MEREWETHER HIGH SCHOOL
4 Unit Mathematics
1999 Trial HSC Examination

Question 1

a) Find $\int \frac{x^2\ dx}{(1-x)(1+x^2)}$

b) **(i)** If $I_n = \int \tan^n x\ dx$ show that $I_n = \frac{1}{n-1} \cdot \tan^{n-1} x - I_{n-2}$ for $n \geq 2$.

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^7 x\ dx$.

c) Find $\int x^2 e^x\ dx$

d) Find $\int \cos^5 x \sin^3 x\ dx$

Question 2

a) Express $\frac{(1+2i)^2}{2+i}$ in the form $a + ib$ where a and b are real numbers.

b) Sketch on an Argand diagram the region where $-\frac{\pi}{4} \leq \arg(z-1) \leq \frac{\pi}{4}$ and $|z-2| \leq 1$ are simultaneously true.

c) **(i)** Express $z = -3\sqrt{3} + 3i$ in mod-arg form.

(ii) Hence find the smallest positive integer n such that z^n is real.

d) Solve the equation $z^4 - (z+1)^4 = 0$.

e) z is a complex number such that $|z-1| = 1$ and $\arg z = \theta$

(i) Explain why $\arg(z-1) = 2\theta$

(ii) Hence find $\arg(z^2 - 3z + 2)$ in terms of θ.

f) **(i)** On an Argand diagram show vectors to represent complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$

(ii) If $|z_1| = |z_2|$ show that $\frac{z_1 - z_2}{z_1 + z_2}$ is purely imaginary.

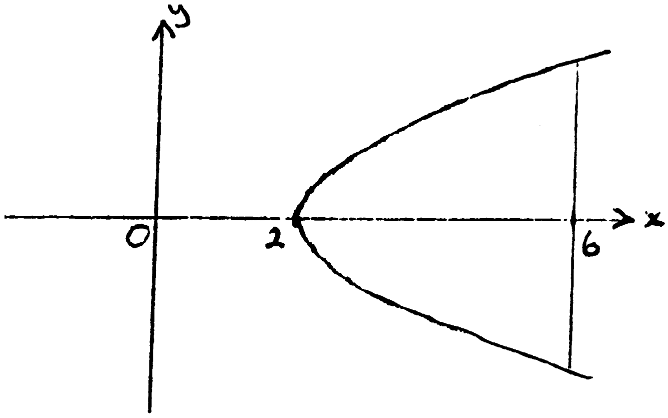
Question 3

a) Sketch the curve $y = x^2(3 - x)$, clearly showing intercepts with the axes and stationary point.

b) By using the sketch drawn in part **a)**, sketch (on separate diagrams) each of the following

(i) $y = |x^2(3 - x)|$ **(ii)** $|y| = x^2(3 - x)$ **(iii)** $y = \frac{1}{x^2(3-x)}$ **(iv)** $y^2 = x^2(3 - x)$

c) The diagram shows the region bounded by the curve $y^2 = 4(x - 2)$ and the line $x = 6$. Use the method of cylindrical shells to find the volume of the solid formed by rotating the given region about the line $x = 1$.



Question 4

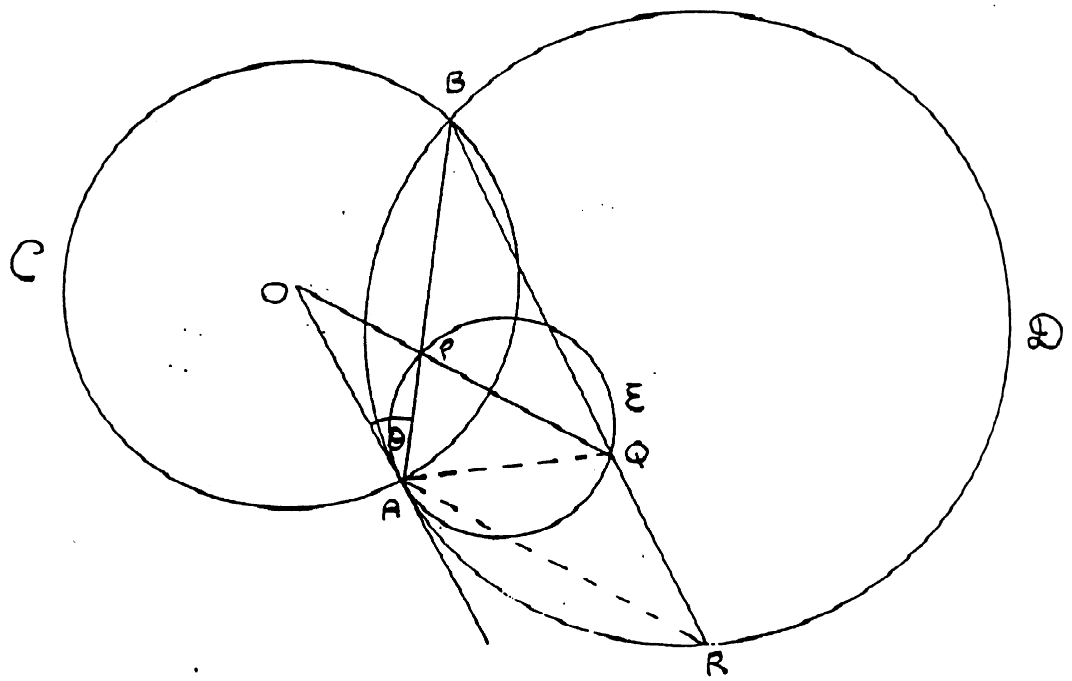
a) The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x - 1)^2$, and α, β are constants. Show that $\alpha = n$ and $\beta = -(1 + n)$.

b) The quartic polynomial $f(x) = x^4 + px^3 + qx^2 + rx + s$ has four zeros $\alpha, \beta, \gamma, \delta$ such that the sum of α and β equals the sum of γ and δ. Let $C = \alpha + \beta = \gamma + \delta$, let $P = \alpha\beta$ and let $Q = \gamma\delta$.

(i) Find p, q, r and s in terms of C, P and Q

(ii) It is given that the polynomial $g(x) = x^4 - 18x^3 + 79x^2 + 18x - 440$ has the property that the sum of two of the zeros equals the sum of the other two zeros. Find all four zeros of $g(x)$.

c)



In the diagram above, OA is a radius of a circle $\mathcal{C}$ with centre O, and two circles $\mathcal{D}$ and $\mathcal{E}$ are drawn touching the line OA at A, with circle $\mathcal{E}$ inside circle $\mathcal{D}$. The larger circle $\mathcal{D}$ meets circle $\mathcal{C}$ again at B, and the line AB meets the smaller circle $\mathcal{E}$ again at P. The line OP meets circle $\mathcal{E}$ again at Q, and the line BQ meets circle D again at R.

(i) Let $\angle OAP = \theta$. Explain why $\angle PQA = \theta$.

(ii) Prove thet the points O, B, Q and A are concyclic.

(iii) Prove that OQ bisects $\angle BQA$.

(iv) Prove that $OQ||AR$ and that $AQ = RQ$.

(v) Prove that $BQ : QA = BP : PA$.

<u>Question 5</u>

a) The normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$ meets the axes at Q and R.

(i) Show that the equation of the normal at P is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

(ii) Find the area of triangle OQR in terms of a, b, θ

(iii) Find the position(s) of P for which this area is a maximum.

b) On Wednesday morning at 5 a.m. a plane crashes into a harbour. The rescue team and its equipment are only effective when the depth of the water is no more than 7 metres. At low tide the depth of water is 5 metres, and at high tide the depth is 10 metres. Low tide occurs at 4 a.m. and high tide at 10.15 a.m. Assume that the movement of the tide is simple harmonic motion.

(i) Find the period and amplitude of the motion.

(ii) If the deadline for the rescue operation is 6 p.m. on Wednesday evening, find the periods of time between 5 a.m. and 6 p.m. during which the rescue team can work.

Question 6

a) Let α, β, γ be the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are real. The equation $x^3 + ax^2 + bx + c = 0$ has roots $\alpha^2, \beta^2, \gamma^2$. Find a, b, c in terms of p, q, r.

b) (i) Find, in mod-arg form, the seventh roots of unity and show them on an Argand diagram.

(ii) If α is one of the complex roots, show that the quadratic equation whose roots are $\alpha + \alpha^2 + \alpha^4$ and $\alpha^3 + \alpha^5 + \alpha^6$ is $x^2 + x + 2 = 0$

(iii) Find the cubic equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$.

c) $P\left(cp, \frac{c}{p}\right)$ lies on the rectangular hyperbola $xy = c^2$

(i) Show that the equation of the normal at P is $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$.

(ii) The normal at P meets the x axis at A, and the tangent at P meets the y axis at B. M is the midpoint ofr AB. The equation of the tangent at P is $x + p^2y = 2cp$. Show that the equation of the locus of M, as P moves on the hyperbola, is $2c^2xy = c^4 - y^4$.

Question 7

a) (i) Sketch the parabola $y = x^2 - 4x + 3$.

(ii) Show that the curve $y = e^{x^2-4x+3}$ has only one stationary point and find its coordinates.

(iii) Sketch the curve $y = e^{x^2-4x+3}$ for $0 \leq x \leq 3$.

(iv) Hence show that $\frac{3}{e} < \int_0^3 e^{x^2-4x+3}\, dx < 3e^3$.

b) From a point on the ground an object of mass m kg is projected vertically upwards with an initial speed of u metres per sec. It reaches a maximum height of h metres before falling back to ground. Throughout the flight air resistance is equal to mkv^2, where v is the velocity. Acceleration due to gravity is g.

(i) Show that $h = \frac{1}{2k} \ln \left(\frac{g+ku^2}{g} \right)$

(ii) Q is a point d metres above the ground. Let V be the speed of the object at Q on its upward path. Show that $d = \frac{1}{2k} \ln \left(\frac{g+ku^2}{g+kV^2} \right)$.

(iii) On the object's downward path it passes Q with a speed of $\frac{V}{2}$ metres per second. Show that $V = \sqrt{\frac{3g}{k}}$.

Question 8

a) (i) State de Moivre's Theorem.

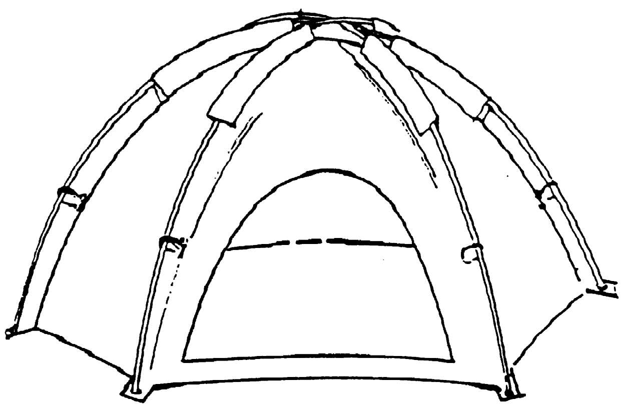
(ii) If z is a complex number prove that $z + \overline{z} = 2\Re(z)$.

(iii) Prove that $(1 + i \tan \theta)^n + (1 - i \tan \theta)^n = 2 \sec^n \theta \cos n\theta$.

(iv) Hence prove that $\Re\Big(\big(1 + i \tan \frac{\pi}{8}\big)^8 \Big) = 64\big(12\sqrt{2} - 17\big)$.

b) (i) Show that the area of a regular hexagon of side s is given by $A = \frac{3\sqrt{3}s^2}{2}$.

(ii) The diagrams below illustrate a dome tent. When erected, the base is a regular hexagon which measures 2 metres from corner to adjacent corner (internal measurement). Flexible exterior poles extend between opposite corners in semi-circle arcs to support the tent. By taking slices parallel to the base of the tent find the volume enclosed by the tent.



Top View