MEREWETHER HIGH SCHOOL

YEAR 12 TRIAL HSC EXAMINATIONS 2000

MATHEMATICS

4 Unit ADDITIONAL PAPER

Time allowed: Three hours plus 5 minutes reading time.

INSTRUCTIONS: *All questions may be attempted
*All questions are of equal value
*In every question all necessary working should be shown - full marks may not be awarded for answers only.
*Approved silent calculators may be used.
***Standard integrals are printed at the back of this exam. paper.**
***Start each question on a new sheet of paper.**

Question 1

Marks

a (i) Find $\int \frac{dx}{x.\ln x}$ 1

(ii) Find $\int \frac{dx}{4+3\cos x}$ 2

(iii) Evaluate $\int_{-1}^{0} \frac{(x-1)dx}{x^2+2x+2}$ 3

b Use integration by parts to evaluate $\int_0^{\frac{1}{2}} \cos^{-1} x dx$ 3

c (i) Find A, B and C so that

$$\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2} \text{ for all x, } x \neq -3$$ 3

(ii) Hence evaluate $\int_0^1 \frac{10}{(3+x)(1+x^2)} dx$ 3

Question 2 (START A NEW SHEET OF PAPER)

a Find the cube roots of 27i 4

NOTE: parts (b), (c) and (d) are NOT related

b On an Argand diagram, shade in the region defined by

$$\text{Im}(z) \leq 1 \text{ and } \frac{\pi}{3} \leq \arg(z+i) \leq \frac{\pi}{2}$$ 3

c The complex number z and its its conjugate $\bar{z}$ satisfy the equation

$$z.\bar{z} - 2iz = -3 - 2i$$

Find the possible values of z. 4

d (i) Sketch the graph specified by $|z - 2 - i\sqrt{3}| = \sqrt{7}$ 3

(ii) Hence find the maximum value of $|z|$ 1

Question 3 (START A NEW SHEET OF PAPER)

a (1 - i) is a root of the equation $x^4 - 3x^3 + 3x^2 - 2 = 0$. Find all the other roots. 3

b Consider the cubic equation $P(x) = x^3 + ax + b$

Show that if a > 0, then P(x) = 0 has exactly one real root. 3

c Sketch the following curves on SEPARATE sets of axes, showing clearly all the main features:

(i) $y = (x+1)(3-x)$ 1

(ii) $y = \frac{1}{(x+1)(3-x)}$ 2

(iii) $y = \left|\frac{1}{(x+1)(3-x)}\right|$ 2

(iv) $y = \log_e(x+1)(3-x)$ 4

Question 4 (START A NEW SHEET OF PAPER)

a (i) Show that the normal to the hyperbola $xy = c^2$ at the point $P(ct, \frac{c}{t})$ has the equation $y = t^2x + \frac{c}{t} - ct^3$ 2

(ii) If the normal at P meets the line $y = x$ at N, and the tangent at P meets $y = x$ at T, find the co-ordinates of N and T. 2

(iii) If O is the origin, prove that $OT.ON = 4c^2$ 3

b The ellipse E has cartesian equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(i) State E's eccentricity, the co-ordinates of its foci S and S' and the equations of its directrices. 3

(ii) Sketch the curve neatly, showing essential features. 3

(iii) If P is any point on E, and the length of the interval PS is 2units, find the length of the interval PS' 2

Question 5 (START A NEW SHEET OF PAPER)

a (i) Sketch on the number plane the circle $(x - 1)^2 + y^2 = 4$, labelling all intercepts on the x and y axes. 2

(ii) On this diagram shade the region

$$\{ (x,y) : (x - 1)^2 + y^2 \leq 4 \} \cap \{ (x,y) : x \geq 0 \}$$ 1

(iii) Your shaded region in part (ii) forms the base of a solid with every cross-section perpendicular to the x-axis forming a square, one side of which lies on the base. Find the volume of the solid. 5

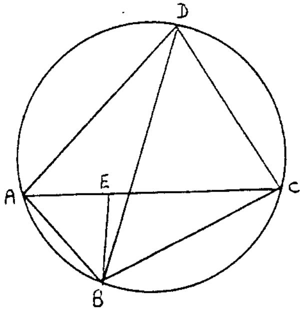
b (i) Given $I_n = \int_0^1 x^n e^{2x} dx$ where n is a positive integer, use integration by parts to show that:

$$I_n = ½ (e^2 - n \times I_{n-1})$$ 4

(ii) Hence evaluate $\int_0^1 x^3 e^{2x} dx$ 3

Question 6 (START A NEW SHEET OF PAPER)

a



The figure above shows a cyclic quadrilateral ABCD with diagonals AC and BD. E is a point on AC such that $\angle ABE = \angle DBC$.

(i) Prove that:

(α) $\Delta ABE /// \Delta DBC$ 2

(β) $\Delta ABD /// \Delta EBC$ 2

(ii) Hence prove Ptolemy's Theorem, which is that:

$$BA \times DC + AD \times BC = AC \times BD$$ 3

b If the circular disc with centre (3,0) and radius 2 is rotated about the y-axis, then a doughnut-shaped solid is formed.

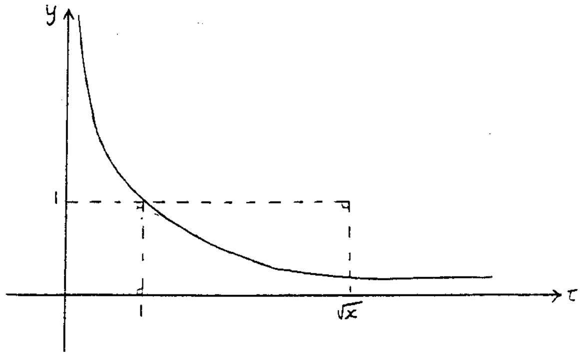
(i) Use the method of cylindrical shells to show clearly that the volume of this solid is given by:

$$V = 4\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx$$ 4

(ii) Hence find the volume of the solid. 4

Question 7 (START A NEW SHEET OF PAPER)

a



This diagram shows that $0 < \int_1^{\sqrt{x}} \frac{dt}{t} < \sqrt{x}$, for all x > 1

Evaluate this integral, and then use this inequality to show that:

$$\lim_{x \to \infty} \left(\frac{\ln x}{x} \right) = 0$$ 2

b (i) Find in exact form all turning points and points of inflexion on the curve $y = \frac{\ln x}{x}$,

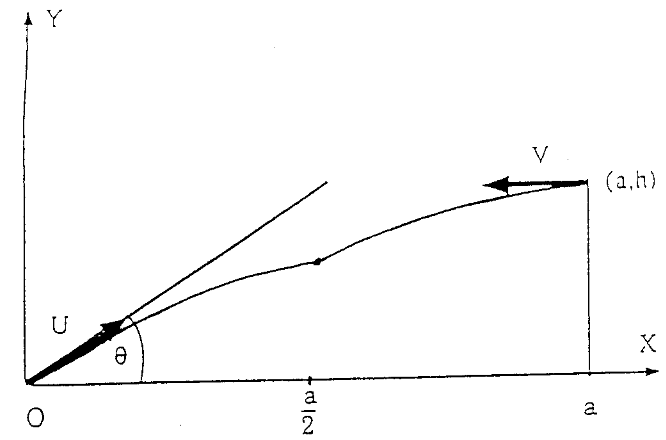
given $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$ 2

(ii) Sketch $y = \frac{\ln x}{x}$ 3

Question 7 continued on the next page.

Question 7 continued

c



A gun is so aimed that the shell it fires strikes a target released simultaneously from an aeroplane flying horizontally towards the gun at a speed of V ms^{-1} and at a height 'h' metres. The aeroplane was at a horizontal distance 'a' metres from the gun when the target was released, and the shell strikes the target at half this horizontal distance 'a', as shown on the diagram. The initial velocity of the shell is U ms^{-1} and the angle of projection is θ

(i) Show that the equations of motion of the target are:

$\dot{x} = -V$ $\dot{y} = -gt$

$x = a - Vt$ $y = h - \frac{1}{2}gt^2$ 3

(iii) Show that the gun was aimed at a point h metres vertically above the aeroplane at the instant of release, and that

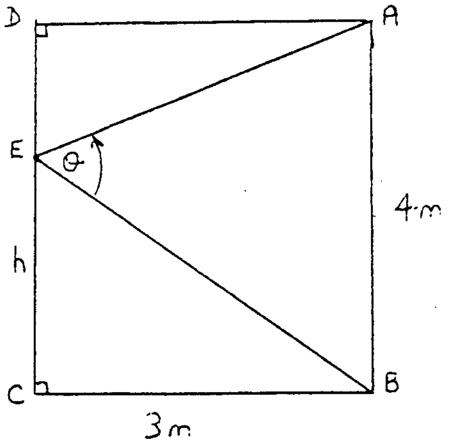
$$U = \frac{V}{\underset{a}{\cancel{3}}}.\sqrt{a^2 + 4h^2}$$ 5

Question 8 (START A NEW SHEET OF PAPER)

a (i) sin(A+B) = sinAcosB + sinBcosA and cos(A+B) = cosAcosB - sinAsinB

Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ 2

(ii)



ABCD is a rectangle. AB is 4 metres long and BC is 3 metres. E is a variable point on side CD. Let ∠AEB be θ and EC be h metres in height.

(i) Show that $\tan\theta = \dfrac{12}{9 - 4h + h^2}$ 3

(ii) What value of h makes θ a maximum? 4

b On a certain day the depth of water in a bay at high tide was 11m. At low tide, 6¼ hours later, the depth of water was 7m. If the next high tide is due at 3.20pm, what is the earliest time that a ship, which needs a depth of at least 10m, can enter the bay? (Assume that the rise and fall of the tide is Simple Harmonic) 6

2000 - 4UNIT TRIAL HSC - SOLUTIONS

1 a) (i) $I = \int \frac{dx}{x\ln x}$ let $u = \ln x$, $du = \frac{1}{x}dx$

$= \int \frac{du}{u}$

$= \ln u + c$

$= \ln(\ln x) + c$

1

(ii) $I = \int \frac{dx}{4+3\cos x}$ let $t = \tan\frac{x}{2}$

$= \int \frac{\frac{2dt}{1+t^2}}{4+3\frac{(1-t^2)}{1+t^2}}$

$= \int \frac{2dt}{4+4t^2+3-3t^2}$

$= \int \frac{2dt}{t^2+7}$

$= \frac{2}{\sqrt{7}}\tan^{-1}\frac{t}{\sqrt{7}} + c$

2

$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{7}}\right) + C$

(iii) $I = \frac{1}{2}\int_{-1}^{0}\frac{(2x-2)\,dx}{x^2+2x+2}$

$= \frac{1}{2}\int_{-1}^{0}\frac{(2x+2)dx}{x^2+2x+2} - \frac{1}{2}\int_{-1}^{0}\frac{4\,dx}{x^2+2x+2}$

$= \frac{1}{2}\Big[\ln(x^2+2x+2\Big]_{-1}^{0} - \int_{-1}^{0}\frac{2\,dx}{(x+1)^2+1}$

$= \frac{1}{2}(\ln 2 - \ln 1) - 2\left[\tan^{-1}(x+1)\right]_{-1}^{0}$

$= \frac{1}{2}\ln 2 - 2(\tan^{-1}1 - \tan^{-1}0)$

$= \frac{1}{2}\ln 2 - 2(\frac{\pi}{4} - 0)$

$= \frac{1}{2}\ln 2 - \frac{\pi}{2}$

3

b) $I = \int_0^{\frac{1}{2}} \cos^{-1}x \times 1.\,dx$

$\int uv'dx = uv - \int vu'dx$

$= \left[\cos^{-1}x\,.\,x\right]_0^{\frac{1}{2}} - \int x.\frac{-1}{\sqrt{1-x^2}}dx$

$= \left(\frac{1}{2}\cos^{-1}\frac{1}{2} - 0\right) - \frac{1}{2}\int_0^{\frac{1}{2}}\frac{-2x\;dx}{\sqrt{1-x^2}}$

$= \frac{1}{2}\times\frac{\pi}{3} - \frac{1}{2}\left[\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

$= \frac{\pi}{6} - \left(\sqrt{1-\frac{1}{4}} - \sqrt{1-0}\right)$

3

$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$

c) (i) $\frac{10}{(3+x)(1+x^2)} = \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$

$\therefore\ 10 \equiv A(1+x^2) + (3+x)(Bx+C)$

let $x=-3$ $\quad 10 = 10A \Rightarrow A = 1$

let $x=0$ $\quad 10 = A + 3C \Rightarrow C = 3$

let $x=1$ $\quad 10 = 2A + 4B + 4C \Rightarrow B = -1$

$\therefore A = 1, B = -1, C = 3$

3

(ii) $I = \int_0^1 \frac{10.\,dx}{(3+x)(1+x^2}$

$= \int_0^1 \left(\frac{1}{3+x} + \frac{3-x}{1+x^2}\right)dx$

$= \int_0^1 \frac{1}{3+x} + \frac{3}{1+x^2} - \frac{x}{(1+x^2})dx$

$= \left[\ln(3+x) + 3\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \ln 4 + 3\tan^{-1}1 - \frac{1}{2}\ln 2 - \ln 3 - 3\tan^{-1}0 + \frac{1}{2}\ln 1$

$= \ln 4 - \ln\sqrt{2} - \ln 3 + \frac{3\pi}{4}$

$= \ln\frac{4}{3\sqrt{2}} + \frac{3\pi}{4}$

3

2. a) Let $x^3 = 27i$

$x^3 - 27i = 0$

$x^3 + 27i^3 = 0$

$(x+3i)(x^2 - 3ix + 9i^2) = 0$

$(x+3i)(x^2 - 3ix - 9) = 0$

$x = -3i \qquad x = \frac{3i \pm \sqrt{9i^2+36}}{2}$

$= \frac{3i \pm \sqrt{27}}{2}$

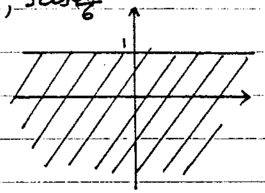
$= \frac{3i \pm 3\sqrt{3}}{2}$

4

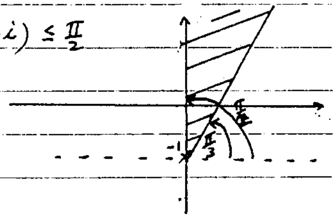
$\therefore$ cube roots of $27i$ are $-3i$, $\frac{3i \pm 3\sqrt{3}}{2}$

OR $3\operatorname{cis}\frac{3\pi}{2}$, $3\operatorname{cis}\frac{\pi}{6}$, $3\operatorname{cis}\frac{5\pi}{6}$

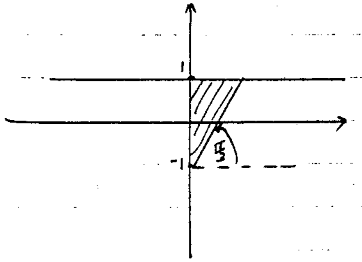
b) $\operatorname{Im}(z) \le 1$



$\frac{\pi}{3} \le \arg(z+i) \le \frac{\pi}{2}$



$\therefore \operatorname{Im}(z) \le 1$ and $\frac{\pi}{3} \le \arg(z+i) \le \frac{\pi}{2}$



3

c) $z\bar{z} - 2iz = -3 - 2i$

let $z = x + iy$, so $\bar{z} = x - iy$

$\therefore (x+iy)(x-iy) - 2i(x+iy) = -3 - 2i$

$x^2 + y^2 - 2ix - 2i^2y = -3 - 2i$

$x^2 + y^2 + 2y - 2ix = -3 - 2i$

equating reals & imaginaries,

$x^2 + y^2 + 2y = -3$, $x = 1$

$\therefore y^2 + 2y + 1 + 3 = 0$

$y^2 + 2y + 4 = 0$

$y = \frac{-2 \pm \sqrt{4-16}}{2}$

$= \frac{-2 \pm \sqrt{12}.i}{2}$

$= -1 \pm \sqrt{3}i$

$\therefore z = 1 + i(-1 \pm \sqrt{3}i)$

$= 1 - i \pm \sqrt{3}i^2$

$= 1 \pm -\sqrt{3} - i$

$= 1 - \sqrt{3} - i,\ 1 + \sqrt{3} - i$

4

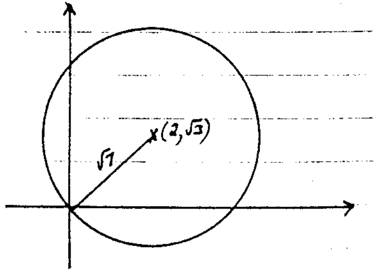
d) $|z - 2 - i\sqrt{3}| = \sqrt{7}$

circle, centre $2 + i\sqrt{3}$, radius $\sqrt{7}$

Note however $|2 + i\sqrt{3}| = \sqrt{2^2 + (\sqrt{3})^2}$

$= \sqrt{7}$

$\therefore$ circle passes through $(0,0)$



3

max value of $|z|$ = max dist of any point on circle from $(0,0)$

$\therefore \max|z| = 2\sqrt{7}$

1

3. a) Since $(1-i)$ is a root, so is $(1+i)$

$\therefore (x-(1-i))(x-(1+i)$ is a factor of $P(x)$

ie $(x-1)^2 - i^2 = x^2-2x+2$

$$\begin{array}{r} x^2 - x - 1 \\ x^2-2x+2 \overline{) x^4-3x^3+3x^2+0x-2} \\ \underline{x^4-2x^3+2x^2} \downarrow \quad \downarrow \\ -x^3+x^2+0x \\ \underline{-x^3+2x^2-2x} \downarrow \\ -x^2+2x-2 \\ \underline{-x^2+2x-2} \\ \end{array}$$

$\therefore$ eq'n becomes $(x^2-2x+2)(x^2-x-1)=0$

$\swarrow \qquad \searrow$

$x = 1-i, 1+i$

$x = \frac{1 \pm \sqrt{1+4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2}$

3

$\therefore x = 1 \pm i, \frac{1 \pm \sqrt{5}}{2}$

b) $P(x) = x^3+ax+b$

$P'(x) = 3x^2+a$

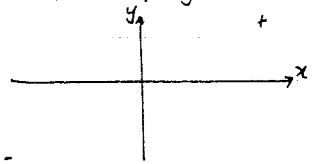
if $a>0$, $3x^2+a>0$ $\therefore P(x)>0$

$\therefore P(x)$ is monotonic increasing throughout

If $x \to -\infty$, $P(x) \to -\infty$

and If $x \to \infty$, $P(x) \to \infty$

and $P(x)$ is a polynomial

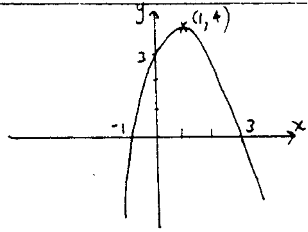


$\therefore P(x)$ must cut x-axis once only

$\therefore P(x)=0$ has exactly one root if $a>0$

3

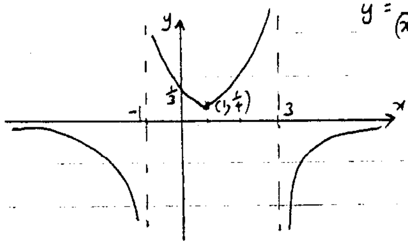
c) (i)



$y=(x+1)(3-x)$

1

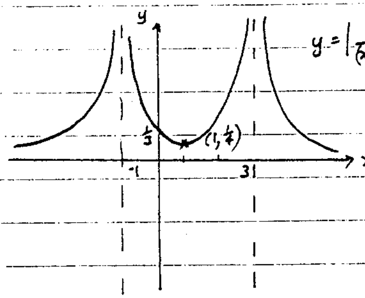
(ii)



$y = \frac{1}{(x+1)(3-x)}$

2

(iii)

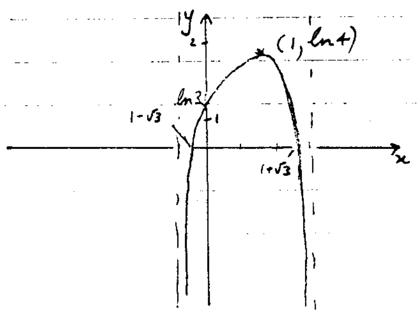


$y = \left| \frac{1}{(x+1)(3-x)} \right|$

2

(iv)

$(x+1)(3-x)>0 \quad \therefore \quad -1<x<3$



4

4 a) $xy = c^2$

$\therefore xy' + y = 0$ by implicit diff.

$\therefore y' = \frac{-y}{x}$

at $(ct, \frac{c}{t})$ m of tan $= -\frac{c/t}{ct} = \frac{-1}{t^2}$

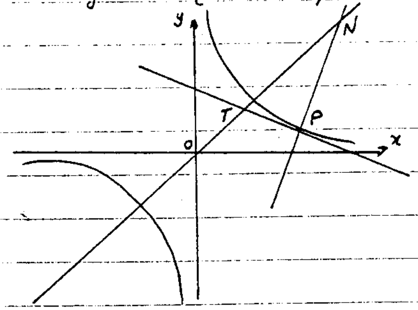
$\therefore$ m of norm $= t^2$

$y - y_1 = m(x-x_1)$

$y - \frac{c}{t} = t^2(x-ct)$

2

$\therefore \quad y = t^2x + \frac{c}{t} - ct^3$ is eq'n of norm at P



to find N, let $y=x$

$\therefore x = t^2x + \frac{c}{t} - ct^3$

$(t^2-1)x = c(t^3-\frac{1}{t})$

$x = \frac{c}{t}(\frac{t^4-1}{t^2-1}) = \frac{c}{t}(t^2+1)$

1

$\therefore$ N is $(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1))$

to find T, we need the eqn of tangent at P

$y - y_1 = m(x-x_1)$

$y - \frac{c}{t} = -\frac{1}{t^2}(x-ct)$

$t^2y - ct = -x + ct$

$\therefore x + t^2y - 2ct = 0$

let $y=x$

$x + t^2x = 2ct$

$x(t^2+1) = 2ct$

$x = \frac{2ct}{t^2+1}$

1

$\therefore$ T is $\left(\frac{2ct}{(t^2+1)}, \frac{2ct}{t^2+1} \right)$

(iii)

$\therefore OT = \sqrt{\left(\frac{2ct}{1+t^2}-0\right)^2 + \left(\frac{2ct}{1+t^2}-0\right)^2}$

$= \sqrt{2 \cdot \frac{4c^2t^2}{(1+t^2)^2}}$

$= \sqrt{2} \cdot \frac{2ct}{1+t^2}$

$ON = \sqrt{\left(\frac{c}{t}(t^2+1)-0\right)^2 + \left(\frac{c}{t}(t^2+1)-0\right)^2}$

$= \sqrt{2 \cdot \frac{c^2}{t^2}(t^2+1)^2}$

$= \sqrt{2} \cdot \frac{c}{t}(t^2+1)$

$\therefore OT \cdot ON = \sqrt{2} \cdot \frac{2c\cancel{t}}{\cancel{1+t^2}} \times \frac{c}{\cancel{t}}(\cancel{t^2+1}) \cdot \sqrt{2}$

3

$= 4c^2$

b)(i) $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$\therefore e^2 = \frac{9}{25}$

$e = 3/5$

4

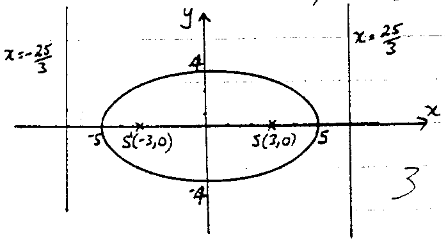
3

$\therefore ae = 5 \times \frac{3}{5} = 3$

$\frac{a}{e} = \frac{5}{3/5} = \frac{25}{3}$

$\therefore$ eccentricity is $3/5$, foci $S(3,0)$ and $S'(-3,0)$

and directrices are $x = \frac{25}{3}$, $x = \frac{-25}{3}$



3

(ii) $PS + PS' = 2a$

$= 10$

$\therefore 2 + PS' = 10$

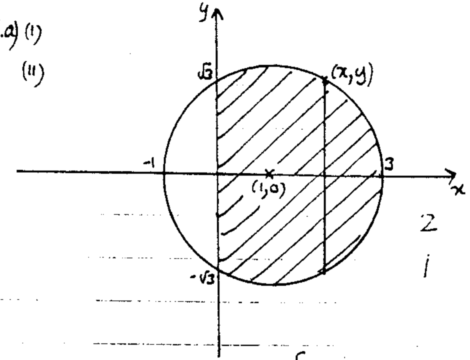
$PS' = 8$

2

$\therefore$ length of PS' is 8 units

5.a) (I)

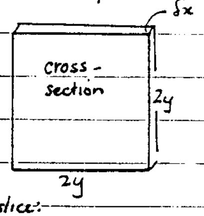
(II)



2

i

(III)



~~volume of slice:~~

$A = 4y^2$

$\therefore \delta V = 4y^2 . \delta x$

$= 4(4-(x-1)^2)dx$

$= 4(4-x^2+2x-1)dx$

$= 4(3+2x-x^2)dx$

summing all slices

$\therefore V = \lim_{\delta x \to 0}\left(4.\sum_0^3 (3+2x-x^2)\delta x\right.$

$= 4\int_0^3 (3+2x-x^2)\,dx$

$= 4\left[3x+x^2-\frac{x^3}{3}\right]_0^3$

$= 4(9+9-9-0-0+0)$

$= 36$

$\therefore$ volume of solid is 36 unit^3

5

b (I) $I_n = \int_0^1 x^n . e^{2x} . dx$

$\int uv'dx = uv - \int vu'dx$

$= \left[x^n . \frac{e^{2x}}{2}\right]_0^1 - \int_0^1 \frac{e^{2x}}{2} . nx^{n-1} . dx$

$= \frac{e^2}{2} - \frac{n}{2}\int_0^1 x^{n-1} . e^{2x}\,dx$

$= \frac{e^2}{2} - \frac{n}{2} \times I_{n-1}$

$= \frac{1}{2}(e^2 - n \times I_{n-1})$

4

(II) $I_0 = \int_0^1 x^0 . e^{2x}dx$

$= \int_0^1 e^{2x}dx$

$= \left[\frac{e^{2x}}{2}\right]_0^1$

$= \frac{1}{2}(e^2-1)$

$I_1 = \frac{1}{2}(e^2 - 1 \times I_0)$

$= \frac{1}{2}(e^2 - \frac{1}{2}(e^2-1))$

$= \frac{1}{4}(2e^2-e^2+1)$

$= \frac{1}{4}(e^2+1)$

$I_2 = \frac{1}{2}(e^2 - 2 \times I_1)$

$= \frac{1}{2}(e^2 - 2 \times \frac{1}{4}(e^2+1))$

$= \frac{1}{4}(2e^2-e^2-1)$

$= \frac{1}{4}(e^2-1)$

$I_3 = \frac{1}{2}(e^2 - 3 \times I_2)$

$= \frac{1}{2}(e^2 - 3 \times \frac{1}{4}(e^2-1))$

= [illegible]

= [illegible]

3

$= \frac{1}{8}(4e^2-3e^2+3)$

$= \frac{1}{8}(e^2+3)$

6 a) (α) In Δ's ABE and DBC

∠ABE = ∠DBC (data)

∠BAC = ∠BDC (∠'s at circumf. subt same arc are equal)

∴ Δ ABE ||| Δ DBC (2 prs corr. ∠'s equal)

(β) In Δ ABD and BCE

∠ABD = ∠ABE + ∠EBD

∠EBC = ∠DBC + ∠EBD

But ∠ABE = ∠DCB

∴ ∠ABD = ∠EBC

Also ∠BDA = ∠BCE (∠'s at circumf. subt. by same arc equal)

∴ ΔABD ||| ΔEBC (2 prs corr. ∠'s equal)

(II) In Δ's ABE and BCD

$\frac{AB}{DB} = \frac{BE}{BC} = \frac{AE}{DC}$

$\therefore AB \times DC = DB \times AE$ .... (1)

In Δ's ABD and ΔEBC

$\frac{AB}{EB} = \frac{BD}{BC} = \frac{AD}{EC}$

$\therefore AD \times BC = BD \times EC$ .... ②

① + ②

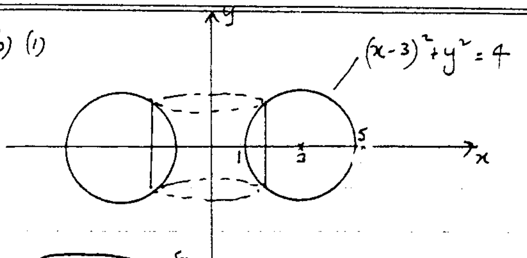
$AB \times DC + AD \times BC = DB \times AE + BD \times EC$

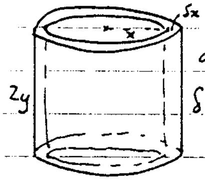
$= DB(AE + EC)$

$= DB \times AC$

$\therefore AB \times DC + AD \times BC = AC \times BD$

b) (I)





a typical shell

$\delta V = \pi((x+\delta x)^2 - x^2) . 2y$

$= 2\pi y(\cancel{x^2} + 2x\delta x + (\delta x)^2 - \cancel{x^2})$

$= 2\pi y(2x\delta x + (\delta x)^2)$

we may ignore $(\delta x)^2$ as insignificant

$\therefore \delta V = 2\pi y . 2x\delta x$

$= 4\pi xy\,\delta x$

since $y^2 = 4-(x-3)^2$

$y = \sqrt{4-(x-3)^2}$

$\therefore \delta V = 4\pi x\sqrt{4-(x-3)^2} . \delta x$

summing the shells,

$V = \lim_{\delta x \to 0}\sum_1^5 4\pi x\sqrt{4-(x-3)^2} . \delta x$

$= 4\pi\int_1^5 x\sqrt{4-(x-3)^2}\,dx$

(II) let $x-3 = 2\sin\theta$ $\therefore dx = 2\cos\theta\,d\theta$

and $x = 3+2\sin\theta$

$$\boxed{\begin{aligned}\therefore V &= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+2\sin\theta) . \sqrt{4-4\sin^2\theta} . 2\cos\theta\,d\theta \\ &= 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+2\sin\theta) . \cos\theta . \cos\theta\,d\theta\end{aligned}}$$

at $x=5$, $2 = 2\sin\theta$ $\therefore \theta = \frac{\pi}{2}$

at $x=1$ $-2 = 2\sin\theta$ $\therefore \theta = -\frac{\pi}{2}$

$V = 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3\cos^2\theta + 2\cos^2\theta\sin\theta)\,d\theta$

$= 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{3}{2}(1+\cos 2\theta) - 2\cos^2\theta . (-\sin\theta)\,d\theta$

6 cont'd.

$= 16\pi\left[\frac{3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - 2\frac{\cos^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 16\pi\left(\frac{3}{2}\left(\frac{\pi}{2}+0\right) - \cancel{\frac{2}{3}} - \frac{3}{2}\left(\frac{-\pi}{2}-0\right) + \cancel{\frac{2}{3}}\right)$

$= 16\pi\left(\frac{3}{4}\pi + \frac{3\pi}{4}\right)$

$= 24\pi^2$ units ^{3}

Q7(a) $I = \int_1^{\sqrt{x}} \frac{dt}{t}$

$= \left[\ln t\right]_1^{\sqrt{3}}$

$= \ln\sqrt{x} - \ln 1$

$= \frac{1}{2}\ln x$

since $0 < \frac{1}{2}\ln x < \sqrt{x}$

$\frac{0}{x} < \frac{\frac{1}{2}\ln x}{x} < \frac{\sqrt{x}}{x}$

$0 < \frac{\ln x}{x} < \frac{2}{\sqrt{x}}$

as $x \to \infty$, $\frac{2}{\sqrt{x}} \to 0$

$\therefore 0 < \frac{\ln x}{x} < 0$ as $x \to \infty$

ie. $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$

b) for turning points, $\frac{dy}{dx} = 0$

$\frac{1 - \ln x}{x^2} = 0$

$\therefore \ln x = 1$

$x = e^1$

at $x = e$, $y = \frac{\ln e}{e} = \frac{1}{e}$

and $\frac{d^2y}{dx^2} = \frac{2e-3}{e^3} < 0$

$\therefore$ max turning point at $(e, \frac{1}{e})$

for pts of infl. $\frac{d^2y}{dx^2} = 0$ + changes sign

$\frac{2\ln x - 3}{x^3} = 0$

$\ln x = 3/2$

$x = e^{3/2}$

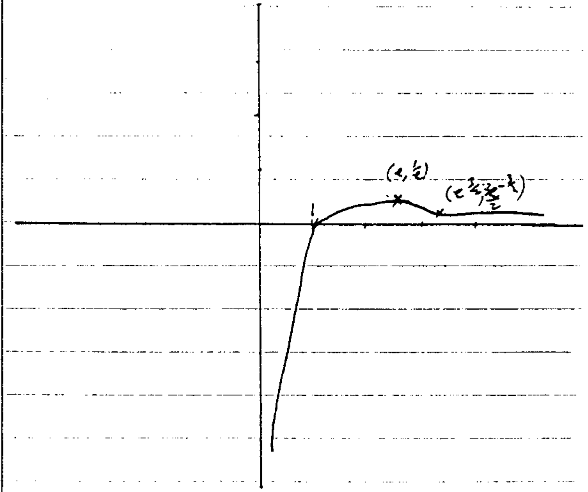
at $x = e^{\frac{3}{2}}$, $y = \frac{\ln e^{\frac{3}{2}}}{e^{\frac{3}{2}}} = \frac{3e^{-\frac{3}{2}}}{2}$

and

x	4.4	$e^{\frac{3}{2}}$	4.5
$\frac{d^2y}{dx^2}$	−		+

$\therefore$ changes sign

$\therefore$ pt of infl at $(e^{\frac{3}{2}}, \frac{3e^{-\frac{3}{2}}}{2})$



7 c)(i) target: angle of proj. is 0, speed −V ms⁻¹

horizontal:

$\ddot{x} = 0$ *

$\dot{x} = \int 0\, dt$

$= c$

at $t = 0$, $\dot{x} = V\cos 0$

$= V$

$\therefore \dot{x} = -V$ * ①

$x = \int -V\, dt$

$= -Vt + c_1$

at $t = 0$, $x = a$

$a = 0 + c_1$

$\therefore x = -Vt + a$

$= a - Vt$ * ②

vertical

$\ddot{y} = -g$ *

$\dot{y} = \int -g\, dt$

$= -gt + c_2$

at $t = 0$, $\dot{y} = -V\sin 0$

$= 0$

$\therefore 0 = 0 + c_2$

$\therefore \dot{y} = -gt$ * ③

$y = \int -gt\, dt$

$= -\frac{gt^2}{2} + c_3$

at $t = 0$, $y = h$

$\therefore h = 0 + c_3$

$\therefore y = -\frac{1}{2}gt^2 + h$

$= h - \frac{1}{2}gt^2$ * ④

(ii) shell

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g$

$\dot{x} = U\cos\theta$ ⑤ $\qquad$ $\dot{y} = -gt + U\sin\theta$ ⑦

$x = Ut\cos\theta$ ⑥ $\qquad$ $y = -\frac{1}{2}gt^2 + Ut\sin\theta$ (

at $x = \frac{a}{2}$

~~$\frac{a}{2} = a - Vt$ from ②~~

~~$\therefore Vt = \frac{a}{2}$~~

~~$t = \frac{a}{2V}$~~

$\frac{a}{2} = Ut\cos\theta$ from ⑥

$\therefore t = \frac{a}{2U\cos\theta}$

So at $t = \frac{a}{2U\cos\theta}$

height of target = height of shell

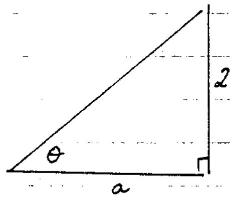
$-\frac{1}{2}gt^2 + h = -\frac{1}{2}gt^2 + Ut\sin\theta$, ④ = (

$h = Ut\sin\theta$

$= \cancel{U} \cdot \frac{a}{2\cancel{U}\cos\theta} \cdot \sin\theta$

$\therefore \frac{2h}{a} = \tan\theta$ ⑨

So the gun is aimed thus:



$\therefore$ gun is aimed at a point 2h metres above ground

ie aimed h metres above aeroplane

7b(ii) To prove $U = \frac{V}{a}\sqrt{a^2+4h^2}$

from ⑨ $h = \frac{a\tan\theta}{2}$

$\therefore RHS = \frac{V}{a}\sqrt{a^2+a^2\tan^2\theta}$

$= \frac{V}{a}.a\sqrt{1+\tan^2\theta}$

$= V.\sqrt{\sec^2\theta}$

$= V.\sec\theta$

at $x = \frac{a}{2}$

$\frac{a}{2} = a - Vt$ from ②

$\therefore t = \frac{a}{2V}$

also $\frac{a}{2} = Ut\cos\theta$

$\therefore t = \frac{a}{2U\cos\theta}$

$\therefore \frac{a}{2V} = \frac{a}{2U\cos\theta}$

$\therefore \frac{2V}{a} = \frac{2U\cos\theta}{a}$

$\frac{V}{U} = \cos\theta$

$\therefore \sec\theta = \frac{U}{V}$

So $RHS = V.\frac{U}{V}$

$= U$

$= LHS$

$\therefore U = \frac{V}{a}\sqrt{a^2+4h^2}$

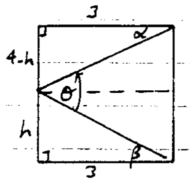
8. a (i) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A-B}$

$= \frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B}$

$= \dfrac{\frac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \frac{\sin B\cancel{\cos A}}{\cancel{\cos A}\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$

$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(ii)



$\theta = \alpha + \beta$

$\tan\alpha = \frac{4-h}{3}$, $\tan\beta = \frac{h}{3}$

$\therefore \tan\theta = \tan(\alpha+\beta)$

$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{4-h}{3} + \frac{h}{3}}{1 - \frac{4-h}{3}.\frac{h}{3}}$

$= \frac{12}{9-(4-h)h}$

$= \frac{12}{9-4h+h^2}$

$\therefore \theta = \tan^{-1}\frac{12}{9-4h+h^2}$

$\frac{d\theta}{dh} = \frac{1}{1+\left(\frac{12}{9-4h+h^2}\right)^2} \times -12(9-4h+h^2)^{-2} \times (-4+2h$

$= \dfrac{1}{\frac{(9-4h+h^2)^2+144}{\cancel{(9-4h+h^2)^2}}} \times \frac{-24(h-2)}{\cancel{(9-4h+h^2)^2}}$

Q8 cont'd

a) ii) $\frac{d\theta}{dh} = \frac{-24(h-2)}{(9-4h+h^2)^2+144}$

for max/min values of θ, $\frac{d\theta}{dh} = 0$

ie $-24(h-2) = 0$, $(9-4h+h^2)^2+144 > 0$

$h = 2$

Check

h	2^-	2	2^+
$\frac{d\theta}{dh}$	$+$		$-$
	/		\

$\therefore$ max value of θ occurs if $h = 2$

b) period $= 2 \times 6\frac{1}{4}$ hrs

$= 750$ mins

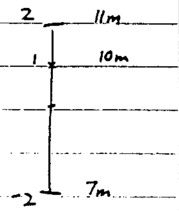
$P = \frac{2\pi}{n}$

$\therefore \frac{2\pi}{n} = 750$

$n = \frac{2\pi}{750} = \frac{\pi}{375}$

difference between high + low tides is 4 m

$\therefore$ amp $= 2$



since it is SHM,

$\ddot{x} = -n^2x$

$\therefore x = a\cos(nt+\alpha)$

$= 2\cos\left(\frac{\pi t}{375}+\alpha\right)$

at $t=0$, $x=2$ (at previous H Tide, 2.50 am)

$2 = 2\cos\alpha$

$\cos\alpha = 1$

$\therefore \alpha = 0$

$\therefore x = 2\cos\frac{\pi}{375}t$

at $x = 1$, depth of water is 10m

$\therefore 1 = 2\cos\frac{\pi}{375}t$

$\cos\frac{\pi t}{375} = \frac{1}{2}$

$t = \frac{375}{\pi}.\cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{375}{\pi} \times \frac{\pi}{3}$

$= 125$ mins, 625 min, etc

$\therefore$ depth of 10 m next occurs 625 mins after 2.50 ~~pm~~ am, ie 1.15 PM

ie pm is the earliest time

(125 mins is before next Low tide ∴ too early)