc FG crcle $ABFG$) $\therefore \angle DBF = \angle EBF = x$ ΔDBE is isos ($\perp$ to Δ bisects $\angle DBE$) DE=DF ($\perp$ from apex to base bisects "DE") OR $\Delta DBF \equiv \Delta EBF\ (AAS)$ DE=DF ($corr$ sides in congruent Δ)	3 marks- correct solution with correct reasons 2 marks- only one error at least two correct and relevant theorems 1 mark- one correct and relevant theorem

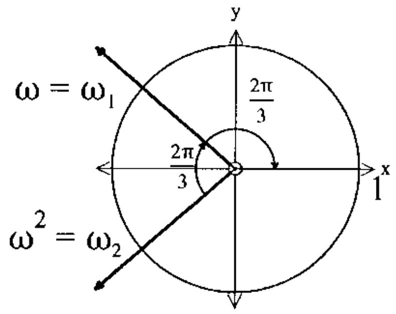
QUESTION 14

a-i	$xy = \frac{1}{2}\text{ab}$ $y = \frac{ab}{2x} \Rightarrow y^2 = \frac{a^2b^2}{4x^2}$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{x^2}{a^2} + \frac{1}{b^2} \times \frac{a^2b^2}{4x^2} = 1$ $\frac{x^2}{a^2} + \frac{a^2}{4x^2} = 1$ $4x^4 + a^4 = 4x^2a^2$ $4x^4 - 4x^2a^2 + a^4 = 0$ $(2x^2 - a^2)^2 = 0$ $x^2 = \frac{a^2}{2}$ $x = \pm\frac{a}{\sqrt{2}}$ $x = \frac{a}{\sqrt{2}} \Rightarrow y = \frac{ab}{2} \times \frac{\sqrt{2}}{a}$ $P = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ $Q = \left(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right)$	2 marks
a-ii	distance $S\,S' = 2ae$ Area of $\Delta\ S'SP = \frac{1}{2} \times 2ae \times \frac{b}{\sqrt{2}}$ $= \frac{a\,b\,e}{\sqrt{2}}$ $\therefore$ quad $QS'PS$ $= 2 \times \frac{a\,b\,e}{\sqrt{2}}$ $= \sqrt{2}\,a\,b\,e$ $\frac{Area\ QS'\ PS}{Area\ E} = \frac{\sqrt{2}\,a\,b\,e}{\pi ab} = \frac{\sqrt{2}\,e}{\pi}$ $\therefore$ Ratio $\Rightarrow \sqrt{2}\,e : \pi$	2 marks – correct solution for general case 1 mark - Correct solution for specific solution only - finding a ratio in e only

b	$\int \frac{x^2+6}{x^2+x-6}\,dx$ $\frac{x^2+6}{x^2+x-6} = \frac{x^2+x-6-x+12}{x^2+x-6}$ $= 1 - \frac{x-12}{x^2+x-6}$ $\therefore \frac{x-12}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$ $x - 12 = A(x-2) + B(x+3)$ Let $x = 2$ $^{-}10 = 5B \Rightarrow B = {}^{-}2$ Let $x = {}^{-}3$ $^{-}15 = {}^{-}5A \Rightarrow A = 3$ $\int \frac{x^2+6}{x^2+x-6}$ $= \int 1 - \left[\frac{3}{x+3} - \frac{2}{x-2}\right] dx$ $= x - 3\ln\lvert x+3\rvert + 2\ln\lvert x-2\rvert$ $= x + \ln\left\lvert \frac{(x-2)^2}{(x+3)^3} \right\rvert + C$	3 marks – correct solution 2 marks – correct use of partial fractions 1 mark – correct rearrangement of initial algebraic fraction
c	$\int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right) + \sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$ $\therefore 2 \times \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\cos x - \sin x}\,dx$ $2I = 0$ $\therefore \quad I = 0$	3 marks – correct solution fully demonstrated. 2 marks – use of result but failure to achieve the correct solution and/or insufficient demonstration 1 mark – use fo correct trig identify to simplify. Nb question did require the use of the specified result

	 $x^2 + y^2 = 16$ $x = \pm\sqrt{16 - y^2}$ $\therefore \quad R = 9 + \sqrt{16 - y^2}$ $r = 9 - \sqrt{16 - y^2}$ $A = \pi(R^2 - r^2)$ $= \pi(R + r)(R - r)$ $= \pi(18)\left(2\sqrt{16 - y^2}\right)$ $= 36\pi\left(\sqrt{16 - y^2}\right)$	2 marks
	$\delta V = A\ \delta y$ $V = \lim_{\delta y \to 0} \sum_{-4}^{4} A\ \ \delta y$ $= 36\ \pi \int_{-4}^{4} \sqrt{16 - y^2}\ dy$ $= 36\pi\ \times$ semicircle $= 36\pi\ \times \frac{1}{2}\pi\ 4^2$ $= 288\pi^2\ units^3$	3 marks – correct solution 2 marks – 1 mark

QUESTION 15

a-i	$\omega = \text{cis}\left(\frac{2\pi}{3}\right)$ $\omega_2 = \text{cis}\left(\frac{4\pi}{3}\right) = \left(\text{cis}\left(\frac{2\pi}{3}\right)\right)^2 = \omega^2$ or $\omega^3 = 1$ $\therefore (\omega^3)^2 = 1^2 = 1$ $\therefore (\omega^2)^3 = 1$ $\therefore \omega^2$ is a root  	1 mark – correct explanation
a-ii	Roots of the Polynomial are $1, \omega, \omega^2$ $\Sigma\alpha = -\frac{b}{a} = 0$ $\therefore 1 + \omega + \omega^2 = 0$	1 mark – correct explanation
a-iii	$P(x) = x^3 + bx^2 + cx + d$ $\Sigma\alpha = -\frac{b}{a} =$ -b as $a = 1$ $= \alpha + \beta + a\,\omega + b\omega^2 + a\omega^2 + b\omega$ $= \alpha(1 + \omega + \omega^2) + \beta(1 + \omega + \omega^2)$ $= \alpha \times 0 + \beta \times 0 = 0$ $\therefore \quad b = 0$ $\Sigma\alpha\beta = \frac{c}{a} = c$ as $a = 1$ $= (\alpha + \beta)(\alpha\omega + \beta\omega^2) + (\alpha + \beta)(a\omega^2 + \beta\omega) + (\alpha\omega + \beta\omega^2)(\alpha\omega^2 + \beta\omega)$ $= (\alpha^2\omega + \alpha\beta\omega + \alpha\beta\omega^2 + \beta^2\omega^2) + (\alpha^2\omega^2 + \alpha\beta\omega + \alpha\beta\omega^2 + \beta^2\omega)$ $+ (\alpha^2\omega^3 + \alpha\beta\omega^2 + \alpha\beta\omega^4 + \beta^2\omega^3)$ $= \alpha^2(\omega + \omega^2 + \omega^3) + \beta^2(\omega + \omega^2 + \omega^3) + 3\alpha\beta(\omega + \omega^2)$* *$(\alpha\beta\omega^4 = \alpha\beta\omega^3\omega = \alpha\beta\omega)$ also $\omega + \omega^2 = {}^{-}1$ $c = 0 + 0 - 3\alpha\beta = {}^{-}3\alpha\beta$	1 mark per correct pronumeral.

$$\Sigma\,\alpha\,\beta\,\gamma = -\frac{d}{a} = -d$$

$$\begin{aligned}
-d &= (\alpha+\beta)(\alpha\omega+\beta\omega^2)(\alpha\omega^2+\beta\omega)\\
&= (\alpha+\beta)(\alpha^2\omega^3+\alpha\beta\omega^4+\alpha\beta\omega^2+\beta^2\omega^3)\\
&= (\alpha+\beta)(\alpha^2+\beta^2+\alpha\beta(\omega^4+\omega^2))\\
&= (\alpha+\beta)(\alpha^2+\beta^2+\alpha\beta(\omega+\omega^2))\\
&= (\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)\\
&= (\alpha^3+\beta^3)
\end{aligned}$$

$$\begin{aligned}
\therefore\quad P(x) &= x^3+0x^2-3\alpha\beta x-(\alpha^3+\beta^3)\\
&= x^3-3\alpha\beta x-(\alpha^3+\beta^3)
\end{aligned}$$

---

b

$$\tan\left(\frac{x}{4}\right)=t \qquad x=0 \Rightarrow t=\tan 0=0 \qquad t=\tan\left(\frac{x}{4}\right)$$

$$\cos\left(\frac{x}{2}\right)=\frac{1-t^2}{1+t^2} \qquad x=\pi \Rightarrow t=\tan\frac{\pi}{4}=1 \qquad x=4\tan^{-1}(t)$$

$$dx=\frac{4}{1+t^2}\,\mathrm{dt}$$

$$\sin\left(\frac{x}{2}\right)=\frac{2t}{1+t^2}$$

$$\int_0^1 \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\cdot\frac{4}{1+t^2}\,\mathrm{dt}$$

$$=\int_0^1 \frac{1+t^2}{1+t^2+1-t^2+2t}\times\frac{4}{1+t^2}\,\mathrm{dt}$$

$$=\int_0^1 \frac{4}{2+2t}\,\mathrm{dt}$$

$$=2\int_0^1 \frac{1}{1+t}\,\mathrm{dt}$$

$$=2\,[\ln(1+t)]_0^1$$

$$=2\{\ln 2-\ln 1\}$$

$$=2\ln 2 = \ln 4$$

4 marks – correct solution

3 marks

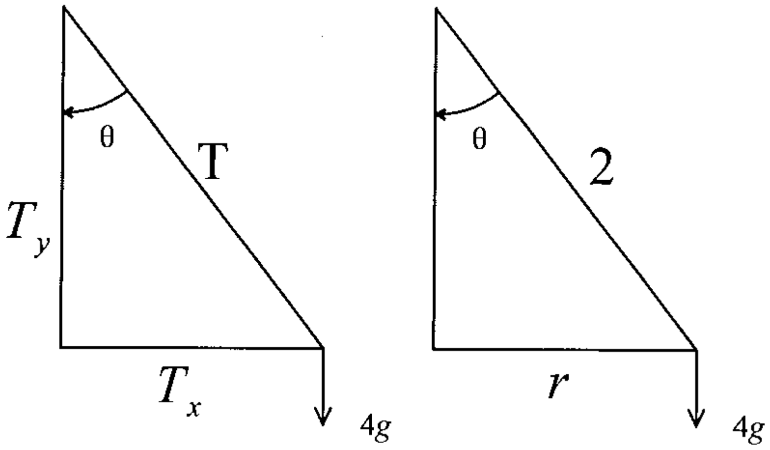
- incorrect final simplification
- one error in final integration process

2

- correct t substitution and limits

1

- correct t substitution or new limts

<table>
<tr>
<td>c-i</td>
<td>


$$T_y = T\cos\theta = 4g$$

$$T_x = T\sin\theta = \frac{mv^2}{r} = \frac{4(\sqrt{g})^2}{r} = \frac{4g}{r}$$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{4g}{r} \div 4g = \frac{1}{r}$$
</td>
<td>2 – fully demonstratetd

1 mark – limited demonstration</td>
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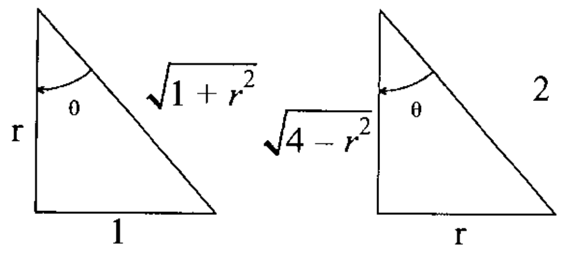
</table>

c-ii



$$\cos\theta = \frac{r}{\sqrt{1+r^2}} \text{ also } \cos\theta = \frac{\sqrt{4-r^2}}{2}$$

$$\therefore \quad \frac{r}{\sqrt{1+r^2}} = \frac{\sqrt{4-r^2}}{2}$$

$$\frac{r^2}{1+r^2} = \frac{4-r^2}{4}$$

$$4r^2 = (1+r^2)(4-r^2)$$

$$4r^2 = 4 + 4r^2 - r^2 - 4r^4$$

$$4r^4 + r^2 - 4 = 0$$

$$r^2 = \frac{^-1 \pm \sqrt{17}}{2}$$

but $r^2 \geq 0$

$$r = \frac{^-1 + \sqrt{17}}{2}$$

$$\cos\theta = \frac{r}{\sqrt{1+r^2}}$$

$$\cos^2\theta = \frac{r^2}{1+r^2}$$

$$= \frac{\frac{\sqrt{17}-1}{2}}{1 + \frac{\sqrt{17}-1}{2}}$$

$$= \frac{\sqrt{17}-1}{\sqrt{17}+1}$$

$$= \frac{(\sqrt{17}-1)^2}{17-1}$$

$$\therefore \cos\theta = \sqrt{\frac{(\sqrt{17}-1)^2}{16}}$$

$$= \frac{\sqrt{17}-1}{4}$$

4marks – correct solution fully demonstrated.

3 marks

- derivation of positive r with reasons

2 marks – 2 correct expressions for cosθ and production of quadratic or similar

1 marks – 2 correct expressions for cosθ

c-iii	$T\cos\theta = 4g$ $T\left[\dfrac{\sqrt{17}-1}{4}\right] = 4g$ $T = \dfrac{16 \times 9{\cdot}8}{\sqrt{17}-1}$ $= 50{\cdot}206\,..$ $\cong\ 50{\cdot}2\,N$	1 mark – correct answer

Question 16

(a) (i) tangent at P: $x + p^2 y = 2cp$

$$\Rightarrow \quad y = \frac{-1}{p^2}x + \frac{2c}{p}$$

$$m_1 = -1/p^2$$

$$\Rightarrow \text{ gradient of ON}, \; m_2 = p^2$$

equation of ON: $y = p^2 x$

$$\Rightarrow \quad x + p^2(p^2 x) = 2cp$$

$$x(1+p^4) = 2cp$$

$$x = \frac{2cp}{1+p^4}$$

$$\Rightarrow \quad y = p^2\left(\frac{2cp}{1+p^4}\right)$$

$$y = \frac{2cp^3}{1+p^4}$$

$$\therefore N \text{ is } \left\{\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right\}$$

2 Marks: correct answer
1 Mark: correct derivation of either x or y coordinate.

(ii) $x = \frac{2cp}{1+p^4} \Rightarrow x(1+p^4) = 2cp \Rightarrow x^2(1+p^4)^2 = 4c^2p^2$ (1)

$y = p^2 x \Rightarrow p^2 = y/x \Rightarrow p^4 = y^2/x^2$ (2)

Substitute (2) into (1):

$$x^2\left(1+\left(\frac{y}{x}\right)^2\right)^2 = 4c^2\left(\frac{y}{x}\right)$$

$$x^2\left(\frac{x^2+y^2}{x^2}\right)^2 = \frac{4c^2y}{x}$$

$$x^2\left(\frac{[x^2+y^2]^2}{x^4}\right) = \frac{4c^2y}{x}$$

$$\frac{[x^2+y^2]^2}{x^2} = \frac{4c^2y}{x}$$

$\times x^2$

$$[x^2+y^2]^2 = 4c^2xy$$

2 Marks: correct answer

1 Mark: significant progress beyond elimination of p.

b) i) $I_n = \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \cos^{n-1} x \cdot \cos x \, dx$

$u = \cos^{n-1} x \qquad dv = \cos x$

$du = (n-1)\cos^{n-2} x \cdot -\sin x \qquad v = \sin x$

$= -(n-1)\sin x \cos^{n-2} x$ ①

$I_n = \left[\sin x \cos^{n-1} x\right]_0^{\pi/2} - \int_0^{\pi/2} -(n-1)\cos^{n-2} x \sin x \cdot \sin x \, dx$

$= \left[0 - 0\right] + (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx$ ②

$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \left\{1 - \cos^2 x\right\} dx$

$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \, dx - (n-1)\int_0^{\pi/2} \cos^n x \, dx$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$\underline{I_n = \frac{(n-1)}{n} I_{n-2}}$

3 Marks: correct answer
2 Marks: significant progress up to and including ② and beyond
1 Mark: correctly applies IBP to obtain ① and makes additional relevant progress.

ii) $$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

$$\Rightarrow \frac{I_{2n}}{I_0} = \frac{I_{2n}}{1} \times \frac{1}{I_0}$$

$$= \frac{I_{2n}}{I_{2n-2}} \times \frac{I_{2n-2}}{I_{2n-4}} \times \frac{I_{2n-4}}{I_{2n-6}} \times \cdots \times \frac{I_4}{I_2} \times \frac{I_2}{I_0}$$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \quad (1)$$

$$= \left(\frac{2n}{2n}\right) \times \left(\frac{2n-1}{2n}\right) \times \left(\frac{2n-2}{2n-2}\right) \times \left(\frac{2n-3}{2n-2}\right) \times \left(\frac{2n-4}{2n-4}\right) \times \left(\frac{2n-5}{2n-4}\right) \times \cdots \times \left(\frac{2}{2}\right)$$

$$= \frac{2n \times (2n-1) \times (2n-2) \times (2n-3) \times \cdots \times 3 \times 2 \times 1}{(2n)(2n) . 2(n-1) . 2(n-1) . 2(n-2) . 2(n-2) \cdots 2(2) . 2(2) . 2(1) \times 2} \quad (2)$$

$$= \frac{(2n)!}{\underbrace{\{2 . 2 . 2 \ldots 2\}}_{2n \text{ times}} . \{n^2 . (n-1)^2 . (n-2)^2 \ldots 2^2 . 1^2\}}$$

$$= \frac{(2n)!}{2^{2n} \{n(n-1)(n-2) \ldots 2 . 1\}^2}$$

$$\frac{I_{2n}}{I_0} = \frac{(2n)!}{2^{2n} (n!)^2}$$

3 Marks : correct answer

2 Marks : obtains a correct version of expression (2)

1 Mark : Obtains expression (1) or equivalent progress

iii) $\displaystyle\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx$

Let $u = a - x \Rightarrow x = a - u$

$\therefore dx = -du \qquad x = 0 \Rightarrow u = a, \quad x = \frac{a}{2} \Rightarrow u = \frac{a}{2}, \quad x = a \Rightarrow u = 0$

$$\int_{a/2}^a f(x)\,dx = \int_{a/2}^0 f(a-u)(-du) \qquad ①$$

$$= -\int_{a/2}^0 f(a-u)\,du$$

$$= \int_0^{a/2} f(a-u)\,du$$

$$= \int_0^{a/2} f(a-x)\,dx$$

$$\therefore \int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

$$= \int_0^{a/2} \{f(x) + f(a-x)\}\,dx$$

2 Marks : correct answer

1 Mark : makes significant progress beyond ①

(iv) $$\int_0^{\pi} x\cos^6 x\,dx = \int_0^{\pi/2}\left\{x\cos^6 x + (\pi - x)\cos^6(\pi - x)\right\}dx$$

$$= \int_0^{\pi/2}\left\{x\cos^6 x + \pi\cos^6(\pi - x) - x\cos^6(\pi - x)\right\}dx$$

$$\boxed{\begin{aligned}\cos(\pi - x) &= \cos\pi\cos x + \sin\pi\sin x\\ &= -1\times\cos x + 0\times\sin x\\ &= -\cos x\end{aligned}}$$

$$= \int_0^{\pi/2}\left\{\cancel{x\cos^6 x} + \pi\cos^6(\pi - x) - \cancel{x\cos^6 x}\right\}dx$$

$$= \pi\int_0^{\pi/2}\cos^6 x\,dx$$

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2} \Rightarrow I_6 = \pi\left[\frac{5}{6}I_4\right]$$

$$= \pi\left[\frac{5}{6}\right]\left[\frac{3}{4}I_2\right]$$

$$= \pi\left[\frac{5}{6}\right]\left[\frac{3}{4}\right]\left[\frac{1}{2}\right]I_0$$

$$I_0 = \int_0^{\pi/2}(\cos x)^0\,dx = \int_0^{\pi/2}1\,dx = \Big[x\Big]_0^{\pi/2} = \pi/2$$

$$\underline{\underline{\int_0^{\pi} x\cos^6 x\,dx = \pi\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\cdot\pi/2 = 5\pi^2/32}}$$

3 Marks: correct answer from specified method
2 Marks: Obtains 1 mark AND determines the sequence $5/6, 3/4, 1/2$
1 Mark: evaluates I_0 or obtains $\pi\int^{\pi/2}\cos^6 x\,dx$