Total marks - 120
Attempt Questions 1-8
All questions are of equal value

Answer each question on a NEW PAGE.

Marks

Question 1 (15 marks) Start a NEW page.

(a) Find $\int \frac{\cos^3 x}{\sin^2 x} dx$. **3**

(b) Find $\int \operatorname{cosec} x \, dx$. **3**

(c) Find $\int \frac{2x-1}{x^2-6x+10} dx$. **4**

(d) Let $I_n = \int_1^e x(\ln x)^n dx$ for $n = 0,1,2,3,\ldots$ **5**

(i) Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$.

(ii) Hence find $\int_1^e x(\ln x)^2 dx$.

Marks

Question 2 (15 marks) Start a NEW page.

(a) If $z = -1 + i\sqrt{3}$, express each of the following in the form $a+ib$ where a and b are real **5**

(i) $\bar{z}$.

(ii) z^2 .

(iii) $\frac{1}{z}$.

(iv) z^6 .

(b) Given $z_1 = -1 - i$ and $z_2 = 3 + i$, draw neat labeled sketches to show the locus of z where: **5**

(i) $|z - z_1| \le |z - z_2|$.

(ii) $0 \le \arg(z - z_1) \le \frac{\pi}{4}$.

(iii) $\arg(z - z_1) = \arg(z - z_2)$.

(c) The complex number z satisfies the equation $z\bar{z} + 2iz = 12 + 6i$.
Find all possible values of z. **3**

(d) The quadratic equation $z^2 - (1 + i)z + 2i = 0$ has roots α, β.
Find, in simplest form, the value of $\alpha^{-2} + \beta^{-2}$. **2**

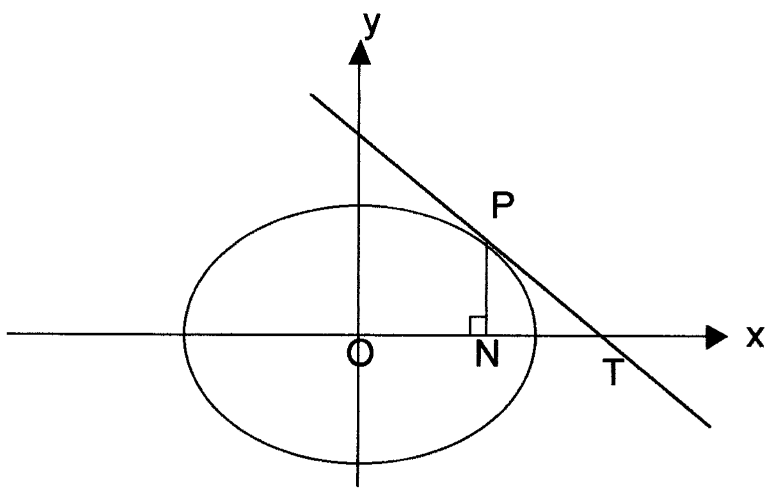
Marks

Question 3 (15 marks) Start a NEW page.

(a) For the conic $9x^2 - 16y^2 = 144$, sketch the curve, showing foci, directrices and asymptotes. **5**

(b) The tangent at $P(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ cuts the x axis at T. The perpendicular PN is drawn to the x axis.

Prove that ON.OT = a^2. **5**



Question 3 continues on page 5

Marks

Question 3 continued

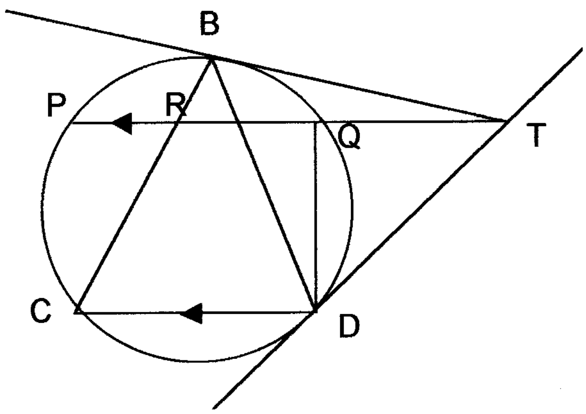
(c) PQ and CD are parallel chords of a circle.
The tangent at D cuts PQ extended at T.
B is the point of contact of the other tangent from T to the circle.
BC meets PQ at R. **5**

Copy the diagram onto your answer sheet.

(i) Prove that $\angle BDT = \angle BRT$.

(ii) Prove that B, T, D, R are concyclic.

(iii) Prove that $\angle BRT = \angle DRT$.



Marks

Question 4 (15 marks) Start a NEW page.

(a) The cubic equation $x^3 - 2x^2 - 3x - 4 = 0$ has roots α, β, γ **5**

(i) Find the value of $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\alpha\gamma} + \dfrac{\gamma}{\alpha\beta}$

(ii) Form the equation with integer coefficients whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$

(b) The polynomial $P(z) = z^4 - 2z^3 - 7z^2 + 26z - 20$ has a zero at $z = 2 + i$. Find all of the zeros of $P(z)$ **4**

(c) $\mathrm{P}\left(cp, \dfrac{c}{p}\right)$, $p > 0$, and $\mathrm{Q}\left(cq, \dfrac{c}{q}\right)$, $q > 0$, are two points on the rectangular hyperbola $xy = c^2$. The tangents at P and Q intersect at R. Given that the equation of chord PQ is $x + pqy = c(p + q)$. **6**

(i) Find the equation of the tangent at P.

(ii) Find the coordinates of R.

(iii) If the secant PQ passes through $(3c, 0)$, find the locus of R and state any restrictions on the locus.

Marks

Question 5 (15 marks) Start a NEW page.

(a) The region bounded by the curve $y = 2x - x^2$ is rotated about the line $x = 2$ to form a solid of revolution. By taking slices perpendicular to the line $x = 2$, find the volume of the solid. **5**

(b) The Great Pyramid of Cheops is approximately 150 metres high and its base is a square of approximate area 5 hectares. **4**

(i) Show that the area of the cross-section of a square pyramid at height y metres above the base is given by $A(y) = \left(\frac{h-y}{h}\right)^2 \times A$, where A is the area of the base, and h is the height of the pyramid.

(ii) Use the slice technique to find the volume of the Great Pyramid of Cheops

(c) On a certain day, the depth of water in a harbour at high tide at 5 am is 9 metres. At the following low tide at 11.20 am the depth is 3 metres. Find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 metres of water is required. Assume that the tides are undergoing simple harmonic motion. **6**

Marks

Question 6 (15 marks) Start a NEW page.

(a) (i) Let OABC be a square on the Argand diagram where O is the origin and neither A nor C is on the axes.
The points A and C represent the complex numbers z and iz respectively.

Show that B represents the complex number $z(1+i)$ **4**

(ii) The square is now rotated about O through 45^0 in an anticlockwise direction to OA'B'C'.

Find the complex number represented by the point B'.

(b) (i) Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2$

5

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$, using the substitution $u=\frac{\pi}{2}-x$

(c) If $f(x)=(x-1)(x-3)$ sketch the following curves, showing all intercepts, asymptotes and turning points. Draw a separate graph for each. **6**

(i) $y=\frac{1}{f(x)}$

(ii) $y=[f(x)]^2$

(iii) $y^2=f(x)$

(iv) $|y|=f(x)$

Marks

Question 7 (15 marks) Start a NEW page.

(a) (i) Sketch $y = \dfrac{1}{x^2 + 1}$ and $y = \dfrac{x^2}{x^2 + 1}$ showing the coordinates of their points of intersection.

9

(ii) The region bounded by these curves is rotated about the y axis to form a solid of revolution. By considering the solid as the sum of cylindrical shells, find the volume.

(b) The Fibonacci Sequence, F_n , is defined by :

$F_1 = 1$ $\qquad F_2 = 1$ $\qquad F_{n+2} = F_{n+1} + F_n$, for all $n \geq 1$ 6

(i) Prove that $F_8 = 3 \times F_5 + 2 \times F_4$

(ii) Prove, by mathematical induction, that F_{4n} is divisible by 3, for all positive integers n.

Marks

Question 8 (15 marks) Start a NEW page.

(a) (i) Use DeMoivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) Use this result to solve the equation $8x^3 - 6x + 1 = 0$ **5**

(iii) Deduce that $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

b) A curve has parametric equations $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$ **10**

(i) Show that $\frac{dy}{dx} = \cot\frac{\theta}{2}$

(ii) Hence show that $\frac{d^2y}{dx^2} = -\frac{1}{y^2}, y \neq 0$

(iii) Show that the curve has stationary points at $(n\pi, 2)$, for n odd.

(iv) Determine the nature of these stationary points.

(v) Sketch the curve, showing stationary points and intercepts on the axes.

(vi) Discuss the nature of the points at which the curve intersects the x axis.

End of the paper

MEREWETHER HIGH TRIAL HSC EXT. 2 MATHS 2002

Question 1

a $\int \frac{\cos^3 x}{\sin^2 x}\,dx = \int \frac{(1-\sin^2 x)}{\sin^2 x}.\cos x\,dx$ 1 mark

let $u = \sin x$
$du = \cos x\,dx$

$= \int (u^{-2} - 1)\,du$ 1 mark

$= -\frac{1}{u} - u$

$= -\operatorname{cosec} x - \sin x + c$ 1 mark

b $\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x}$

$t = \tan \frac{x}{2}$

$dx = \frac{2\,dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}$ 1 mark

$= \int \frac{dt}{t}$ 1 mark

$= \ln|t|$

$= \ln\left|\tan \frac{x}{2}\right| + c$ 1 mark.

OR $-\ln|\operatorname{cosec} x + \cot x| + c$

OR $\ln|\operatorname{cosec} x - \cot x| + c$

c $\int \frac{2x+1}{x^2-6x+10}\,dx = \int \frac{2x-6}{x^2-6x+10}\,dx + \int \frac{7}{x^2-6x+10}\,dx$ 1 mark

$= \ln|x^2-6x+10| + 7\int \frac{dx}{(x-3)^2+1}$ 2 marks

$= \ln|x^2-6x+10| + 7\tan^{-1}(x-3) + c$ 1 mark.

d (i) $I_n = \int_1^e x(\ln x)^n\,dx$

$u = (\ln x)^n$ $\quad v = \frac{x^2}{2}$

$u' = \frac{n}{x}(\ln x)^{n-1}$ $\quad v' = x$

1 mark u v u' v'

$\therefore I_n = \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$ 1 mark

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ 1 mark.

(ii) $\int_1^e x(\ln x)^2\,dx = I_2 = \frac{e^2}{2} - I_1$

$= \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2}I_0\right)$ 1 mark.

$= \frac{1}{2}\int_1^e x\,dx$

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$ 1 mark

$= \frac{1}{4}(e^2 - 1)$

Question 2

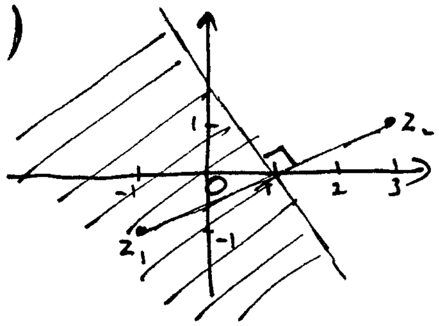
a (i) $\overline{z} = -1 - i\sqrt{3}$ 1 mark (ii) $z^2 = -2 - 2\sqrt{3}\,i$ 1 mark

(iii) $\frac{1}{z} = -\frac{1}{4} - \frac{\sqrt{3}}{4} i$ 1 mark (iv) $z^6 = \left(2 \operatorname{cis} \frac{2\pi}{3}\right)^6$ 1 mark

$= 64 \operatorname{cis} 4\pi$

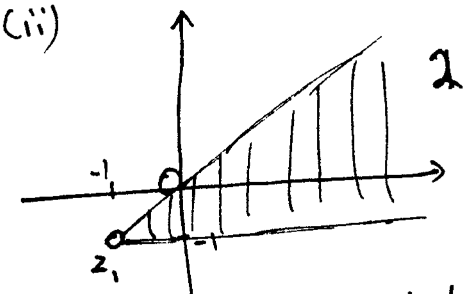
$= 64$ 1 mark.

b (i)



1 mark

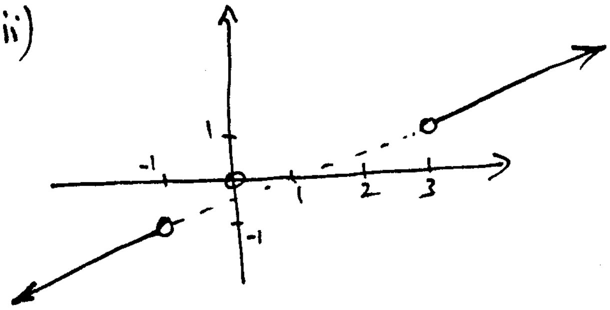
(ii)



2 marks

(lose 1 mark if z_1 included)

(iii)



2 marks

(given 1 mark if interval $z_1 z_2$ shown)

(lose 1 mark if z_1, z_2 included).

c Let $z = x + iy$ 1 mark

$\therefore x^2 + y^2 + 2ix - 2y = 12 + 6i$

Equating real & imaginary parts $\Rightarrow x = 3$ 1 mark for equating real, imaginary parts

$\therefore 9 + y^2 - 2y = 12$

$y = 3, -1$

$\therefore z = 3 + 3i$ or $z = 3 - i$ 1 mark.

d $z^2 - (1+i)z + 2i = 0$

$\left.\begin{aligned} \therefore \alpha + \beta &= 1 + i \\ \alpha\beta &= 2i \end{aligned}\right\}$ 1 mark.

Now $\alpha^{-2} + \beta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{(1+i)^2 - 4i}{-4}$

$= \frac{i}{2}$ 1 mark.

Question 3

a $9x^2 - 16y^2 = 144$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

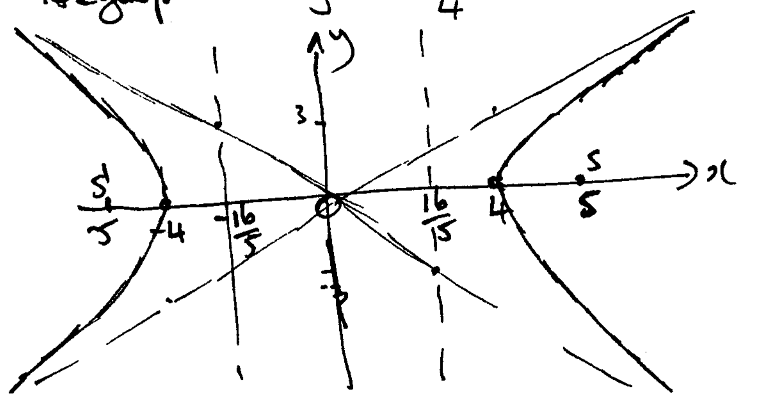
$a = 4,\ b = 3$ $\quad 9 = 16(e^2 - 1)$

$e = \frac{5}{4}$ 1 mark.

Foci $(\pm 5, 0)$ 1 mark

Directrices $x = \pm\frac{16}{5}$ 1 mark

Asymptotes $y = \pm\frac{3x}{4}$ 1 mark.



1 mark.

b $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0$

$y' = -\frac{b^2x}{a^2y}$ 1 mark

At P, grad of tangent $= \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

Eq^n of tangent at P is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ 1 mark.

At T, $y=0 \implies$ $-ab\sin^2\theta = -b\cos\theta(x - a\cos\theta)$

$x - a\cos\theta = \frac{a\sin^2\theta}{\cos\theta}$

$x = a\cos\theta + \frac{a\sin^2\theta}{\cos\theta}$

$= \frac{a}{\cos\theta}(\cos^2\theta + \sin^2\theta)$

$= \frac{a}{\cos\theta}$

$\therefore$ T is $\left(\frac{a}{\cos\theta}, 0\right)$ 1 mark.

N is $(a\cos\theta, 0)$ 1 mark.

Thus $ON.OT = a^2$ 1 mark.

c (i) $\angle BDT = \angle BCD$ (angle in alternate segment) 1 mark

$\angle BCD = \angle BRT$ (corresponding angles in // lines) 1 mark

$\therefore \angle BDT = \angle BRT$

(ii) B, T, D, R are concyclic (BT subtends equal angles at R, D on same side of it) 1 mark.

(iii) $TB = TD$ (tangents from external point equal) 1 mark

$\therefore \angle BRT = \angle DRT$ (equal chords subtend equal angles at circumference) 1 mark.

Question 4

a $x^3 - 2x^2 - 3x - 4 = 0$

(i) $\alpha + \beta + \gamma = 2$
$\alpha\beta + \alpha\gamma + \beta\gamma = -3$
$\alpha\beta\gamma = 4$ } 1 mark

Now $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$ 1 mark

$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)}{\alpha\beta\gamma}$ 1 mark

$= \frac{4+6}{4}$

$= \frac{5}{2}$ 1 mark

(ii) Required eq'n is $\left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) - 4 = 0$

$1 - 2x - 3x^2 - 4x^3 = 0$

$4x^3 + 3x^2 + 2x - 1 = 0$ 1 mark

b $2+i$ is a root, real coefficients $\therefore 2-i$ is a root 1 mark

Together they form the quadratic factor $z^2 - 4z + 5$ 1 mark

$$\begin{array}{r} z^2+2z-4 \\ z^2-4z+5 \overline{)\, z^4 - 2z^3 - 7z^2 + 26z - 20} \end{array}$$ 1 mark

Now $z^2 + 2z - 4 = 0$ has roots $z = -1 \pm \sqrt{5}$ 1 mark

$\therefore$ the 4 zeros are $2 \pm i, -1 \pm \sqrt{5}$

c (i) Chord PQ given $x + pqy = c(p+q)$

As $q \to p$, eq'n tangent at P is $x + p^2y = 2cp$ 1 mark

(ii) $x + p^2y = 2cp \quad \ldots (1)$

$x + q^2y = 2cq \quad \ldots (2)$

(1) − (2) $(p^2 - q^2)y = 2c(p-q)$

$y = \frac{2c}{p+q}$ 1 mark

Sub. into (1) $x = \frac{2cpq}{p+q}$ 1 mark

$\therefore$ R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

(iii) PQ through $(3c, 0)$, $\therefore 3c = c(p+q)$

$p+q = 3$

Hence, at R, since $y = \frac{2c}{p+q}$, then $y = \frac{2c}{3}$ is the locus of R 1 mark.

with the restriction that R is in the first quadrant (as $p,q > 0$) and that R is between the hyperbola and the coordinate axes.

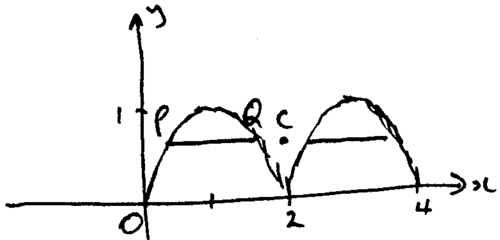
Now $y = \frac{2c}{3}$ cuts $xy = c^2$ when $x = \frac{3c}{2}$

$\therefore$ restriction is $0 < x < \frac{3c}{2}$

1 mark for $x > 0$
1 mark for $x < \frac{3c}{2}$

Question 5

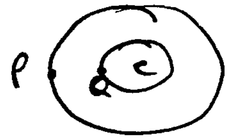
a



$$y = 2x - x^2$$
$$x^2 - 2x + y = 0$$
$$x = \frac{2 \pm \sqrt{4-4y}}{2}$$
$$x = 1 \pm \sqrt{1-y}$$

For P as shown $x = 1 - \sqrt{1-y}$

Take slice through $P(x,y)$ on $y = 2x - x^2$; thickness δy



$CP = 2 - x$
$CQ = x$

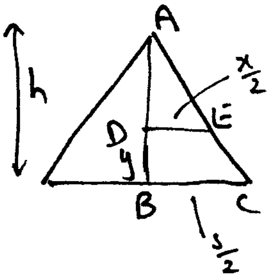
Area of cross-section is $A = \pi\left[(2-x)^2 - x^2\right]$
$= \pi(4 - 4x)$

Vol. of slice $\delta V = \pi(4-4x)\delta y$

$$\text{Volume} = \lim_{\delta y \to 0} \sum_{y=0}^{1} \pi(4-4x)\delta y$$
$$= 4\pi \int_0^1 (1-x)\,dy$$
$$= 4\pi \int_0^1 (1-y)^{\frac{1}{2}}\,dy$$
$$= 4\pi \left[-\tfrac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$$
$$= \frac{8\pi}{3} \text{ cu. units}$$

Marking
Correct radii of cross-section 1 mark
Correct area of cross-section 1 mark
$V = 4\pi \int_0^1 (1-x)\,dy$ 1 mark
Obtaining x as function of y 1 mark
Correct answer 1 mark

b (i) Let base of square pyramid be s units square & let x be side length of square cross-section at height y



$\triangle ADE \,|||\, \triangle ABC$

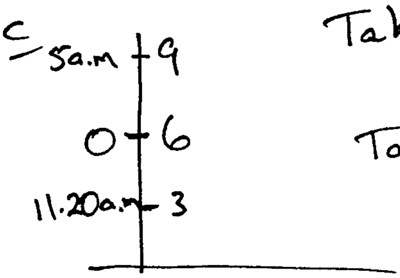
$$\therefore \frac{\frac{x}{2}}{\frac{s}{2}} = \frac{h-y}{h}$$
$$x = \frac{(h-y)}{h}.s$$

$\therefore$ Area of square at height y is $\left(\frac{h-y}{h}\right)^2 \times s^2$
$= \left(\frac{h-y}{h}\right)^2 \times A$ 1 mark

(ii) Vol. slice $\delta V = \left(\frac{h-y}{h}\right)^2 .A.\delta y$ 1 mark

Total volume $V = \int_0^h \left(\frac{h-y}{h}\right)^2 A\,dy$ 1 mark
$$= \frac{50000}{150^2}\int_0^{150}(150-y)^2\,dy$$ 1 mark
$$= 2\,500\,000 \text{ m}^3$$

c



Take O at 6m level as centre of motion

$\therefore a = 3$ — 1 mark

Take $t = 0$ at 5 a.m

Period $T = 12$ hr 40 mins

$= \frac{38}{3}$ hrs.

$\therefore n = \frac{3\pi}{19}$ — 1 mark

Now $x = a\cos(nt + \alpha)$

$x = 3\cos\left(\frac{3\pi t}{19} + \alpha\right)$

But $x = 3$ when $t = 0$ $\therefore 3 = 3\cos\alpha$

$\alpha = 0$ — 1 mark for <u>proving</u> $\alpha = 0$

Using $x = 3\cos\left(\frac{3\pi t}{19}\right)$

require t when $x = 1.5$ — 1 mark.

$\cos\frac{3\pi t}{19} = \frac{1}{2}$

$\frac{3\pi t}{19} = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

$t = \frac{19}{9}, \frac{95}{9}, \ldots$

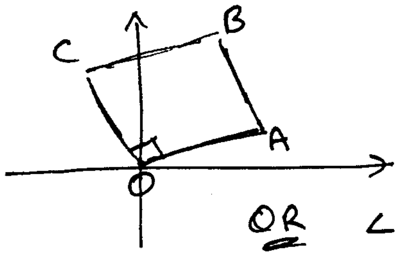
$= 2\frac{1}{9}$ hrs, $10\frac{5}{9}$ hrs, ... — 1 mark

Thus depth is 7.5m or more between 5 a.m and 7.07 am and then from 3.33 p.m until . . .

$\therefore$ Latest time before noon is 7.07 a.m. — 1 mark

Question 6

a (i)



$\vec{OB} = \vec{OA} + \vec{AB}$

$= \vec{OA} + \vec{OC}$ 1 mark

$\therefore B$ represents $z + iz = z(1+i)$ 1 mark

OR $\angle BOA = 45°$
$BO = \sqrt{2} \times AO$ } 1 mark

$\therefore B$ represents $z \times \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

$= z \times \sqrt{2} \times \frac{1}{\sqrt{2}}(1+i)$ 1 mark.

$= z(1+i)$

(ii) B' represents $z(1+i)\operatorname{cis}\frac{\pi}{4} = z(1+i)\frac{1}{\sqrt{2}}(1+i)$ 1 mark

$= \frac{z}{\sqrt{2}}(1+i)^2$

$= \sqrt{2}\, z i$ 1 mark

b (i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$ 1 mark

Let $t = \tan\frac{x}{2}$

$= \int_0^1 \frac{2dt}{2+2t}$

$= \int_0^1 \frac{dt}{1+t}$

$= \Big[\ln|1+t|\Big]_0^1$

$= \ln 2$ 1 mark.

(ii) $I = \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$

Let $u = \frac{\pi}{2} - x$

$\therefore du = -dx$

$\therefore I = -\int_{\frac{\pi}{2}}^{0} \frac{(\frac{\pi}{2}-u)}{1+\cos(\frac{\pi}{2}-u)+\sin(\frac{\pi}{2}-u)}\,du$ 1 mark

$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\,du$ 1 mark

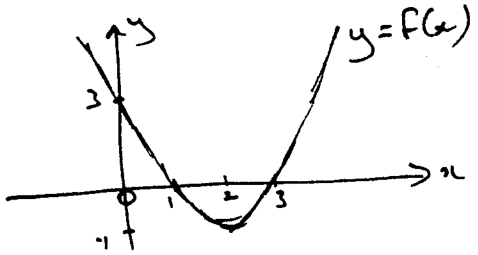
$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{1+\sin u+\cos u}\,du - I$ 1 mark.

$2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{du}{1+\sin u+\cos u}$

$= \frac{\pi}{2}\ln 2$ 1 mark

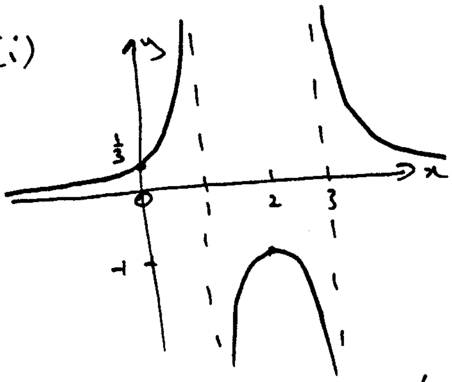
$\therefore I = \frac{\pi}{4}\ln 2$

c



$y = f(x)$

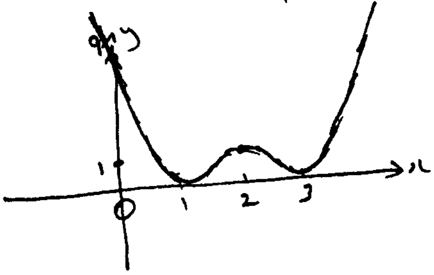
(i)



$y = \frac{1}{f(x)}$

1 mark.

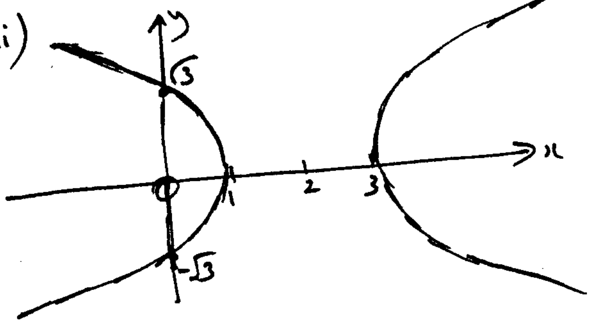
(ii)



$y = [f(x)]^2$

2 marks

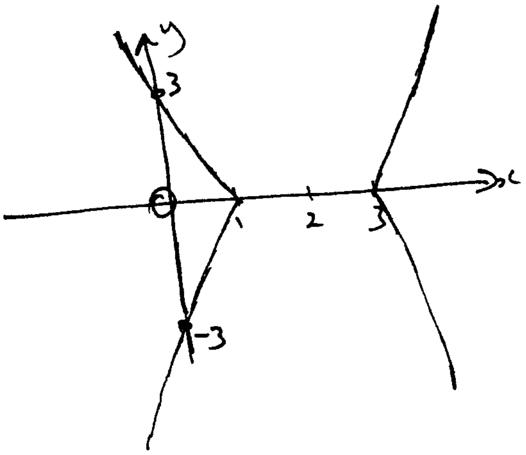
(iii)



$y^2 = f(x)$

2 marks

(iv)

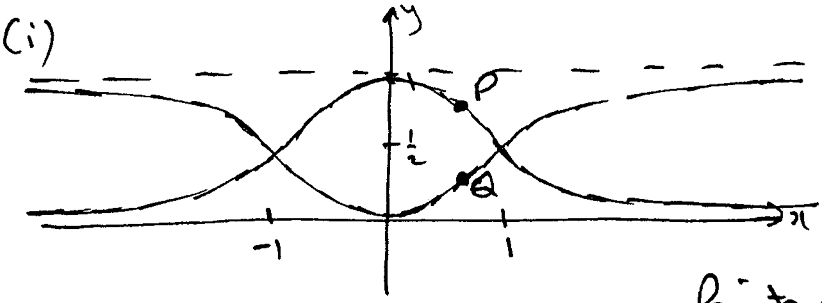


$|y| = f(x)$

1 mark.

Question 7

a (i)



$y = \frac{x^2}{x^2+1}$ 1 mark

$y = \frac{1}{x^2+1}$ 1 mark

Points of intersection 1 mark

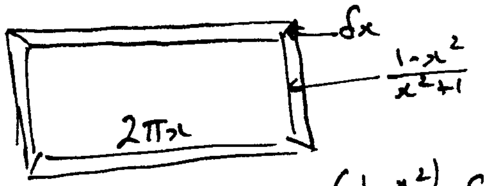
(ii) Take thin slice, through $P(x,y)$ on $y = \frac{1}{x^2+1}$ as shown, thickness δx. Rotate slice about y axis to form a thin-walled hollow cylindrical shell of: inner radius x

height $PQ = \frac{1}{x^2+1} - \frac{x^2}{x^2+1}$

$= \frac{1-x^2}{x^2+1}$ 1 mark

Vol. of shell

thickness δx



$\delta V = 2\pi x \frac{(1-x^2)}{x^2+1} \delta x$ 1 mark

$$\text{Volume} = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x \left(\frac{1-x^2}{x^2+1}\right) \delta x$$

$= 2\pi \int_0^1 \frac{-x^3+x}{x^2+1}\, dx$ 1 mark

$= 2\pi \int_0^1 \left(-x + \frac{2x}{x^2+1}\right) dx$ 1 mark

$= 2\pi \left[-\frac{x^2}{2} + \ln(x^2+1) \right]_0^1$ 1 mark

$= \cancel{2}\pi(2\ln 2 - 1)$ cu. units 1 mark

b (i) $F_8 = F_7 + F_6$

$= (F_6 + F_5) + (F_5 + F_4)$ 1 mark

$= (F_5 + F_4) + 2F_5 + F_4$

$= 3F_5 + 2F_4$ 1 mark

(ii) Prove true for $n=1$

$F_4 = 3$, which is div. by 3 1 mark

Assume true for $n=k$

ie assume F_{4k} is div. by 3

ie assume $\exists$ integer M such that $F_{4k} = 3M$

Prove true for $n=k+1$ if true for $n=k$ 1 mark

ie prove F_{4k+4} is div. by 3 if F_{4k} is div. by 3

Now $F_{4k+4} = F_{4k+3} + F_{4k+2}$

$= (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k})$

$= (F_{4k+1} + F_{4k}) + 2 \times F_{4k+1} + F_{4k}$

$= 3 \times F_{4k+1} + 2 \times F_{4k}$

$= 3 \times F_{4k+1} + 6M$

$= 3(F_{4k+1} + 2M)$, which is div. by 3 as $F_{4k+1} + 2M$ is an integer   1 mark.

Conclusion

True for $n = k+1$ if true for $n = k$

But true for $n = 1$ 1 mark.

$\therefore$ true for all integer $n \geq 1$

Question 8

a (i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\therefore \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$

Equating real parts $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$ 1 mark.

(ii) $8x^3 - 6x + 1 = 0$

$4x^3 - 3x = -\frac{1}{2}$

Let $x = \cos\theta$

Eq'n becomes $4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$ 1 mark

$\cos 3\theta = -\frac{1}{2}$

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

Hence $8x^3 - 6x + 1 = 0$ has solution

$\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$ 1 mark.

(iii) Using prod. of roots $= -\frac{d}{a}$ 1 mark

$\cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} \cdot \cos\frac{8\pi}{9} = -\frac{1}{8}$

$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\cos\frac{\pi}{9}\right) = -\frac{1}{8}$

$\therefore \sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$ 1 mark.

b (i) $x = \theta - \sin\theta$ $\quad$ $y = 1 - \cos\theta$

$\frac{dx}{d\theta} = 1 - \cos\theta$ $\quad$ $\frac{dy}{d\theta} = \sin\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$ 1 mark

$= \frac{\sin\theta}{1-\cos\theta}$

let $t = \tan\frac{\theta}{2}$ $\quad$ $= \dfrac{\frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}$

$= \frac{2t}{2t^2}$

$= \frac{1}{t}$

$= \cot\frac{\theta}{2}$ 1 mark

(ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right)$

$$= \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$ 1 mark

$$= -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{1-\cos\theta}$$

$$= \frac{-1}{2\sin^2\frac{\theta}{2}\,(1-\cos\theta)}$$

$$= \frac{-1}{(1-\cos\theta)(1-\cos\theta)}$$

$$= -\frac{1}{y^2}$$ 1 mark.

(iii) Stat. pts when $\frac{dy}{dx} = 0$

$\cot\frac{\theta}{2} = 0$

$\frac{\theta}{2} = \frac{n\pi}{2}$, n odd

$\therefore \theta = n\pi$, n odd 1 mark

At these points, $x = n\pi - \sin(n\pi)$
$= n\pi$
and $y = 1 - \cos(n\pi)$
$= 2$

$\therefore$ Stat. pts at $(n\pi, 2)$ for n odd 1 mark.

(iv) Since $\frac{d^2y}{dx^2} = -\frac{1}{y^2}$, then for each stat. point, $\frac{d^2y}{dx^2} = -\frac{1}{4}$

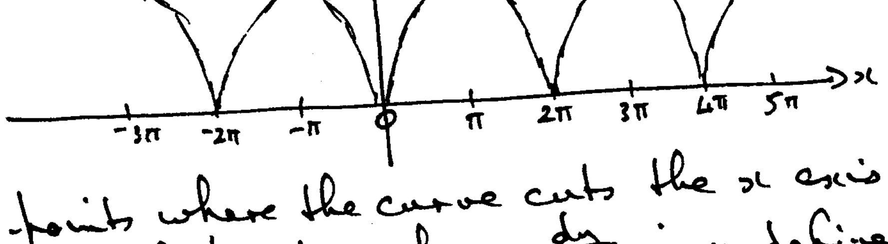
$\therefore$ <u>all</u> are max. turning points 1 mark.

(v) Cuts x axis when $y=0$

$\therefore 1-\cos\theta = 0$

$\theta = 0, \pm n\pi$ where n even 1 mark

Range is $0 \le y \le 2$ (as $y = 1-\cos\theta$ and $-1 \le \cos\theta \le 1$)



1 mark.

(vi) The points where the curve cuts the x axis are critical points, where $\frac{dy}{dx}$ is undefined and tangents are vertical. 1 mark