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Name: ....

2015

TRIAL
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

- **General Instructions**
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I

Pages 3 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II

Pages 7 – 13

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. Evaluate $\int \frac{dx}{x^2-4x+13}$

(A) $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c$

(B) $\frac{2}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c$

(C) $\frac{1}{3}\tan^{-1}\left(\frac{2x-4}{3}\right)+c$

(D) $\frac{2}{3}\tan^{-1}\left(\frac{2x-4}{3}\right)+c$

2. Find the equations of the directrices of the ellipse $\frac{x^2}{4}+y^2=1$

(A) $x=\pm\frac{2}{\sqrt{5}}$

(B) $x=\pm\frac{4}{\sqrt{3}}$

(C) $x=\pm\sqrt{3}$

(D) $x=\pm\frac{\sqrt{5}}{2}$

3. What is the gradient of the curve $xy - x^2 + 3 = 0$ at the point when $x = 1$?

(A) – 4

(B) – 1

(C) 1

(D) 4

4. The region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant is rotated about the y–axis. Which integral could be used to find the volume of the solid of revolution formed?

(A) $V = \pi \int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}}\right) dy$

(B) $V = \pi \int_0^1 \left(y^{\frac{1}{2}} - y^{\frac{1}{3}}\right) dy$

(C) $V = \pi \int_0^1 \left(y^{\frac{2}{3}} - y\right) dy$

(D) $V = \pi \int_0^1 (x^4 - x^6)\ dx$

5. What are the five fifth roots of $1 + \sqrt{3}i$?

(A) $2^{\frac{1}{5}}\, cis \left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

(B) $2^5\, cis \left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

(C) $2^{\frac{1}{5}}\, cis \left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

(D) $2^5\, cis \left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

6. Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x \, dx$

(A) 0

(B) 2

(C) $\frac{\pi}{8}$

(D) $\frac{3\pi}{8}$

7. A, B and C are three consecutive terms in an arithmetic progression.

Which of the following is a simplification of $\frac{\sin(A+C)}{\sin B}$?

(A) $2\cos B$

(B) $\sin 2B$

(C) $\cot B$

(D) 1

8. Consider the graph of the function $x^3 - y^3 = 1$.

Which of the following statements is NOT true?

(A) The graph has a vertical tangent at $x = 1$

(B) The graph has a horizontal tangent at $x = 0$

(C) The line $y = -x$ is an axis of symmetry

(D) There is a least one point $P(a, b)$ on the graph such that $b > a$

9. What is the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$?

(A) $-3i - 2$

(B) $3i - 2$

(C) $3i + 2$

(D) $2 - 3i$

10. Solve the inequality: $\frac{x+1}{x-3} \leq \frac{x+3}{x-2}$.

(A) $x < 2$ and $x > 3$

(B) $x < 2$ and $3 < x \leq 7$

(C) $2 < x < 3$

(D) $2 < x < 3$ and $x \geq 7$

End of Section I

Section II
90 marks
Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$.

Find the value of:

(i) $\bar{B}$ **1**

(ii) $\dfrac{A}{B}$ **2**

(iii) The square roots of B **2**

(iv) The modulus and argument of A **2**

(v) A^4 **1**

(b) The roots of the polynomial equation $2x^3 - 3x^2 + 4x - 5 = 0$ are α, β and γ.
Find the polynomial equation which has roots:

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ **2**

(ii) $2\alpha,\ 2\beta$ and 2γ **2**

(c) Find $\displaystyle\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$ **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} 3x\ \sin(x^2)\,dx$ **3**

(b) (i) Find the values of A, B, and C such that: **2**

$$\frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

(ii) Hence evaluate $\displaystyle\int_2^4 \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x}\,dx$ **2**

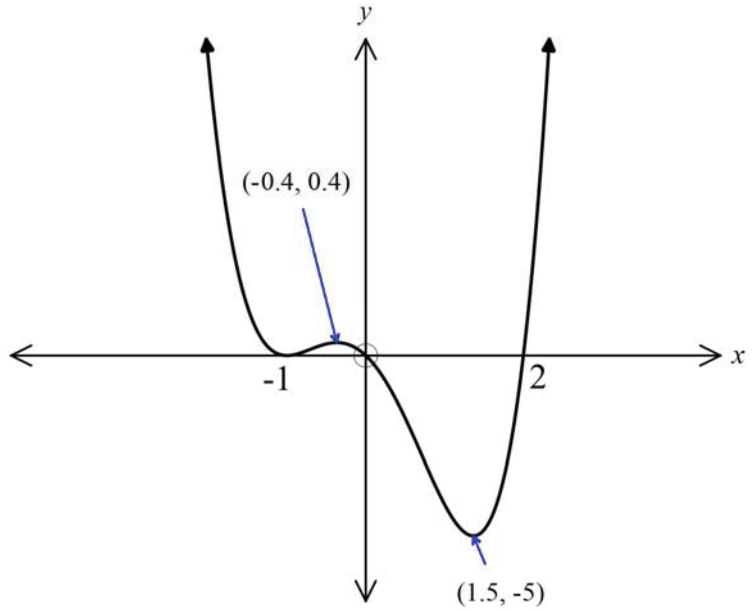
(c) Solve the equation $x^4 - 7x^3 + 17x^2 - x - 26 = 0$, given that $x = (3 - 2i)$ is a root of the equation. **4**

(d) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by $y = 3x^2 - x^3$ and the x axis around the y-axis. **4**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is shown below.



Draw separate $\frac{1}{3}$ page sketches for each of the following:

(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) Show that: $\frac{\cos A - \cos(A + 2B)}{2 \sin B} = \sin(A + B)$ **3**

(c) A particle is projected vertically upwards. The resistance to the motion is proportional to the square of the velocity. The velocity of projection is V m/s.

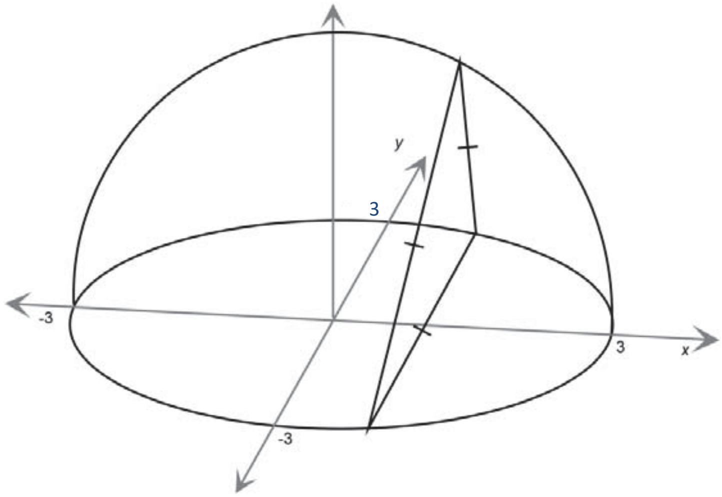
(i) Show that the acceleration is given by: $\ddot{x} = -(g + Kv^2)$ where K is a constant. **1**

(ii) Find the maximum height reached and the time taken to reach this height, expressing your answer in terms of V and K. **4**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) **4**



The diagram above shows a solid which has the circle $x^2 + y^2 = 9$ as its base.

The cross-section perpendicular to the x axis is an equilateral triangle.

Calculate the volume of the solid.

(b) Given that $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, has a double root at $x = \alpha$, find the value of α. **3**

(c) A sequence is defined such that $u_1 = 1, u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. **4**

Prove by induction that $u_n < \left(\frac{7}{4}\right)^n$ for integers $n \geq 1$.

(d) The point $P\,(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the normal at P cuts the hyperbola again at Q with coordinates $(-\frac{c}{t^3}, -ct^3)$ **3**

(ii) Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) z represents the complex number $x + iy$. Sketch the regions:

(i) $|\arg z| < \dfrac{\pi}{4}$ **1**

(ii) $\mathrm{Im}\,(z^2) = 4$ **2**

(b) The complex roots of $z^3 = 1$ are 1, ω and ω^2.

(i) Find the value of $(1+\omega^2)^6$ **1**

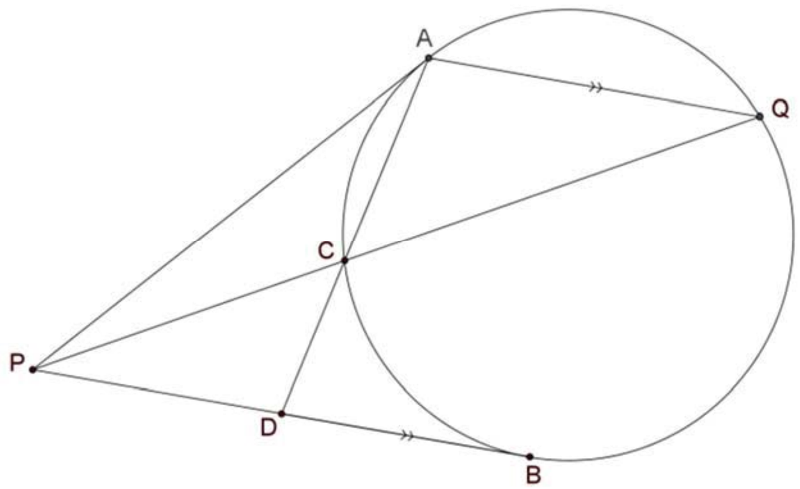
(ii) Hence show that $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$ **2**

(c) In the diagram below, *PA* and *PB* are tangents to the circle. The chord *AQ* is parallel to the tangent *PB*. *PCQ* is a secant to the circle and chord *AC* produced meets *PB* at *D*.

(i) Show that ΔCDP is similar to ΔPDA. **2**

(ii) Hence show that $PD^2 = AD \times CD$. **1**

(iii) Hence, or otherwise, prove that *AD* bisects *PB*. **2**



Question 15 continues on the page 11

Question 15 (continued)

(d) Derive the reduction formula: **4**

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$$

and use this reduction formula to evaluate $\int_0^1 x^5 e^{-x^2} dx$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. P is the point $(a\sec\theta, b\tan\theta)$.

(i) Show that the equation of the tangent at P is $bx\sec\theta - ay\tan\theta = ab$. **2**

(ii) Find the equation of the normal at *P*. **2**

(iii) Find the coordinates of the points *A* and *B* where the tangent and normal respectively cut the *y*-axis. **2**

(iv) Show that AB is the diameter of the circle that passes through the foci of the hyperbola. **3**

(b) Suppose n is a positive integer.

(i) Show that $1 - x^2 + x^4 - x^6 + \dots\dots\dots\dots + (-1)^{n-1}x^{2n-2} = \frac{1-(-1)^n x^{2n}}{1+x^2}$ **1**

(ii) Hence show that **2**

$$-x^{2n} \le \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \dots\dots\dots\dots + (-1)^{n-1}x^{2n-2}\right) \le x^{2n}$$

(iii) By integrating over suitable values of x, deduce that **2**

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots\dots + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}$$

(iv) Explain why $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots$ **1**

End of Examination.

MATHS EXT^n 2 : TRIAL 2015

SECTION I:

① A

$$\int \frac{dx}{x^2 - 4x + 13}$$

$$= \int \frac{dx}{(x-2)^2 + 9}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + c$$

② B.

$$\frac{x^2}{4} + y^2 = 1$$

$a = 2, \quad b = 1$

$b^2 = a^2(1 - e^2)$

$1 - e^2 = \frac{1}{4} \Rightarrow e^2 = \frac{3}{4}$

$e = \frac{\sqrt{3}}{2}$

$x = \pm \frac{a}{e} = \pm \frac{2}{\frac{\sqrt{3}}{2}}$

$= \pm \frac{4}{\sqrt{3}}$

③ D

$xy - x^2 + 3 = 0$

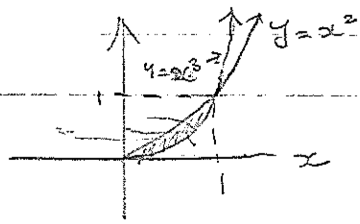
$x\frac{dy}{dx} + y - 2x = 0$

$\frac{dy}{dx} = \frac{2x - y}{x}$

$x = 1, \Rightarrow y = -2$

$\therefore m = \frac{2(1) + 2}{1} = 4$

④ C



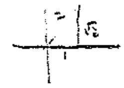
$\delta V = \pi(x_2^2 - x_1^2)\,\delta y$

$V = \pi \int_0^1 (y^{\frac{1}{3}})^2 - (y^{\frac{1}{2}})^2\,dy$

$= \pi \int_0^1 y^{\frac{2}{3}} - y\,dy$

⑤ C

$1 + \sqrt{3}i = 2\,\text{cis}\frac{\pi}{3}$



Let $z = r\,\text{cis}\,\theta$

$z^5 = 2\,\text{cis}\frac{\pi}{3}$

$r^5\,\text{cis}\,5\theta = 2\,\text{cis}\frac{\pi}{3}$

$r = 2^{\frac{1}{5}}$

and $5\theta = \frac{\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{15} + \frac{2k\pi}{5}$

⑥ A

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\,dx$$

odd function

$\therefore \int_{-a}^{a} \_\_ = 0$

⑦ A

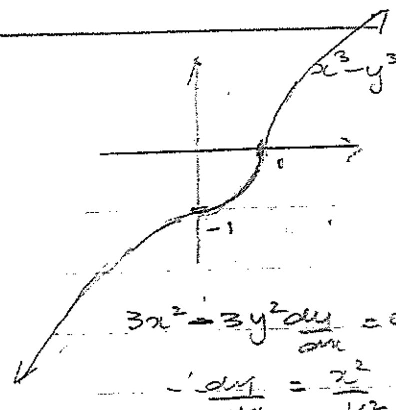
$\frac{A+C}{2} = B$

$\therefore A + C = 2B$

$\frac{\sin 2B}{\sin B} = \frac{2\sin B\cos B}{\sin B}$

$= 2\cos B$

⑧ D



$3x^2 - 3y^2\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{x^2}{y^2}$

A: $x = 1, y = 0, \Rightarrow$ vertical tangent ✓

B: $x = 0, y' = 0 \Rightarrow$ horizontal tangent ✓

C: symmetric about $y = -x$ ✓

D: $P(a, b), b > a \Rightarrow y > x$

$y^3 = x^3 - 1 \quad \therefore y^3 < x^3$

$\Rightarrow y < x$ ✗

(9) B

$$P(1+i) = (1+i)^3 + (1+i)^2 - (1+i) + 1$$
$$= 1 + 3i - 3 - i + 1 + 2i - 1 - 1 - i + 1$$
$$= -2 + 3i$$

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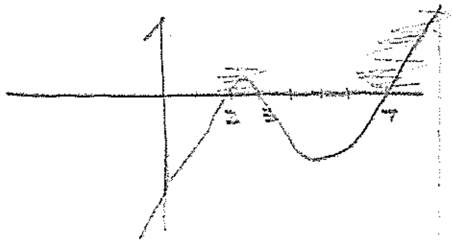
(10) D

$$\frac{x+1}{x-3} \le \frac{x+3}{(x-2)}$$

$$(x+1)(x-3)(x-2)^2 \le (x+3)(x-2)(x-3)^2$$
$$(x+3)(x-2)(x-3)^2 - (x+1)(x-3)(x-2)^2 \ge 0$$
$$(x-2)(x-3)\left[(x+3)(x-3) - (x+1)(x-2)\right] \ge 0$$
$$(x-2)(x-3)\left[x^2 - 9 - x^2 + x + 2\right] \ge 0$$
$$(x-2)(x-3)(x-7) \ge 0$$



$2 < x < 3, \quad x \ge 7$

SECTION II

Question 11

(a) (i) $\overline{B} = -5 + 12i$

(ii)
$$\frac{A}{B} = \frac{3 + 3\sqrt{3}i}{-5 - 12i} \times \frac{-5 + 12i}{-5 + 12i}$$
$$= \frac{-15 + 36i - 15\sqrt{3}i - 36\sqrt{3}}{25 + 144}$$
$$= \frac{(-15 - 36\sqrt{3}) + (36 - 15\sqrt{3})i}{169}$$

(a) cont

(iii) Let $(x + iy)^2 = -5 - 12i$

$x^2 - y^2 = -5$ —— ①

$2xy = -12$

$xy = -6$ —— ②

sub $y = -\frac{6}{x}$ in ①

$$x^2 - \left(-\frac{6}{x}\right)^2 = -5$$
$$x^4 - 36 = -5x^2$$
$$x^4 + 5x^2 - 36 = 0$$
$$(x^2 + 9)(x^2 - 4) = 0$$

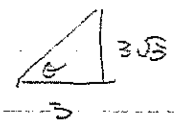
$\therefore x = \pm 2 \quad (x \in R)$

$x = 2, y = -3$ or $x = -2, y = 3$

$\therefore$ Roots are $2 - 3i$ or $-2 + 3i$

$[\pm(2 - 3i)]$

(iv) $r \operatorname{cis} \theta = 3 + 3\sqrt{3}i$



$r = 3(2) = 6$

$\theta = \frac{\pi}{3}$

$\therefore 3 + 3\sqrt{3}i = 6 \operatorname{cis} \frac{\pi}{3}$

$\left[\text{mod } A = 2, \ \arg A = \frac{\pi}{3}\right]$

(v) $A^4 = 6^4 \operatorname{cis} \frac{4\pi}{3}$

$$= -1296 \operatorname{cis} \frac{\pi}{3}$$
$$= -1296\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$
$$= -648\left[1 + i\sqrt{3}\right]$$

QUESTION 11 (cont)

(b) $2x^3 - 3x^2 + 4x - 5 = 0$

(i) Equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

is $2\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 5 = 0$

$$2 - 3x + 4x^2 - 5x^3 = 0$$

$$5x^3 - 4x^2 + 3x - 2 = 0$$

(ii) Equation with roots $2\alpha, 2\beta, 2\gamma$

is

$$2\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 5 = 0$$

$$\frac{\cancel{2}x^3}{\cancel{8}4} - \frac{3x^2}{4} + \frac{4x}{2} - 5 = 0$$

$$x^3 - 3x^2 + 8x - 20 = 0$$

(c) $$\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$$

$$= \int \frac{dx}{\sqrt{9 - [4x^2 - 16x]}}$$

$$= \int \frac{dx}{\sqrt{9 - 4[x-2]^2 + 16}}$$

$$= \int \frac{dx}{2\sqrt{\frac{25}{4} - (x-2)^2}}$$

$$= \frac{1}{2} \sin^{-1} \frac{2}{5}(x-2) + C$$

QUESTION 12

(a) $$\int_0^{\frac{\sqrt{\pi}}{2}} 3x \sin(x^2)\,dx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} \frac{3}{2} \sin(x^2)\, 2x\,dx \qquad u = x^2, \quad du = 2x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sin u\,du$$

$$= \left[-\cos u\right]_0^{\frac{\pi}{4}}$$

$$= \left[-\frac{1}{\sqrt{2}} + 1\right] = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

(b) $$\frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$A(x-1)(x+2) + Bx(x+2) + Cx(x-1) = 4x^2 - 3x - 4$$

$$(A + B + C)x^2 + (A + 2B - C)x - 2A = 4x^2 - 3x - 4$$

$A + B + C = 4$ —— (1)

$A + 2B - C = -3$ —— (2)

$-2A = -4 \Rightarrow A = 2$

in (1) $B + C = 2$ —— (1A)

in (2) $2B - C = -5$ —— (2A)

(1A) + (2A) $\Rightarrow 3B = -3$

$B = -1$

$\therefore C = 3$

$\therefore \boxed{A = 2, B = -1, C = 3}$

QUESTION 12 (continued)

(b) (ii) $\displaystyle\int_2^4 \frac{4x^2-3x-4}{x^3+x^2-2x}\,dx = \int_2^4 \frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\,dx$

$= \Big[2\ln x + \ln(x-1) + 3\ln(x+2)\Big]_2^4$

$= \left[2\ln 4 + \ln 3 + 3\ln 6 - 2\ln 2 - \ln 1 - 3\ln 4\right]$

$= \ln\left[\dfrac{4^2 \times 3 \times 6^3}{2^2 \times 4^3}\right] = \ln\left[\dfrac{81}{2}\right]$

(c) $x^4 - 7x^3 + 17x^2 - x - 26 = 0$

real coeff. $x = 3 - 2i$ is a root $\qquad \sum\alpha = 6$

$\therefore \quad x = 3 + 2i$ is a root (1) $\qquad \alpha\beta = 13$

$\therefore x^2 - 6x + 13 = 0$ is a factor

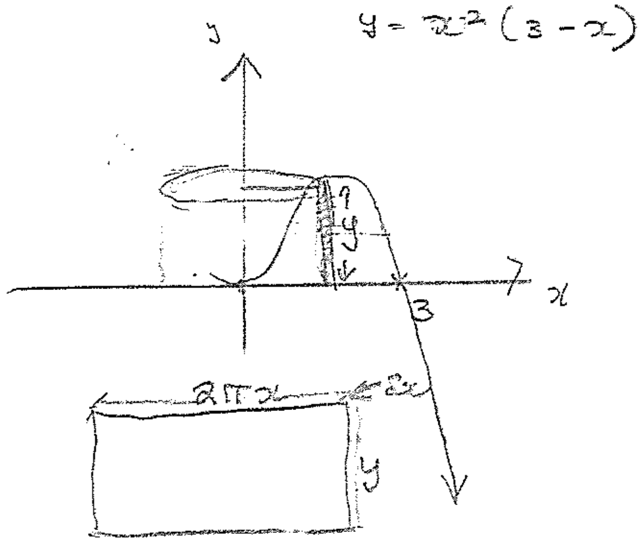
By observation (or long division)

$x^4 - 7x^3 + 17x^2 - x - 26 = (x^2 - 6x + 13)(x^2 - x - 2)$

$= (x^2 - 6x + 13)(x-2)(x+1)$

$\therefore$ Roots are $x = -1, 2$ and $3 \pm 2i$

(d) $V = \displaystyle\int_0^3 2\pi x y\,dx$



$= \displaystyle\int_0^3 2\pi(3x^3 - x^4)\,dx$

$= 2\pi\left[\dfrac{3}{4}x^4 - \dfrac{x^5}{5}\right]_0^3$

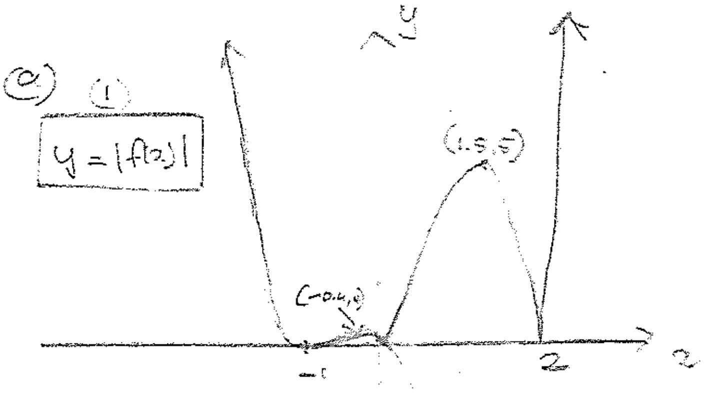
$= 2\pi\left[\dfrac{3}{4}\times 81 - \dfrac{3}{5}\times 81\right]$

$= 2\pi \times \dfrac{3}{20} \times 81 = \dfrac{243}{10}\pi\ u^3$

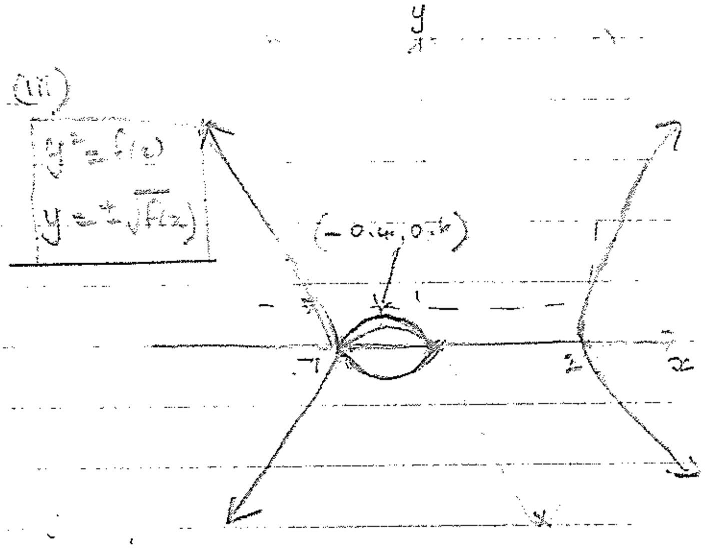
$\delta V = 2\pi x y\,\delta x$

QUESTION 13:

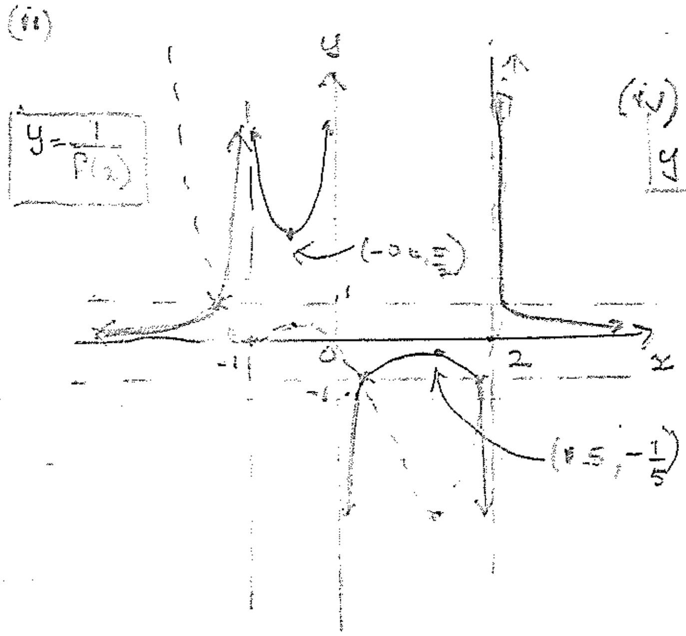
(a) (i)



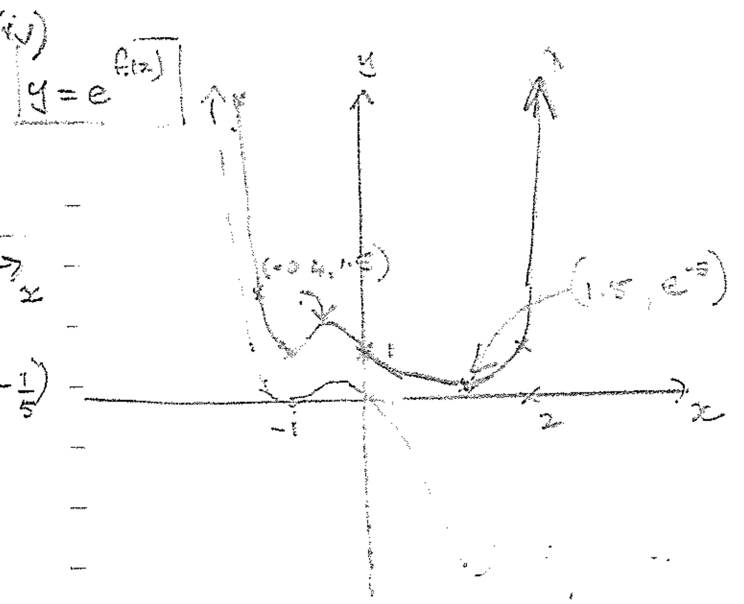
(iii)



(ii)



(iv)



(b) $$\text{LHS} = \frac{\cos A - [\cos A \cos 2B - \sin A \sin 2B]}{2\sin B}$$

$$= \frac{\cos A - [\cos A(1-2\sin^2 B) - \sin A \cdot 2\cos B \sin B]}{2\sin B}$$

$$= \frac{\cancel{\cos A} - \cancel{\cos A} + \cancel{2}\cos A \sin^2 B + \cancel{2}\sin A \cos B \sin B}{\cancel{2}\sin B}$$

$$= \cos A \sin B + \sin A \cos B$$

$$= \sin(A+B)$$

QUESTION 13

(c) (i) $F = ma$

$\uparrow +ve$ $\quad g \downarrow$ $\quad mg + kv^2$

$$-(mg + kv^2) = m\ddot{x}$$

$$\ddot{x} = -\left(g + \frac{k}{m}v^2\right)$$

$$= -(g + Kv^2) \text{ where } K \text{ is a constant}$$

(ii) Need v, x $\qquad t=0,\ v=V,\ x=0.$

$$v\frac{dv}{dx} = -(g + Kv^2)$$

$$\frac{dv}{dx} = \frac{-(g+Kv^2)}{v}$$

$$\therefore \frac{dx}{dv} = -\frac{v}{g+Kv^2}$$

$$x = -\frac{1}{2K}\int \frac{2Kv}{g-Kv^2}\,dv$$

$$= -\frac{1}{2K}\ln(g+Kv^2) + c$$

$$x=0,\ v=V \Rightarrow c = \frac{1}{2K}\ln(g+KV^2)$$

$$\therefore x = -\frac{1}{2K}\ln\left(\frac{g+Kv^2}{g+KV^2}\right)$$

when $v = 0$, $\quad x_{max} = -\frac{1}{2K}\ln\left(\frac{g}{g+KV^2}\right)$

$$= \frac{1}{2K}\ln\left(1 + \frac{K}{g}V^2\right) \text{ m}$$

Need v, t. $\quad \frac{dv}{dt} = -(g+Kv^2) \qquad t=0,\ v=V$

$$\frac{dt}{dV} = -\frac{1}{g+Kv^2}$$

$$= -\frac{1}{K\left(\frac{g}{K}+v^2\right)}$$

QUESTION 13 (continued)

(c)(ii) (cont) $\therefore t = -\frac{1}{K}\sqrt{\frac{K}{g}}\tan^{-1}\left(\sqrt{\frac{K}{g}}v\right) + C$ (1)

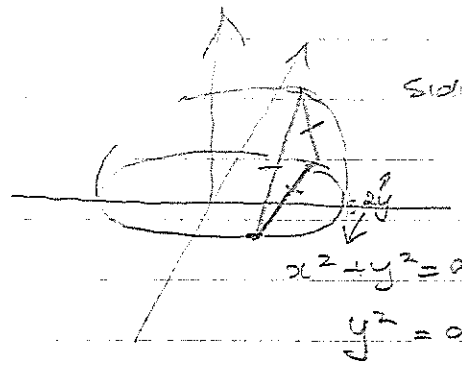
$t=0,\ v=V \Rightarrow C = \frac{1}{\sqrt{Kg}}\tan^{-1}\sqrt{\frac{K}{g}}V$

$$\therefore t = \frac{1}{\sqrt{Kg}}\left[\tan^{-1}\sqrt{\frac{K}{g}}V - \tan^{-1}\sqrt{\frac{K}{g}}v\right]$$

$$v=0 \Rightarrow T = \frac{1}{\sqrt{Kg}}\tan^{-1}\sqrt{\frac{K}{g}}V \text{ secs}$$

QUESTION 14

(a)



side of $\Delta = 2y$ (1)

$x^2+y^2=9$

$y^2 = 9 - x^2$

$A_\Delta = \frac{1}{2}(2y)(2y)\sin 60°$

$= 2y^2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}y^2$ (1)

$V = 2\int_0^3 \sqrt{3}y^2\,dx$

$= 2\sqrt{3}\int_0^3 9 - x^2\,dx$ (1) $= 2\sqrt{3}\left[9x - \frac{x^3}{3}\right]_0^3 = 2\sqrt{3}\left[27 - \frac{27}{3}\right]$

$= 36\sqrt{3}$ u^3

(b) $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

$f'(x) = 4x^3 - 18x^2 + 18x + 4$

$f'(x) = 0 \Rightarrow 2x^3 - 9x^2 + 9x + 2 = 0$ $\quad f(1) \neq 0,\ f(-1) = 0$

$f'(2) = 2(8) - 9(4) + 9(2) + 2 = 0$

$\therefore x=2$ is a root of $f'(x)$

Also $f(2) = 16 - 6(8) + 9(4) + 4(2) - 12 = 0$

$\therefore x=2$ is at least a double root. But there can't be a triple root as we know there is a double root & there are only 4 roots

$\therefore x=2$ is a double root and $\alpha = 2$

QUESTION 14 (continued)

(c) $u_1 = 1$, $u_2 = 1$, $u_n = u_{n-1} + u_{n-2}$ $n \geq 3$

To prove: $u_n < \left(\frac{7}{4}\right)^n$ $n \geq 1$

(A): $n=1$, $u_1 = 1 < \left(\frac{7}{4}\right)^1$ $\therefore$ S(1) true.

$n=2$ $u_2 = 1 < \left(\frac{7}{4}\right)^2$ $\therefore$ S(2) true

(B): Assume S(k) and S(k-1) true

$\therefore$ $u_k < \left(\frac{7}{4}\right)^k$ and $u_{k-1} < \left(\frac{7}{4}\right)^{k-1}$

RTP: $u_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

$$\begin{aligned} LHS = u_{k+1} &= u_k + u_{k-1} \\ &< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\ &= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}+1\right) \\ &= \left(\frac{7}{4}\right)^{k-1}\left(\frac{11}{4}\right) \\ &= \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{16}\right) \\ &< \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2 \\ &= \left(\frac{7}{4}\right)^{k+1} \end{aligned}$$

$\left(\frac{7}{4}\right)^2 = \frac{49}{16}$

$\therefore$ S(k+1) true

Alternatively (or something similar)

RTP $\left(\frac{7}{4}\right)^{k+1} - u_{k+1} > 0$

$$\begin{aligned} LHS &= \left(\frac{7}{4}\right)^{k+1} - (u_k + u_{k-1}) \\ &> \left(\frac{7}{4}\right)^{k+1} - \left(\frac{7}{4}\right)^k - \left(\frac{7}{4}\right)^{k-1} \\ &= \left(\frac{7}{4}\right)^{k-1}\left[\left(\frac{7}{4}\right)^2 - \frac{7}{4} - 1\right] \\ &= \left(\frac{7}{4}\right)^{k-1}\left(\frac{5}{16}\right) > 0 \end{aligned}$$

(C) So S(1) and S(2) are true $\therefore$ S(3) is true and by the process of mathematical induction, S(n) is true for all n.

(d) (i) $xy = c^2$, $\frac{dy}{dx} = -\frac{c^2}{x^2}$

At $x = ct$ $m_T = -\frac{1}{t^2}$ $\therefore$ $m_N = t^2$

$\therefore$ Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

QUESTION 14 (continued)

(d) (i) (cont) $\quad y - \frac{c}{t} = t^2(x - ct)$ ——— ①

Solve with $y = \frac{c^2}{x}$ ——— ②

$\Rightarrow \frac{c^2}{x} - \frac{c}{t} = t^2(x - ct) \Rightarrow ct^2 - xc = t^3x^2 - ct^4x$

$t^3x^2 + (c - ct^4)x - c^2t = 0$ — You know $x = ct$ is a solution

$(t^3x^2 + c)(x - ct) = 0$

$\therefore$ normal cuts again when $x = -\frac{c}{t^3}$ and $y = \frac{-c^2t^3}{c} = -ct^3$

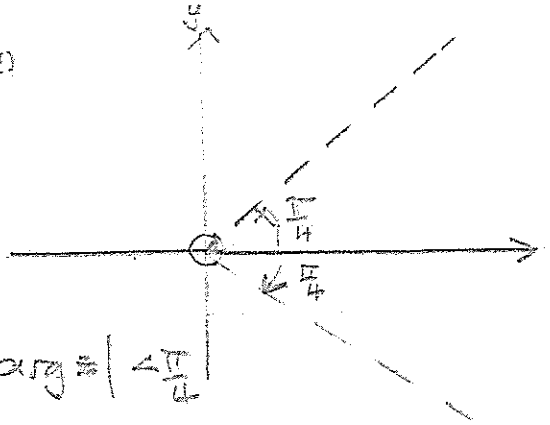
$\therefore Q = \left[-\frac{c}{t^3}, -ct^3\right]$

(ii) "t" $= -\frac{1}{t^3}$ at Q, $\therefore$ Normal at Q cuts at R where $R = \left[\frac{-c}{"t"^3}, -c("t")^3\right]$

$\therefore R = \left[-\frac{c}{\left(-\frac{1}{t^3}\right)^3}, -c\left(-\frac{1}{t^3}\right)^3\right] = \left[ct^9, \frac{c}{t^9}\right]$

QUESTION 15

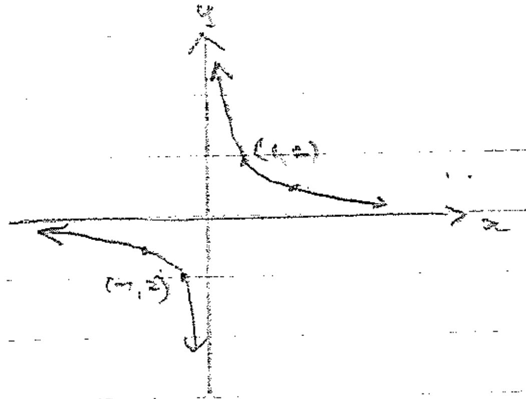
(a) (i)



$|\arg z| < \frac{\pi}{4}$

(ii) $\operatorname{Im}(x+iy)^2 = 4$

$2xy = 4 \Rightarrow xy = 2$



(b) $\omega^3 = 1$ and $(\omega^2)^3 = 1$ $\quad z^3 - 1 = 0$

$\therefore$ sum of roots $= 1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega^2 = -\omega$

$1 = -\omega - \omega^2$

(i) $(1+\omega^2)^6 = (-\omega)^6$ (1)

$= \left[(-\omega)^3\right]^2 = (-1)^2 = 1$

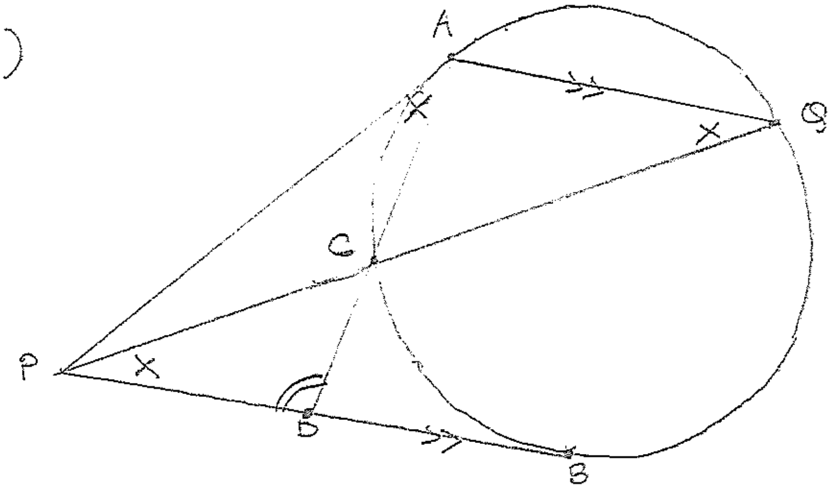
(ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$ $\quad \omega^3 = 1$

$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$ (1)

$= (1-\omega)^2(1-\omega^2)^2 = (1 - \omega - \omega^2 + \omega^3)^2 = (1+1+1)^2 = 9$

QUESTION 15 (continued)

(c)



(i) In Δ PDC and ADP.

$\angle D$ is common

$\angle CPD = \angle CQA$ (alternate $\angle$'s, AQ $\parallel$ PB)

But $\angle PAD = \angle CQA$ ($\angle$ between chord & tangent equals $\angle$ in alternate segment)

$\therefore \angle CPD = \angle PAD$ (equiangular)

(ii) $\dfrac{PD}{DA} = \dfrac{DC}{PD}$ (corresponding sides of similar Δ's)

$\therefore PD^2 = AD \cdot DC$

(iii) But $DB^2 = AD \cdot DC$ (square of tangent equals the product of the intersecting secants)

$\therefore PD^2 = DB^2$

and $PD = DB$.

QUESTION 15. (continued)

$$u = x^{n-1} \qquad v' = -2xe^{-x^2}$$
$$u' = (n-1)x^{n-2} \qquad v = e^{-x^2}$$

(d) $\displaystyle\int x^n e^{-x^2}\,dx = \int \frac{x^{n-1}}{-2}(-2xe^{-x^2})\,dx$

$$= -\frac{1}{2}\int \underset{u}{x^{n-1}}\,\underset{v'}{(-2xe^{-x^2})}\,dx$$

$$= -\frac{1}{2}\left[x^{n-1}\cdot e^{-x^2}\right] + \frac{1}{2}\int (n-1)x^{n-2}\cdot e^{-x^2}\,dx$$

$$= -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{(n-1)}{2}\int x^{n-2}e^{-x^2}\,dx$$

Let $I_n = \left[-\frac{1}{2}x^{n-1}e^{-x^2}\right]_0^1 + \frac{n-1}{2}I_{n-2}$.

$$I_5 = \left[-\frac{1}{2}x^4 e^{-x^2}\right]_0^1 + 2I_3$$

$$= \left(-\frac{1}{2}\cdot e^{-1}\right) + 2I_3$$

$$I_3 = \left[-\frac{1}{2}x^2 e^{-x^2}\right]_0^1 + 1.\,I_1$$

$$= -\frac{1}{2}e^{-1} + I_1$$

$$I_1 = \int_0^1 x e^{-x^2}\,dx$$

$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = -\frac{1}{2}\left[e^{-1} - 1\right]$$

$$\therefore \quad I_3 = -\frac{1}{2}\left[e^{-1} + e^{-1} - 1\right]$$

$$I_5 = -\frac{1}{2}\left[e^{-1}\right] + 2\left[-\frac{1}{2}(e^{-1} + e^{-1} - 1)\right]$$

$$= -\frac{1}{2}e^{-1} - (e^{-1} + e^{-1} - 1)]$$

$$= -\frac{5}{2}e^{-1} + 1$$

QUESTION 16

(a) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(i) $x = a\sec\theta \qquad y = b\tan\theta$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$\therefore$ Eqn of tangent is

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = b\sec\theta\, x - ab\sec^2\theta$$

$$b\sec\theta\, x - a\tan\theta\, y = ab(\sec^2\theta - \tan^2\theta)$$

$\therefore b\sec\theta\, x - a\tan\theta\, y = ab$ since $\sec^2\theta - \tan^2\theta = 1$ —— ①

(ii) Normal at P: $m_N = -\dfrac{a\tan\theta}{b\sec\theta}$

$\therefore$ Eqn of normal is $y - b\tan\theta = -\dfrac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$$b\sec\theta\, y - b^2\tan\theta\sec\theta = -a\tan\theta\, x + a^2\sec\theta\tan\theta$$

$$\therefore a\tan\theta\, x + b\sec\theta\, y = (a^2 + b^2)\sec\theta\tan\theta$$

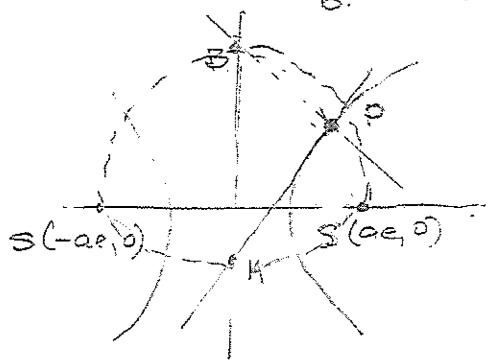
$$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \text{ —— ②}$$

(iii) Put $x = 0$ in ① $\Rightarrow y = -\dfrac{ab}{a\tan\theta}$ $\therefore A = \left(0, -\dfrac{b}{\tan\theta}\right)$

Put $x = 0$ in ② $\Rightarrow y = \dfrac{(a^2+b^2)\tan\theta}{b}$ $\therefore B = \left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$

QUESTION 16 (continued)

(a) (iv) $A = \left(0, \; -\dfrac{b}{\tan\theta}\right)$ $\quad B = \left(0, \; \dfrac{(a^2+b^2)\tan\theta}{b}\right)$



Focus $S(ae, 0)$ $\quad S'(-ae, 0)$

Need to show. $m_{AS} \times m_{BS} = -1$

$$m_{AS} = \frac{\frac{b}{\tan\theta}}{ae} = \frac{b}{ae\tan\theta}$$

$$m_{BS} = \frac{\frac{(a^2+b^2)\tan\theta}{b}}{-ae} = \frac{(a^2+b^2)\tan\theta}{-abe}$$

$$m_{AS} \times m_{BS} = \frac{\cancel{b}}{ae\cancel{\tan\theta}} \times -\frac{(a^2+b^2)\cancel{\tan\theta}}{a\cancel{b}e}$$

$$= -\frac{(a^2+b^2)}{a^2e^2}$$

$$= -\frac{(a^2+a^2(e^2-1))}{a^2e^2} \qquad \text{since } b^2 = a^2(e^2-1)$$

$$= -\frac{a^2e^2}{a^2e^2} = -1$$

$\therefore$ $\angle BSA = 90°$ and AB is the diameter of a circle passing through S (converse of $\angle$ in a semi-circle)

Similarly, $m_{AS'} \times m_{BS'} = \dfrac{-b}{ae\tan\theta} \times \dfrac{(a^2+b^2)\tan\theta}{abe}$

$$= -1$$

and the circle also passes through S'

QUESTION 16 (continued)

(b) (i) $1 - x^2 + x^4 - x^6 + \ldots\ldots + (-1)^{n-1} x^{2n-2}$

is a G.P. $a = 1$, $r = -x^2$ and there are "n" terms

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1\cdot(1-(-x^2)^n)}{1-(-x^2)}$$

$$= \frac{1-(-1)^n x^{2n}}{1+x^2}$$

(ii) $$\frac{1}{1+x^2} - (1 - x^2 + x^4 \ldots + (-1)^{n-1} x^{2n-2})$$

$$= \frac{(-1)^n x^{2n}}{1+x^2}$$

If n is even, $\text{RHS} = \frac{x^{2n}}{1+x^2} \leq x^{2n}$ since $1+x^2 \geq 1$

If n is odd, $\text{RHS} = \frac{-x^{2n}}{1+x^2} \geq -x^{2n}$ since $1+x^2 \geq 1$

$$\therefore \quad -x^{2n} \leq \frac{1}{1+x^2} - (1 - x^2 + x^4 \ldots + (-1)^{n-1} x^{2n-2}) \leq x^{2n}$$

for all n

(iii) $$\int_0^1 -x^{2n}\,dx \leq \int_0^1 \frac{1}{1+x^2} - (1 - x^2 + x^4 \ldots \pm (-1)^{n-1} x^{2n-2})\,dx \leq \int_0^1 x^{2n}\,dx$$

$$\left[-\frac{x^{2n+1}}{2n+1}\right]_0^1 \leq \left[\tan^{-1} x\right]_0^1 - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right]_0^1 \leq \left[\frac{x^{2n+1}}{2n+1}\right]_0^1$$

$$-\frac{1}{2n+1} \leq (\tan^{-1} 1 - \tan^{-1} 0) - \left[1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{n-1}}{2n-1}\right] \leq \frac{1}{2n+1}$$

$$\therefore \quad -\frac{1}{2n+1} \leq \frac{\pi}{4} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\right] \leq \frac{1}{2n+1}$$

(iv) As $n \to \infty$, LHS $\to 0$, RHS $\to 0$ $\therefore$ in the limit,

$$0 = \frac{\pi}{4} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots\right] = 0$$

and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\ldots$