Total marks - 120
Attempt Questions 1-8
All questions are of equal value

Answer each question on a NEW PAGE.

Marks

Question 1 (15 marks) Start a NEW page.

(a) Evaluate (i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1 + \tan x} \, .dx$. **2**

(ii) $\displaystyle\int_0^{\ln \pi} e^x \sin\left(e^x\right) dx$ correct to 2 decimal places. **2**

(b) Find (i) $\displaystyle\int \frac{1 - x}{\sqrt{1 - x^2}} \, .dx$. **3**

(ii) $\displaystyle\int x \cos x \, .dx$ **2**

(c) Find $\displaystyle\int \frac{5t^2 + 3}{t\left(t^2 + 1\right)} \, .dt$. **3**

(d) Use the substitution $t = \tan \dfrac{\theta}{2}$, or otherwise to find

$$\int_0^{\frac{\pi}{2}} \frac{2}{1 + \cos\theta} \, .d\theta$$ **3**

Marks

Question 2 (15 marks) Start a NEW page.

(a) **9**

(i) Sketch the graph of $y = \dfrac{x+3}{x+4}$ showing clearly the coordinates of any point of intersection with the x and the y and the equations of any asymptotes.

(ii) Use the graph of $y = \dfrac{x+3}{x+4}$ in part (i) to find the set of values of x for which the function $y = x - \log_e(x+4)$ is increasing.

(iii) Use the graph of $y = \dfrac{x+3}{x+4}$ in part (i) to sketch on separate axes

(α) the graph of $|y| = \dfrac{x+3}{x+4}$.

(β) the graph of $y = \dfrac{(x+3)^2}{(x+4)(x+3)}$.

(b) Sketch the curve $y = x^2 + \dfrac{2}{x}$ showing all essential features. **6**
Use the graph to determine the nature and number of real roots of the equation $x^3 - kx + 2 = 0$ as 'k' varies.

Question 3 (15 marks) Start a NEW page. **Marks**

(a) (i) On an Argand diagram shade in the region containing all points representing the complex number z such that

$$|z-(1+i)| \leq 1 \text{ and } |z-(1+i)| \leq |z|.$$ **2**

(ii) Find the exact perimeter of the shaded region. **2**

(b) **6**

(i) Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$.

(ii) The tangents to the rectangular hyperbola $xy = c^2$ at the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $pq = 1$, intersect at R. Find the equation of the locus of R and state any restrictions on the values of x for this locus.

(c) **5**

(i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(ii) Hence show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

Marks

Question 4 (15 marks) Start a NEW page.

(a) Let $f(t) = t^3 + ct + d$, where c and d are constants. **6**

Suppose that the equation $f(t) = 0$ has three distinct real roots, t_1, t_2 and t_3.

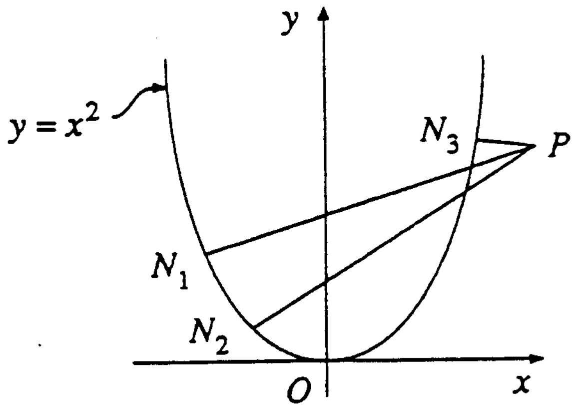
(i) Find $t_1 + t_2 + t_3$.

(ii) Show that $t_1{}^2 + t_2{}^2 + t_3{}^2 = -2c$.

(iii) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$.

Show that $27d^2 + 4c^3 < 0$.

(b) **5**



Consider the parabola $y = x^2$.

Some points (eg P) lie on three distinct normals $(PN_1,\ PN_2,\ \text{and } PN_3)$ to the parabola.

(i) Show that the equation of the normal to $y = x^2$ at the point (t, t^2) may be written as

$$t^3 + \left(\frac{1-2y}{2}\right)t + \left(\frac{-x}{2}\right) = 0$$

Question 4 continues on page 6

Question 4 (continued)

(ii) Suppose that the normals to $y = x^2$ at three distinct points $N_1\left(t_1, t_1^{\,2}\right)$, $N_2\left(t_2, t_2^{\,2}\right)$, and $N_3\left(t_3, t_3^{\,2}\right)$ all pass through $P\left(x_0, y_0\right)$.

Using the result of part (a) (iii), show that the coordinates of P satisfy

$$y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}.$$

(c) For the curve $xy(x + y) + 16 = 0$, show that **4**

$$\frac{dy}{dx} = -\frac{\left(y^2 + 2xy\right)}{x^2 + 2xy}$$

Hence find the equation of the tangent to the curve $xy(x + y) + 16 = 0$ at the point $(-2, -2)$.

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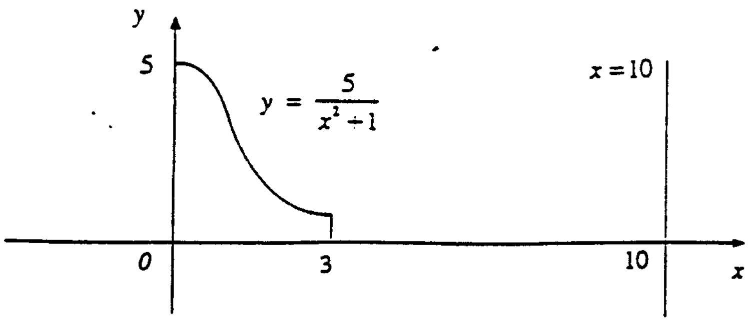
Marks

Question 5 (15 marks) Start a NEW page.

(a) **6**

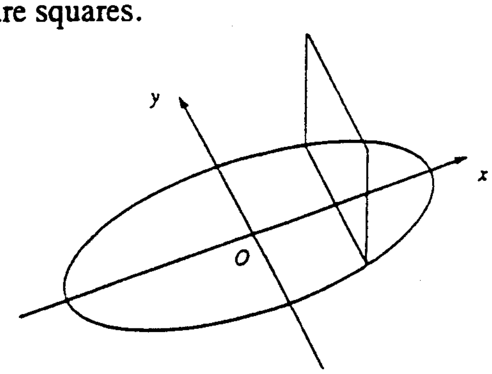


A circular flange is formed by rotating the region bounded by the curve $y = \frac{5}{x^2+1}$, the x axis and the lines $x = 0$ and $x = 3$, through one complete revolution about the line $x = 10$. (All measurements are in centimetres.)

(i) Use the method of cylindrical shells to show that the volume $V cm^3$ of the flange is given by $V = \int_0^3 \frac{(100\pi - 10\pi x)}{x^2+1}\,dx$.

(ii) Hence find the volume of the flange correct to the nearest cm^3.

(b) The base of a tent is in the shape of an ellipse with a major axis of 4 metres and a minor axis of 2 metres. Vertical cross sections taken perpendicular to the major axis of the base are squares. **5**



(i) If the major axis is taken to lie on the x axis and the minor axis is taken to lie on the y axis, show that the ellipse has equation $\frac{x^2}{4} + y^2 = 1$

(ii) Show that the volume $V m^3$ of the tent is given by $V = \int_{-2}^{2} \left(4 - x^2\right) dx$ and hence find the volume of the tent.

Question 5 continues on page 8

Marks

Question 5 (continued)

(c) 4

(i) Find the domain and range of the function $y = \sin\left(\cos^{-1} x\right)$.

(ii) Sketch, showing the important features, the graph of $y = \sin\left(\cos^{-1} x\right)$.

Marks

Question 6 (15 marks) Start a NEW page.

A body of mass one kilogram is project vertically upwards from the ground at a speed of 20 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time. 15

In the following questions, take the acceleration due to gravity to be 10 metres per second per second.

(a) While the body is travelling upwards the equation of motion is

$$\ddot{x} = -\left(10 + \frac{1}{40}v^2\right).$$

(i) Taking $\ddot{x} = v\frac{dv}{dx}$, calculate the greatest height reached by the particle.

(ii) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach this greatest height.

(b) Having reached its greatest height, the particle falls to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

(i) Write down the equation of motion of the particle as it falls.

(ii) Find the speed of the particle when it returns to its starting point.

Marks

Question 7 (15 marks) Start a NEW page.

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ where n is a non-negative integer. **6**

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$ when $n \geq 2$.

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ when $n \geq 2$.

(iii) Evaluate I_4.

(b) **9**

(i) Use de Moivre's theorem and the expansion of $(\cos\theta + i\sin\theta)^3$ to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) Deduce $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$.

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$.

(iv) Hence evaluate $\cos\frac{\pi}{9} \, . \cos\frac{2\pi}{9} \, . \cos\frac{4\pi}{9}$.

Marks

Question 8 (15 marks) Start a NEW page.

(a) The acceleration $a\,ms^{-2}$ of a particle P moving in a straight line is given by **6**

$$a = 3\left(1 - x^2\right),$$

where x metres is the displacement of the particle to the right of the origin. Initially the particle is at the origin and is moving with a velocity of $4\,ms^{-1}$.

(i) Show that the velocity $v\,ms^{-1}$ of the particle is given by

$$v^2 = 16 + 6x - 2x^3.$$

(ii) Will the particle ever return to the origin? Justify your answer.

(b) Let n be a positive integer. **9**

Consider the area bounded by the curve $y = \ln x$, the x axis and $x = n$. Use integration by parts to show that the value of this area is given by

$$\int_1^n \ln x.dx = n\ln n - n + 1.$$

Use the trapezoidal rule and 'n' function values to show that

$$\int_1^n \ln x.dx \doteqdot \frac{1}{2}\ln n + \ln\left[(n-1)!\right].$$

Hence deduce that $n! < e\sqrt{n}\,.\left(\frac{n}{e}\right)^n$.

Question 1

(a) (i) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx = \left[\ln(1+\tan x)\right]_0^{\frac{\pi}{4}}$

$= \ln 2$

(ii) $\int_0^{\ln\pi} e^x \sin e^x\,dx = -\left[\cos e^x\right]_0^{\ln\pi}$

$= \cos e^0 - \cos e^{\ln\pi}$

$= \cos 1 - \cos\pi$

$= 1.54$

(b) (i) $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int\left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right)dx$

$= \sin^{-1}x + \sqrt{1-x^2} + c$

(ii) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + c$

(c) $\frac{5t^2+3}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$

$\therefore 5t^2+3 = A(t^2+1) + Bt^2 + Ct$

$\therefore 5 = A+B$

$3 = A \Rightarrow B = 2$

$0 = C$

$\therefore \int \frac{5t^2+3}{t(t^2+1)}\,dt = \int\left(\frac{3}{t} + \frac{2t}{t^2+1}\right)dt$

$= 3\ln t + \ln(t^2+1) + c$

$= \ln t^3(t^2+1) + c$

(d) $t = \tan\theta/2$; $dt/d\theta = \frac{1}{2}\sec^2\theta/2$

$\therefore d\theta/dt = \frac{2}{1+\tan^2\theta/2}$

$d\theta = \frac{2}{1+t^2}\,dt$

$\int_0^{\frac{\pi}{2}} \frac{2}{1+\cos\theta}\,d\theta = \int_0^1 \frac{2}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{4}{1+t^2+1-t^2}\,dt$

$= \int_0^1 \frac{4}{2}\,dt$

$= \left[2t\right]_0^1$

$= 2$

OR. $\cos\theta = 2\cos^2\theta/2 - 1$

$1+\cos\theta = 2\cos^2\theta/2$

$\int_0^{\frac{\pi}{2}} \frac{2\,d\theta}{1+\cos\theta} = \int_0^{\frac{\pi}{2}} \frac{2}{2\cos^2\theta/2}\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sec^2\theta/2\,d\theta$

$= 2\left[\tan\theta/2\right]_0^{\pi/2}$

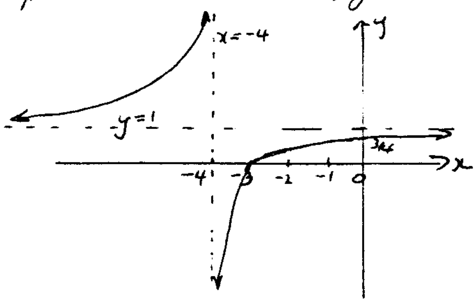
$= 2.$

Question 2

$y = \frac{x+3}{x+4} = \frac{(x+4)-1}{x+4}$

$= 1 - \frac{1}{x+4}$

Asymptotes will be $x=-4$, $y=1$



(ii) $y = x - \log_e(x+4)$ is increasing when $dy/dx > 0$

i.e. $dy/dx = 1 - \frac{1}{x+4}$

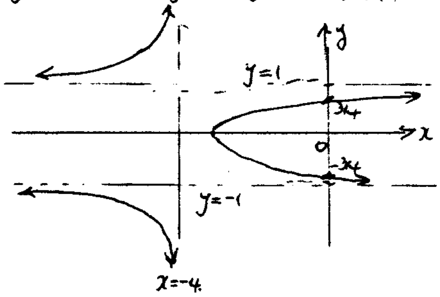
> 0 for $x > -3$, $x < -4$

but $\log(x+4)$ is defined for $x > -4$ only

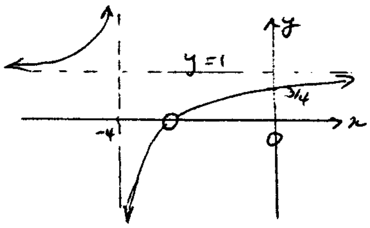
$\therefore$ Curve is increasing for $x > -3$

(iii) if $y \geq 0$ $|y| = y$ i.e. $y = \frac{x+3}{x+4}$

$y < 0$; $|y| = -y$ i.e. $y = -\left(\frac{x+3}{x+4}\right)$



(B) $y = \frac{(x+3)^2}{(x+4)(x+3)}$



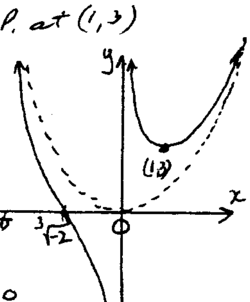
(b) $y = x^2 + \frac{2}{x}$, asymptotes $x=0$, $y = x^2$

$y' = 2x - \frac{2}{x^2}$

$= 0$, $x^3 = 1$ i.e. at $(1,3)$

$y'' = 2 + \frac{4}{x^3} = 0$, $x = \sqrt[3]{-2}$ $\therefore$ P.O.I at $(\sqrt[3]{-2}, 0)$

$y''(1) > 0$ $\therefore$ Min T.P. at $(1,3)$



$x^3 - kx + 2 = 0$

$x^3 + 2 = kx$

$x^2 + \frac{2}{x} = k$

$\therefore$ where $y = k$ intersects $y = x^2 + \frac{2}{x}$ is the solution of $x^3 - kx + 2 = 0$

if $k > 3$, 3 real distinct roots (2 positive 1 negative)

$k = 3$, 3 real roots (2 equal roots at $x=1$, 1 negative root at $x=-2$)

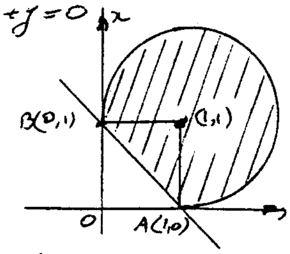
$k < 3$, 1 real root which is negative.

Question 3

$|z-(1+i)| \leq 1$ is the interior and boundary of the circle centre $(1,1)$ radius 1.

$|z-(1+i)| = |z|$ is the perpendicular bisector of $(1,1)$ and $(0,0)$ i.e. $x+y=0$

so $|z-(1+i)| \leq |z|$ is the region on and above $x+y=0$



(ii) PERIMETER

$= AB + \frac{3}{4}\times 2\pi r$

$= \sqrt{2} + \frac{3}{4}\times 2\pi$

$= \sqrt{2} + \frac{3\pi}{2}$

(b) $y = \frac{c^2}{x}$ Eqn of tangent

$dy/dx = \frac{-c^2}{x^2}$ $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

at $x = ct$; $= -\frac{1}{t^2}$ $t^2y - ct = -x + ct$

$x + t^2y = 2ct.$

(ii) $x + p^2y = 2cp$ — tangent at P

$x + q^2y = 2cq$ — tangent at Q

$(p^2-q^2)y = 2c(p-q)$

$y = \frac{2c}{p+q}$

$x + \frac{2cp^2}{p+q} = 2cp$

$x = 2cp - \frac{2cp^2}{p+q}$

$= \frac{2cpq}{p+q}$

$= \frac{2c}{p+q}$ $(pq=1)$

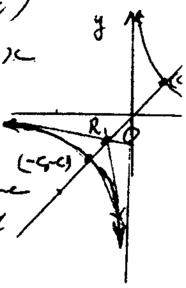
$\therefore R\left(\frac{2c}{p+q}, \frac{2c}{p+q}\right)$

$\therefore R$ lies on $y = x$

$\therefore$ locus of R is $y = x$

where $-c < x < c$, $x \neq 0$

if $pq = 1$, both tangents are drawn from same limb of hyperbola.



(c) let $u = a - x$ $x=0, u=a$

$du = -dx$ $x=a, u=0$

$\int_0^a f(x)\,dx = \int_a^0 f(a-u)\cdot -du$

$= \int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx.$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{4}} \left[\ln 2 - \ln(1+\tan x)\right]dx$

$\therefore 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \left[x\ln 2\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi\ln 2}{4}$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi\ln 2}{8}.$

Question 4

$f(t) = t^3 + ct + d$

$t_1 + t_2 + t_3 = 0$

$\sum t_1 t_2 = c$

$\sum t_1^2 = (\sum t)^2 - 2\sum t_1 t_2$
$= 0 - 2c$

$\therefore t_1^2 + t_2^2 + t_3^2 = -2c$

$f'(t) = 3t^2 + c$
$= 0$ when $t^2 = -\frac{c}{3}$
$t = \pm\sqrt{-c/3}$

if $f(u)\cdot f(v) < 0$, then $f(\sqrt{-\frac{c}{3}})f(-\sqrt{-\frac{c}{3}}) < 0$

$f(\sqrt{-\frac{c}{3}}) = -\frac{c}{3}\sqrt{-\frac{c}{3}} + c\sqrt{-\frac{c}{3}} + d$
$= \frac{2c}{3}\sqrt{-\frac{c}{3}} + d$

$f(-\sqrt{-\frac{c}{3}}) = \frac{c}{3}\sqrt{-\frac{c}{3}} - c\sqrt{-c/3} + d$
$= -\frac{2c}{3}\sqrt{-\frac{c}{3}} + d$

$\left(\frac{2c}{3}\sqrt{-\frac{c}{3}} + d\right)\left(-\frac{2c}{3}\sqrt{-\frac{c}{3}} + d\right) < 0$

$d^2 - \frac{4c^2}{9}\left(-\frac{c}{3}\right) < 0$

i.e. $27d^2 + 4c^3 < 0$

b) $y = x^2$, $dy/dx = 2x = 2t$ at (t, t^2)

Eqn of normal: $y - t^2 = -\frac{1}{2t}(x - t)$

$2ty - 2t^3 = -x + t$

$t^3 + \left(\frac{1-2y}{2}\right)t + \left(-\frac{x}{2}\right) = 0$

$P(x_0, y_0)$ satisfies eqn. of normal so $t^3 + \left(\frac{1-2y_0}{2}\right)t + \left(-\frac{x_0}{2}\right) = 0$, this has 3 distinct roots if

$27d^2 + 4c^3 < 0$ where $d = -\frac{x_0}{2}$, $c = \frac{1-2y_0}{2}$

$27\left(-\frac{x_0}{2}\right)^2 + 4\left(\frac{1-2y_0}{2}\right)^3 < 0$

$(1-2y_0)^3 < -\frac{27x_0^2}{2}$

$1 - 2y_0 < -\frac{3\cdot x_0^{2/3}}{2^{1/3}}$

$y_0 > \frac{3x_0^{2/3}}{2^{4/3}} + \frac{1}{2}$

$y_0 > 3\left(\frac{x_0}{4}\right)^{2/3} + \frac{1}{2}$

(c) $x^2y + xy^2 + 16 = 0$

$2xy + x^2\,dy/dx + y^2 + 2xy\,dy/dx = 0$

$(x^2 + 2xy)\,dy/dx = -(y^2 + 2xy)$

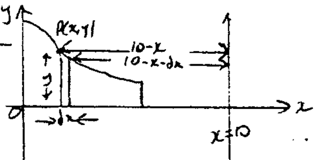
$\therefore dy/dx = -\frac{(y^2 + 2xy)}{x^2 + 2xy}$

at $(-2,-2)$ $= -\frac{(4+8)}{4+8}$
$= -1$

TANGENT: $y + 2 = -1(x + 2)$

i.e. $x + y + 4 = 0$.

Question 5.



Consider a strip of width δx, x units along the x axis, when rotated about $x = 10$, it generates a thin shelled cylinder of volume δV, where

$\delta V \doteq A(x)\cdot h$

$A(x) = \pi(R^2 - r^2)$, $R = 10 - x$, $r = 10 - x - \delta x$

$= \pi(R+r)(R-r)$

$= \pi(20 - 2x - \delta x)\cdot\delta x$

$= 2\pi(10-x)\delta x$, ignoring $(\delta x)^2 \doteq 0$

$\therefore \delta V = 2\pi(10-x)\cdot\delta x\cdot y$

$= 2\pi(10-x)\,\delta x\cdot\frac{5}{x^2+1}$

$\therefore V = \lim_{\delta x \to 0}\sum_{x=0}^{3} \frac{10\pi(10-x)}{x^2+1}\cdot\delta x$

$= \int_0^3 \frac{10\pi(10-x)}{x^2+1}\cdot dx$

$= \int_0^3\left(\frac{100\pi}{x^2+1} - \frac{10\pi x}{x^2+1}\right)dx$

$= \left[100\pi\tan^{-1}x - 5\pi\ln(x^2+1)\right]_0^3$

$= 100\pi\tan^{-1}3 - 5\pi\ln 10$

$\doteq 356\ cm^3$.

(b) General eqn of ellipse

$x^2/a^2 + y^2/b^2 = 1$ where $2a = 4$, $a = 2$ and $2b = 2$, $b = 1$

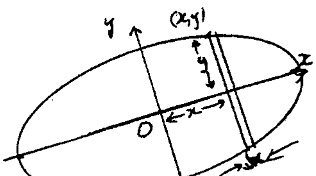
i.e. $\frac{x^2}{4} + y^2 = 1$

$y^2 = 1 - \frac{x^2}{4}$

(ii) Consider a slice of width δx, x units along the x axis and volume δV where $\delta V \doteq A(x)\cdot\delta x = (2y)^2\cdot\delta x$

$= 4y^2\cdot\delta x$

$= (4 - x^2)\,\delta x$



Volume $= \lim_{\delta x \to 0}\sum_{x=-2}^{2} A(x)\cdot\delta x$

$= 2\int_0^2 (4 - x^2)\cdot dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

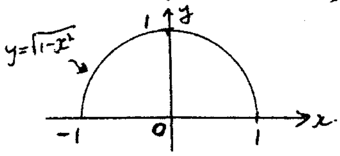
$= \frac{32}{3}$ unit^3.

(c) $y = \sin(\cos^{-1}x)$

D: $-1 \le x \le 1$

R: $0 \le y \le 1$

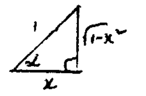
$f(x) = \sin(\cos^{-1}x)$
$= \sin(\sin^{-1}\sqrt{1-x^2})$
$= \sqrt{1-x^2}$ _ a semi-circle



let $\alpha = \cos^{-1}x$

$\cos\alpha = x$

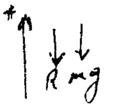
$\sin\alpha = \sqrt{1-x^2}$



$\therefore \alpha = \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$.

Question 6

$F = m\ddot{x}$, $m = 1$



$\therefore \ddot{x} = -g - R$
$= -\left(10 + \frac{1}{40}v^2\right)$

$v\cdot\frac{dv}{dx} = -\left(\frac{400 + v^2}{40}\right)$

$\frac{dv}{dx} = \frac{400 + v^2}{-40v}$

$\therefore dx/dv = \frac{-40v}{400 + v^2}$

$\therefore x = \int\frac{-40v}{400 + v^2}\cdot dv$

MAX. HEIGHT $= \int_{20}^{0}\frac{-40v}{400 + v^2}\cdot dv$

$= 20\left[\ln(400 + v^2)\right]_0^{20}$

$= 20\left[\ln 800 - \ln 400\right]$

$= 20\ln 2$. metres.

(ii) $\frac{dv}{dt} = \frac{400 + v^2}{-40}$

$\frac{dt}{dv} = \frac{-40}{400 + v^2}$

$t = -40\int\frac{dv}{20^2 + v^2}$

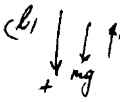
Time $= -40\int_{20}^{0}\frac{dv}{20^2 + v^2}$

$= 40 \times \frac{1}{20}\left[\tan^{-1}\frac{v}{20}\right]_0^{20}$

$= 2\tan^{-1}1$

$= \frac{\pi}{2}$ seconds.

(b)



$\ddot{x} = 10 - \frac{1}{40}v^2$

$v\frac{dv}{dx} = \frac{400 - v^2}{40}$

$\frac{dx}{dv} = \frac{40v}{400 - v^2}$

$x = \int\frac{40v}{400 - v^2}\cdot dv$

When particle hits the ground, it has travelled a distance of $20\ln 2$ and hits ground with velocity V

i.e. $20\ln 2 = \int_0^V \frac{40v}{400 - v^2}\cdot dv$

$20\ln 2 = 20\left[\ln(400 - v^2)\right]_V^0$

$\ln 2 = \ln\frac{400}{400 - V^2}$

$\therefore 2 = \frac{400}{400 - V^2}$

$400 - V^2 = 200$

$V^2 = 200$

$V = 10\sqrt{2}$. ms^{-1}.

QUESTION 7

(a) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \cdot dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x \cdot dx$

$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x (n-1) \sin^{n-2} x \cdot \cos x\, dx$

$= \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \cdot \cos^2 x \cdot dx$

$= (n-1) \int_0^{\frac{\pi}{2}} (1-\sin^2 x) \sin^{n-2} x \cdot dx$

$= (n-1) \left[\int_0^{\frac{\pi}{2}} (\sin^{n-2} x - \sin^n x) dx \right.$

i.e. $I_n = (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} \cdot I_{n-2}$

$I_4 = \frac{3}{4} I_2$; $I_2 = \frac{1}{2} I_0$

$= \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$

$= \frac{3}{8} I_0$; $I_0 = \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= \frac{3\pi}{16}$. $\qquad = \frac{\pi}{2}$

(b) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i \sin 3\theta$

$\therefore \cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta (i\sin\theta)^2 + (i\sin\theta)^3$

Equating reals

$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

(ii) $8x^3 - 6x - 1 = 0$

let $x = \cos\theta$

$8\cos^3\theta - 6\cos\theta - 1 = 0$

$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

i.e. $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore$ Roots are $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$

Product of roots are

$\cos\frac{\pi}{9} \cos\frac{5\pi}{9} \cos\frac{7\pi}{9} = -\frac{(-1)}{8}$

$= \frac{1}{8}$

$\cos\frac{5\pi}{9} = \cos(\pi - \frac{4\pi}{9}) = -\cos\frac{4\pi}{9}$

$\cos\frac{7\pi}{9} = \cos(\pi - \frac{2\pi}{9}) = -\cos\frac{2\pi}{9}$

$\therefore \cos\frac{\pi}{9} \cdot -\cos\frac{2\pi}{9} \cdot -\cos\frac{4\pi}{9} = \frac{1}{8}$

i.e. $\cos\frac{\pi}{9} \cdot \cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} = \frac{1}{8}$

Question 8

(a) $a = 3(1-x^2)$

$\frac{d}{dx}(\frac{1}{2}v^2) = 3 - 3x^2$

$\frac{1}{2}v^2 = \int (3 - 3x^2) dx$

$\frac{1}{2}v^2 = 3x - x^3 + c$

$\left.\begin{array}{l} x=0 \\ v=4 \end{array}\right\}$ $8 = c$

$\therefore \frac{1}{2}v^2 = 3x - x^3 + 8$

$v^2 = 16 + 6x - 2x^3$.

(ii) P starts from $x=0$ and moves to the right, so will it come to rest and change direction?

let $P(x) = 16 + 6x - 2x^3$

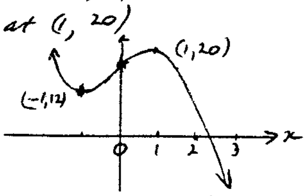
$P'(x) = 6 - 6x^2$

$= 0$, when $x = \pm 1$

$P''(x) = -12x$

$\therefore$ Min T.P. at $(-1, 12)$

Max T.P. at $(1, 20)$

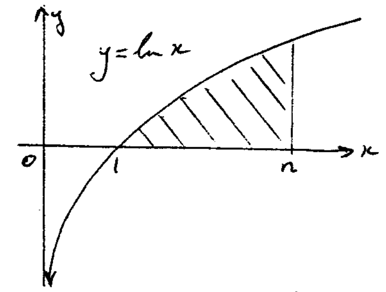


$P(x)$ has only 1 real root

i.e. P comes to rest only once at $2 < x < 3$

so the particle will move to the right where it comes to rest $2 < x < 3$ it then moves to the left without ever stopping and passing through O with velocity of $-4\ ms^{-1}$.

(b)

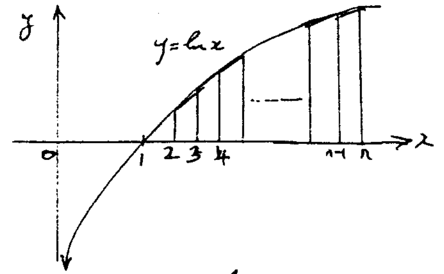


$\int_1^n \ln x \cdot dx = x\ln x - \int x \cdot \frac{1}{x} \cdot dx$

$= \left[x \ln x - x\right]_1^n$

$= n\ln n - n - (-1)$

$= n \ln n - n + 1$



by Trapezoidal rule

$\int_1^n \ln x \cdot dx \doteq \frac{h}{2}[\text{first} + \text{last} + 2\ \text{others}]$

$= \frac{1}{2}\left[\ln 1 + \ln n + 2(\ln 2 + \ln 3 + \cdots + \ln(n-1))\right]$

$= \frac{1}{2}\left[\ln n + 2\ln(2 \cdot 3 \cdot 4 \cdots (n-1))\right]$

$= \frac{1}{2}\ln n + \ln[(n-1)!]$

as the curve is concave downwards

Area by Trapezoidal Rule < Exact area

i.e. $\frac{1}{2}\ln n + \ln(n-1)! < n\ln n - n + 1$

add $\frac{1}{2}\ln n$ to both sides

i.e. $\ln n + \ln(n-1)! < n\ln n + \frac{1}{2}\ln n - n + 1$

$\ln n! < \ln\sqrt{n} \cdot n^n - n + 1$

$< \ln\sqrt{n} \cdot n^n - \ln e^n + \ln e$

$< \ln\left(\frac{\sqrt{n} \cdot n^n \cdot e}{e^n}\right)$

i.e. $n! < \sqrt{n} \cdot \left(\frac{n}{e}\right)^n \cdot e$