Question 1 **15 marks** Start a new page **MARKS**

(a) If $f(x)=(x-1)(x-3)$ then sketch

(i) $y=\dfrac{1}{f(x)}$ 2

(ii) $y=f(|x|)$ 2

(iii) $|y|=f(x)$ 2

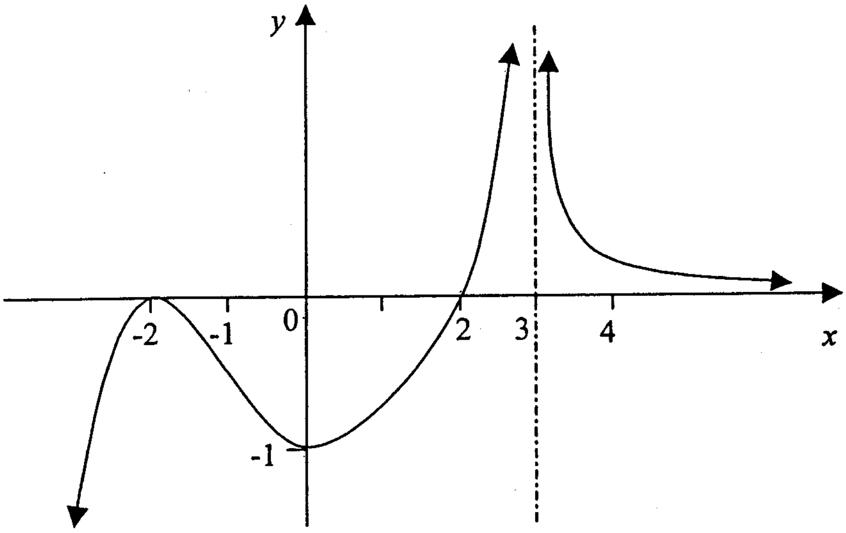
(b) (i) Find the stationary points and the asymptotes of the function 2

$$y=\frac{(x+1)^4}{x^4+1}$$

(ii) Sketch this function labelling all essential features. 1

(iii) Use the graph to find the set of values of k for which 2
$(x+1)^4=k\left(x^4+1\right)$ has two distinct real roots.

(c) Given the graph of $y=f'(x)$ below, sketch the graph of $y=f(x)$. 4
$y=f'(x)$ is the derivative of $y=f(x)$.



Question 2 15 marks Start a new page

MARKS

(a) (i) Find $\int \frac{x}{\sqrt{9-16x^2}}\,dx$ 2

(ii) Find $\int \frac{x^2}{x+1}\,dx$ 2

(iii) Evaluate $\int_0^{\ln 3} xe^x\,dx$ 3

(b) (i) Find real numbers A, B and C such that $\frac{2}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$ 3

(ii) Hence, find $\int_0^1 \frac{2}{(t+1)(t^2+1)}\,dt$. 3

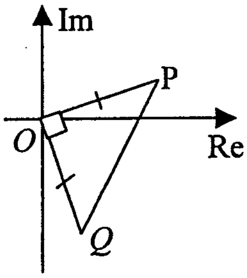
(iii) By using the substitution $t = \tan\left(\frac{x}{2}\right)$ evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x-\cos x}\,dx$. 2

Question 3 15 marks Start a new page

(a) Evaluate $\arg\left((2+i)\overline{w}\right)$ given $w=-1-3i$. 2

(b) Write $x^2-12x+48$ as the product of two linear factors. 2

(c) In the diagram on the right, triangle POQ is right-angled and isosceles. If P represents the complex number $a+bi$, where a and b are real, find the complex number represented by Q. 1



(d) Sketch in the Argand diagram the locus of the complex number z given:

(i) $\arg(z-2) = \arg z + \frac{\pi}{2}$ 2

(ii) $|z+3i| < 2|z|$ 3

(e) (i) Find, in modulus-argument form, the three cube roots of -8. 2

(ii) Write the two unreal cube roots of -8 in the form $a+bi$, where a and b are real. 1

(iii) If w_1 and w_2 are the the unreal cube roots of -8, show that $w_1^{6n} + w_2^{6n} = 2^{6n+1}$ for all integers n. 2

Question 4 **15 Marks** Start a new page **MARKS**

(a) Factorise $P(x)=x^4-5x^3+4x^2+2x-8$ over

(i) Real numbers; **2**

(ii) Complex numbers. **1**

(b) Write down all polynomials that have degree 4 with 3 as a single zero and -1 as a zero of mulitiplicity 3. **1**

(c) If α, β, γ are the roots of $x^3-2x^2+x+3=0$, evaluate:

(i) $\alpha^2+\beta^2+\gamma^2$ **2**

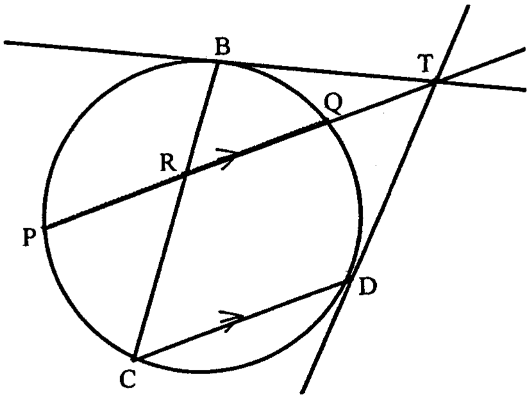
(ii) $\alpha^3+\beta^3+\gamma^3$ **1**

(d) If α, β, γ are the roots of $x^3+2x^2-2x+3=0$, form the equation whose roots are:

(i) $2\alpha, 2\beta, 2\gamma$ **1**

(ii) $\alpha^2, \beta^2, \gamma^2$. **3**

(e)



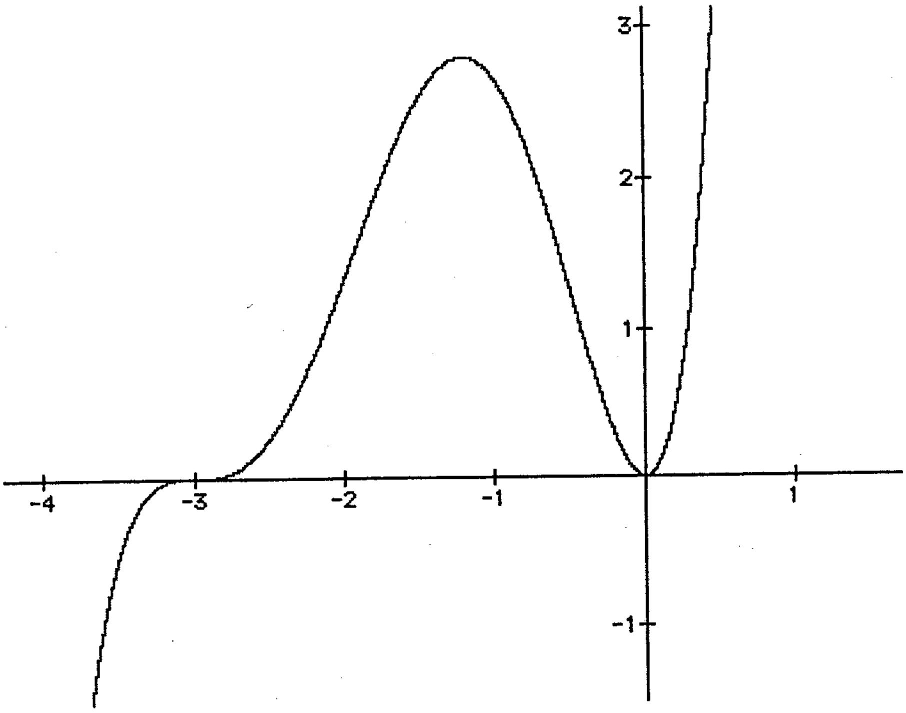
In the diagram, *PQ* and *CD* are parallel chords of a circle. The tangent at *D* meets *PQ* produced externally at *T*. *B* is the point of contact of the other tangent from T to the circle. *BC* meets *PQ* internally at *R*.

Copy or trace this diagram onto your answer page.

(i) Explain why ∠BDT = ∠BRT. **2**

(ii) Show that B, T, D and R are concyclic points. **2**

Question 5 **15 Marks** Start a new page **MARKS**



(a) Consider the graph of $y = f(x)$ as shown above.
On the answer sheet provided, use the graphs of $y = f(x)$ to clearly sketch separately the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f'(x)$ **1**

(b) Suggest a possible polynomial equation for the graph of $y = f(x)$ shown in part (a) of Q5. **1**

(c) A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y axis through 360° about the line $x = 2$.

(i) By slicing perpendicular to the axis of rotation, find the exact volume of S. **4**

(ii) (α) Use the method of cylindrical shells to show that the volume of S is also given by $\displaystyle\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$. **2**

(β) Confirm your answer to part (i) by calculating this definite integral using the substitution $u = 1-x$ **3**

Question 6 15 Marks Start a new page

MARKS

(a) A solid has as its base the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$. 4
If each section perpendicular to the major axis is an equilateral triangle, show that the volume of the solid is $128\sqrt{3}$ units^3.

(b) The region $(x-2R)^2+y^2 \le R^2$ is rotated about the y-axis forming a solid of revolution called a torus. 6
By summing volumes of cylindrical shells, show that the volume of the torus is $4\pi^2R^3$ units^3.

(c) The angles of elevation of the top of a tower P from three points A, B, C are α, β, γ respectively.
A, B, C are in a straight line such that $AB = BC = a$, but the line AC does not pass through S, the base of the tower.

(i) If $\angle ABS = \theta$ and h is the height of the tower, show that $CS^2 = a^2 + h^2\cot^2\beta + 2ah\cot\beta\cos\theta$. 2

(ii) Prove that the height of the tower is $\dfrac{a\sqrt{2}}{\left\{\cot^2\alpha+\cot^2\gamma-2\cot^2\beta\right\}^{\frac{1}{2}}}$. 3

Question 7 15 Marks Start a new page

(a) $P\left(3p,\frac{3}{p}\right)$ and $Q\left(3q,\frac{3}{q}\right)$ are points on the rectangular hyperbola $xy=9$.
The equation of chord PQ is $x+pqy=3(p+q)$.

(i) Find the co-ordinates of N, the midpoint of PQ. 1

(ii) If the chord PQ is a tangent to the parabola $y^2=3x$, prove that the locus of N is $3x=-8y^2$ 3

(b) The equation of a curve is $x^2y^2-x^2+y^2=0$.
(i) Show that the numerical value of y is always less than 1. 2
(ii) Find the equations of the vertical asymptotes. 1
(iii) Show that $\frac{dy}{dx}=\frac{y^3}{x^3}$ 3
(iv) Sketch the curve. 1

Question 7 continues on next page.

Question 7 continued

MARKS

(c) A ball thrown from a point P with velocity V, at an inclination α to the horizontal, reaches a point Q after t seconds.
Show that if PQ is inclined at θ to the horizontal, (where $\alpha > \theta$), then the direction of motion of the ball, when at Q, is inclined to the horizontal at an acute angle of $\tan^{-1}\left[2\tan\theta - \tan\alpha\right]$.
You may use the result without proof
$x = V\cos\alpha \times t$
$y = V\sin\alpha \times t - \frac{1}{2}gt^2$ 4

Question 8 15 Marks Start a new page

(a) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x\,dx$ where n is a positive integer.

(i) Using integration, show that $(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}$. 4

(ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x\,dx$. 3

(b) Consider the polynomial $x^5 - i = 0$.

(i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$. 2

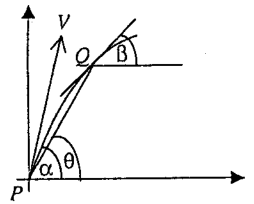
(ii) Show that $(x-i)\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = 0$. 4

(iii) Show that $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$. 2

Question 8 continued

(b) (i) $x^5 - i = (x-i)\left(x^4 + ix^3 + i^2x^2 + i^3x + i^4\right)$ $= (x-i)\left(x^4 + ix^3 - x^2 - ix + 1\right) = 0$ $x \neq i$ so $1 - ix - x^2 + ix^3 + x^4 = 0$	2 Evidence of the factorisation with powers of i needed for **2** marks
(ii) $x^5 - i = 0 \quad \Rightarrow \quad x^5 = i \qquad (rcis\theta)^5 = cis\frac{\pi}{2}$ $r = 1, \quad cis5\theta = cis\frac{\pi}{2}$ $5\theta = \frac{\pi}{2} + 2k\pi$ $\theta = \frac{\pi}{10} + \frac{2k\pi}{5}, \quad k = 0,1,2,3,4$	4
$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$ or $\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{-3\pi}{10}, \frac{-7\pi}{10}$ $(x-i)\left(x - cis\frac{\pi}{10}\right)\left(x - cis\frac{9\pi}{10}\right)\left(x - cis\frac{-3\pi}{10}\right)\left(x - cis\frac{-7\pi}{10}\right) = 0$ Now $cis\frac{\pi}{10} + cis\frac{9\pi}{10} = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10} + \cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}$	1
$= \cos\frac{\pi}{10} + i\sin\frac{\pi}{10} - \cos\frac{\pi}{10} + i\sin\frac{\pi}{10} = 2i\sin\frac{\pi}{10}$ and $cis\frac{\pi}{10} \times cis\frac{9\pi}{10} = \left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\left(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right)$ $= \left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\left(-\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$	1
$= -\cos^2\frac{\pi}{10} - \sin^2\frac{\pi}{10} = -1$	1
Similarly $cis\frac{-3\pi}{10} + cis\frac{-7\pi}{10} = 2i\sin\frac{-3\pi}{10} = -2i\sin\frac{3\pi}{10}$ and $cis\frac{\pi}{10} \times cis\frac{9\pi}{10} = -1$ Hence $(x-i)\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = 0$	1 for these two lines from similarly
(iii) From (i) and (ii) you can write: $\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = x^4 + ix^3 - x^2 - ix + 1$ The coefficient of x^2 will involve the product $\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$.	2
ie. $-x^2 - 4i^2\sin\frac{\pi}{10}\sin\frac{3\pi}{10}x^2 - x^2 = -x^2$	1 for this line with evidence of source
$4\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = 1$ $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$	1 penultimate line

stion 7 continued

(c)	4
	1 for correct diagram if nothing else with no multiple use of θ
$x = Vt\cos\alpha \qquad y = Vt\sin\alpha - \frac{gt^2}{2}$	
$P(0,0), \qquad Q\left(Vt\cos\alpha, Vt\sin\alpha - \frac{gt^2}{2}\right)$	
Gradient of $PQ = \dfrac{Vt\sin\alpha - \frac{gt^2}{2}}{Vt\cos\alpha} = \tan\alpha - \dfrac{gt}{2V\cos\alpha}$	1 Gradient of PQ
Hence $\tan\theta = \tan\alpha - \dfrac{gt}{2V\cos\alpha}$	
At Q, $\dot{x} = V\cos\alpha \qquad \dot{y} = V\sin\alpha - gt$	1 $\dot{x}$ and $\dot{y}$ if nothing else
$\dfrac{dy}{dx} = \dfrac{V\sin\alpha - gt}{V\cos\alpha}$	1 for $\dfrac{dy}{dx}$
$\tan\beta = \tan\alpha - \dfrac{gt}{V\cos\alpha}$	1 for $\tan\beta$
so $\dfrac{gt}{V\cos\alpha} = \tan\alpha - \tan\beta$	
Hence $\tan\theta = \tan\alpha - \dfrac{\tan\alpha - \tan\beta}{2}$ $2\tan\theta = 2\tan\alpha - \tan\gamma + \tan\beta$ $\tan\beta = 2\tan\theta - \tan\alpha$ $\beta = \tan^{-1}(2\tan\theta - \tan\alpha)$	1 evidence of correct simplification to given result

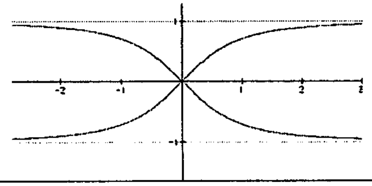
Question 8

(a) (i)	4
$I_n = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^{n-2} x\,\text{cosec}^2 x\,dx$	
$= -\text{cosec}^{n-2} x\cot x \Big\vert_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (n-2)\text{cosec}^{n-3} x(-1)(\sin x)^{-2}\cos x(-\cot x)\,dx$	1
$= 0 + 2^{n-2}\times\sqrt{3} - (n-2)\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x\cos^2 x\,dx$	1 correct evaluation of uv
$= 2^{n-2}\times\sqrt{3} - (n-2)\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x\left(1-\sin^2 x\right)dx$	1
$= 2^{n-2}\times\sqrt{3} - (n-2)\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x\,dx + (n-2)\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^n x\sin^2 x\,dx$	
$= 2^{n-2}\times\sqrt{3} - (n-2)I_n + (n-2)I_{n-2}$ $I_n + (n-2)I_n = 2^{n-2}\times\sqrt{3} + (n-2)I_{n-2}$ $(n-1)I_n = 2^{n-2}\times\sqrt{3} + (n-2)I_{n-2}$	1
(ii) $J = \displaystyle\int_0^{\frac{\pi}{3}} \sec^4 x\,dx = \int_0^{\frac{\pi}{3}} \sec^2 x.\sec^2 x\,dx$ $= \displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\left(1+\tan^2 x\right)dx$ $= \left[\tan x + \dfrac{\tan^3 x}{3}\right]_0^{\frac{\pi}{3}}$ $= \sqrt{3} + \dfrac{3\sqrt{3}}{3} - 0 = 2\sqrt{3}$ OR $\sec x = \text{cosec}\left(\frac{\pi}{2} - x\right)$ $y = \frac{\pi}{2} - x \Rightarrow dy = -dx$ $J = \displaystyle\int_0^{\frac{\pi}{3}} \text{cosec}^4\left(\frac{\pi}{2} - x\right)dx$ $= \displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \text{cosec}^4 y(-dy)$ $I_4 = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^4 y\,dy$ $3I_4 = 4\sqrt{3} + 2I_2$ $I_2 = 2^0\sqrt{3} + 0$ $= \sqrt{3}$ $3I_4 = 4\sqrt{3} + \sqrt{3} = 6\sqrt{3}$ $I_4 = 2\sqrt{3}$	3 **Method 1** 1 correct integrand 1 correct primitive 1 correct substitution **Method 2** Need evidence of change of variable, change of limits before using result from (i)

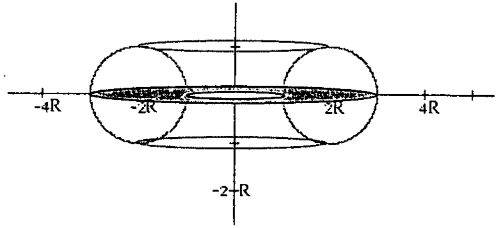
Question 7

(a)	
(i) N is $\left(\dfrac{3p+3q}{2}, \dfrac{\frac{3}{p}+\frac{3}{q}}{2}\right)$ ie $\left(\dfrac{3(p+q)}{2}, \dfrac{3(p+q)}{2pq}\right)$	1
(ii) Since PQ is a tangent to the parabola $y^2 = 3x$, the quadratic equation obtained by solving simultaneously with $x + pqy = 3(p+q)$ will have a double root. $y^2 = 3(3(p+q) - pqy)$ $y^2 + 3pqy - 9(p+q) = 0$	3
$\therefore \quad (3pq)^2 - 4 \times 1 \times [-9(p+q)] = 0$ $(3pq)^2 + 36(p+q) = 0$ OR $\quad (pq)^2 + 4(p+q) = 0$	1 simplification of Δ
From N, $x = \dfrac{3(p+q)}{2}$, $y = \dfrac{3(p+q)}{2pq}$ ie. $p+q = \dfrac{2x}{3}$, $pq = \dfrac{x}{y}$	1 expressions for $p+q$ and pq from N
so $\left(\dfrac{3x}{y}\right)^2 + 36 \times \dfrac{2x}{3} = 0$ $\qquad$ $\left(\dfrac{x}{y}\right)^2 + 4 \times \dfrac{2x}{3} = 0$ $\dfrac{9x^2}{y^2} = -24x$ $\qquad$ OR $\dfrac{x^2}{y^2} = \dfrac{-8x}{3}$ $3x = -8y^2$ is the locus of N. $\qquad$ $3x = -8y^2$ is the locus of N.	1 for $\left(\frac{3x}{y}\right)^2 + 36 \times \frac{2x}{3} = 0$ or equivalent
NB: $\left(y + \dfrac{3pq}{2}\right)^2 = \dfrac{9p^2q^2}{4} + 9(p+q)$	Pay 1
And Gradient of $PQ = \dfrac{-1}{pq}$ $\dfrac{dy}{dx} = \dfrac{3}{2y}$ $\therefore \quad -2y = 3pq$	Pay 1

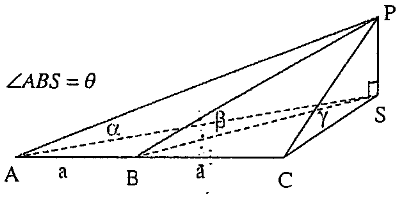
Question 7 continued

(b) (i) $x^2y^2 - x^2 + y^2 = 0$ $x^2y^2 + y^2 = x^2$ $y^2(x^2+1) = x^2$	2
$y^2 = \dfrac{x^2}{x^2+1}$ $y^2 = 1 - \dfrac{1}{x^2+1}$	1
As $x \to \infty$, $y^2 \to 1$ from below and hence $y < 1$	1
(ii) Find the equations of the horizontal asymptotes. $y^2 = 1 - \dfrac{1}{x^2+1}$ As $x \to \infty$, $y^2 \to 1$ Hence equations of horizontal asymptotes are $y = \pm 1$	1
(iii)	3
$x^2y^2 - x^2 + y^2 = 0$ $2xy^2 + x^2 \times 2y\dfrac{dy}{dx} - 2x + 2y\dfrac{dy}{dx} = 0$	1 evidence of partially correct implicit differentiation
$y(x^2+1)\dfrac{dy}{dx} = x(1-y^2) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x(1-y^2)}{y(x^2+1)}$	1 correct expression for $\dfrac{dy}{dx}$
From (i), $x^2(y^2-1) + y^2 = 0 \quad \Rightarrow \quad 1 - y^2 = \dfrac{y^2}{x^2}$ $y^2(x^2+1) - x^2 = 0 \quad \Rightarrow \quad 1 + x^2 = \dfrac{x^2}{y^2}$ $\dfrac{dy}{dx} = \dfrac{x}{y} \times \dfrac{y^2}{x^2} \times \dfrac{y^2}{x^2} = \dfrac{y^3}{x^3}$	1 using these results or equivalent to obtain correct answer
(iv)  	1 based on evidence in earlier parts

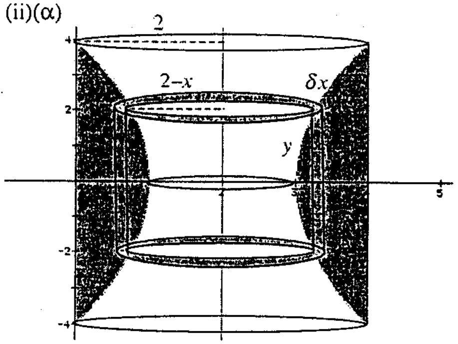
stion 6 continued

(b) 	6
$(x-2R)^2+y^2=R^2$	
$y^2=R^2-(x-2R)^2$	
$y=\pm\sqrt{R^2-(x-2R)^2}$	
$y=\sqrt{R^2-(x-2R)^2}$ is upper boundary.	1 equ'n of upper boundary
$\delta V=2\pi x\times 2y\times\delta x$	
$V=\int_R^{3R}4\pi xy\,dx$	1 correct definite integral for V
$=\int_R^{3R}4\pi x\sqrt{R^2-(x-2R)^2}\,dx$	1 correct substitution for y
Let $x-2R=R\sin\theta$ $\quad x=R,\ \theta=\frac{-\pi}{2}$ $\quad x=3R,\ \theta=\frac{\pi}{2}$	1 set up correct substitution with new limits
$dx=R\cos\theta d\theta$	
$V=4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(2R+R\sin\theta)\sqrt{R^2-R^2\sin^2\theta}\,R\cos\theta d\theta$	
$=4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(2R+R\sin\theta)R^2\cos^2\theta d\theta$	1 simplified integrand
$=4\pi R^3\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(2\cos^2\theta+\sin\theta\cos^2\theta\right)d\theta$	
$=4\pi R^3\left[\theta+\sin 2\theta-\frac{\cos^3\theta}{3}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$	1 correct primitive with correct limits
$=4\pi R^3\left(\frac{\pi}{2}+0-0-\left(\frac{-\pi}{2}+0-0\right)\right)$	
$=4\pi^2R^3$ units^3	

Question 6 continued

(c) 	
(i) $\angle ABS=\theta$ hence $\angle CBS=180-\theta$	2
In ΔCBS, $CS^2=BC^2+BS^2-2\times BC\times BS\cos(180-\theta)$	1 correct cos rule with evidence of $180-\theta$
$\therefore\ CS^2=a^2+BS^2+2\times a\times BS\cos\theta$	1 change of sign and substitution for BS
In ΔBSP, $BS=h\cot\beta$	
$\therefore\ CS^2=a^2+h^2\cot^2\beta+2ah\cot\beta\cos\theta$	
(ii) In ΔASP, $AS=h\cot\alpha$	3
In ΔCSP, $CS=h\cot\gamma$	
$\therefore\ h^2\cot^2\gamma=a^2+h^2\cot^2\beta+2ah\cot\beta\cos\theta$	
In ΔABS, $AS^2=AB^2+BC^2-2\times AB\times BS\cos\theta$	
$\therefore\ h^2\cot^2\alpha=a^2+h^2\cot^2\beta-2ah\cot\beta\cos\theta$	
Hence $\cos\theta=\frac{a^2+h^2\left(\cot^2\beta-\cot^2\alpha\right)}{2ah\cot\beta}$	1 expression for $\cos\theta$
But $h^2\cot^2\gamma=a^2+h^2\cot^2\beta+2ah\cot\beta\cos\theta$ from (i)	
So $\cos\theta=\frac{h^2\left(\cot^2\gamma-\cot^2\beta\right)-a^2}{2ah\cot\beta}$	1 second expression for $\cos\theta$
Hence $\frac{a^2+h^2\left(\cot^2\beta-\cot^2\alpha\right)}{2ah\cot\beta}=\frac{h^2\left(\cot^2\gamma-\cot^2\beta\right)-a^2}{2ah\cot\beta}$	
$a^2+h^2\left(\cot^2\beta-\cot^2\alpha\right)=h^2\left(\cot^2\gamma-\cot^2\beta\right)-a^2$	
$2a^2=h^2\left(\cot^2\gamma-\cot^2\beta-\cot^2\beta+\cot^2\alpha\right)$	1 correct substitution and simplification to expression for $2a^2$
$h^2=\frac{2a^2}{\cot^2\gamma-2\cot^2\beta+\cot^2\alpha}$	
$h=\frac{a\sqrt{2}}{\sqrt{\cot^2\alpha+\cot^2\gamma-2\cot^2\beta}}$	

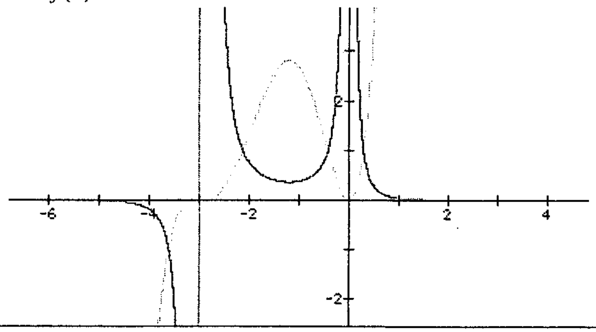
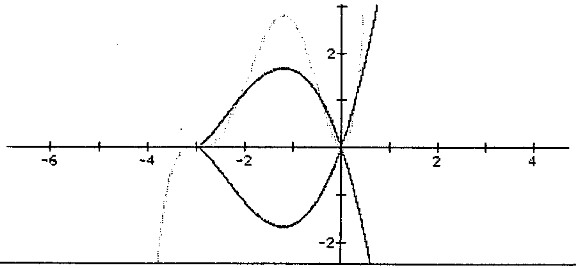
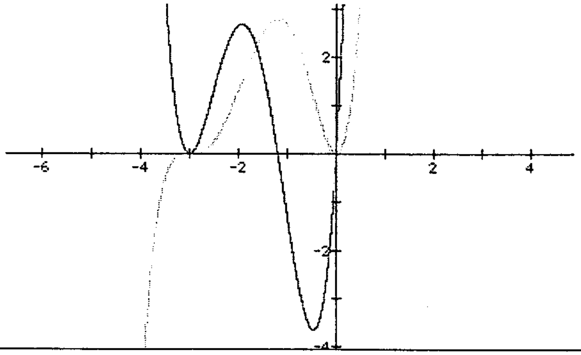
Question 5 continued

(ii)(α)  $\delta V = 2\pi(2-x)\times 2y\times\delta x$ $V = 4\pi\int_0^1 (2-x)y\,dx$ but $y^2 = 16(1-x)$ Use $y = 4\sqrt{1-x}$ as upper branch $V = 4\pi\int_0^1 (2-x)4\sqrt{1-x}\,dx$ $= 16\pi\int_0^1 (2-x)\sqrt{1-x}\,dx$ OR $R - r = 2 - x + \delta x - 2 + x$ $= \delta x$ $R + r = 2 - x + \delta x + 2 - x$ $= 4 - 2x + \delta x$ $(R-r)(R+r)$ $= \delta x(4 - 2x + \delta x)$ $\approx 2(2-x)\delta x$ $\delta V = \pi\left(R^2 - r^2\right)\times 2y$ $= \pi\times 2(2-x)\delta x\times 2y$ $= 4\pi(2-x)y\delta x$	2 Evidence of $y = 4\sqrt{1-x}$ is needed for 1
(β) $\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$ Let $u = 1-x$, $du = -dx$ $= \int_1^0 -16\pi(1+u)\sqrt{u}\,du$ $= 16\pi\int_0^1\left(\sqrt{u} + u^{\frac{3}{2}}\right)du$ $= 16\pi\left[\frac{2}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$ $= 16\pi\left(\frac{2}{3} + \frac{2}{5} - 0\right) = \frac{256\pi}{15}$	3 1 evidence of correct substitution 1 correct primitive 1 correct substitution 2 for $\frac{-256\pi}{15}$ with working

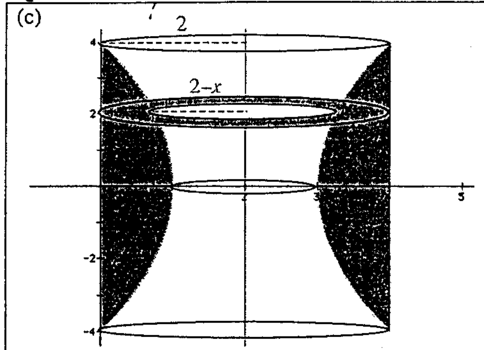
Question 6

(a) Base of each triangle is $2y$ Area of triangle is $\frac{1}{2}\times 2y\times 2y\times\frac{\sqrt{3}}{2} = \sqrt{3}y^2$. $\delta V = \sqrt{3}y^2\delta x$ $V = \int_{-6}^{6}\sqrt{3}y^2 dx$ and $y^2 = \frac{4}{9}\left(36 - x^2\right)$ $= 2\int_0^6 \sqrt{3}\times\frac{4}{9}\left(36 - x^2\right)dx$ $= \frac{8\sqrt{3}}{9}\int_0^6\left(36 - x^2\right)dx = \frac{8\sqrt{3}}{9}\left[36x - \frac{x^3}{3}\right]_0^6$ $= \frac{8\sqrt{3}}{9}(216 - 72) = 128\sqrt{3}$ units^3	4 1 area of Δ 1 indefinite integral + expression for y^2 OR definite integral with correct limits 1 correct integral prior to integration 1 correct primitive + correct subst shown

uestion 5

(a)(i) $y=\dfrac{1}{f(x)}$  	2 1 for two parts correct
(ii) $y^2=f(x)$  	2 Need evidence of graph becoming horizontal at $x=-3$ 1 for half correct
(iii) $y=f'(x)$  	1
(b) $f(x)=kx^2(x+3)^3$ Also accept $f(x)=kx^4(x+3)^3$	1

Question 5 continued

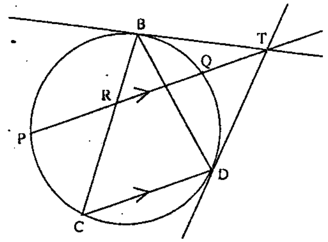
(c)  (i) $\delta V=\pi\left(2^2-(2-x)^2\right)\delta y$ $\delta V=\pi(2-2+x)(2+2-x)\delta y$ $\delta V=\pi x(4-x)\delta y$ $V=\int_{-4}^{4}\pi x(4-x)dy$ but $x=1-\frac{y^2}{16}$ or $x=\frac{16-y^2}{16}$ $=\pi\int_{-4}^{4}\left(1-\frac{y^2}{16}\right)\left(4-1+\frac{y^2}{16}\right)dy$ $=\pi\int_{-4}^{4}\left(3-\frac{y^2}{8}-\frac{y^4}{256}\right)dy$ $=2\pi\left[3y-\frac{y^3}{24}-\frac{y^5}{1280}\right]_0^4=2\pi\left(12-\frac{64}{24}-\frac{4^4\times 4}{256\times 5}-0\right)=\frac{256\pi}{15}$ units^3	4 1 for δV correct 1 evidence of the use of a correct substitution for x 1 correct primitive 1 correct substitution 3 for $\frac{416\pi}{15}$ with working

MHS Trial 2004 Mathematics Extension 2

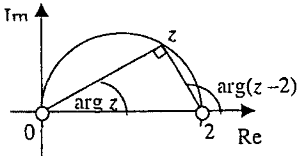
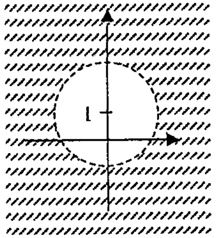
Question 4

(a) (i) Factorise $P(x)=x^4-5x^3+4x^2+2x-8$ over reals $P(x)=x^4-5x^3+4x^2+2x-8$ $P(-1)=1+5+4-2-8=0$ $(x+1)$ is a factor. $\ x^3-6x^2+10x-8$ $x+1\overline{)x^4-5x^3+4x^2+2x-8}$ $\underline{x^4+x^3}$ $-6x^3+4x^2$ $\underline{-6x^3+4x^2}$ $10x^2+2x$ $\underline{10x^2+10x}$ $-8x-8$ $\underline{-8x-8}$ 0 $Q(x)=x^3-6x^2+10x-8$ $Q(4)=64-96+40-8=0$ $(x-4)$ is a factor $\ x^2-2x+2$ $x-4\overline{)x^3-6x^2+10x-8}$ $\underline{x^3-4x^2}$ $-2x^2+10x$ $\underline{-2x^2+8x}$ $2x-8$ $\underline{2x-8}$ 0 $P(x)=(x+1)(x-4)(x^2-2x+2)$ over the reals	2 1 for one factor $x=\frac{2\pm\sqrt{4-4\times1\times2}}{2}$ $=\frac{2\pm\sqrt{-4}}{2}$ $=1\pm i$ $(x-(1+i))(x-(1-i))$ $(x-1-i)(x-1+i)$
(ii) $P(x)=(x+1)(x-4)(x^2-2x+2)$ $x^2-2x+2=(x-1)^2+1$ $=(x-1+i)(x-1-i)$ $P(x)=(x+1)(x-4)(x-1+i)(x-1-i)$ over the complex numbers.	1
(b) $P(x)=k(x-3)(x+1)^3$	1
(c)(i) $x^3-2x^2+x+3=0$ $\alpha+\beta+\gamma=2$ $\quad\alpha\beta+\beta\gamma+\gamma\alpha=1$ $\quad\alpha\beta\gamma=-3$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $=4-2=2$	2
(ii) $\alpha^3-2\alpha^2+\alpha+3=0$ $\beta^3-2\beta^2+\beta+3=0$ $\gamma^3-2\gamma^2+\gamma+3=0$ $\alpha^3+\beta^3+\gamma^3-2\left(\alpha^2+\beta^2+\gamma^2\right)+\alpha+\beta+\gamma+9=0$ $\alpha^3+\beta^3+\gamma^3-4+2+9=0$ $\alpha^3+\beta^3+\gamma^3=-7$	1

Question 4 continued

(d) (i) $2\alpha, 2\beta, 2\gamma$ $\left(\frac{x}{2}\right)^3+2\left(\frac{x}{2}\right)^2-2\left(\frac{x}{2}\right)+3=0$ $x^3+4x^2-8x+24=0$ OR $\frac{x^3}{8}+\frac{x^2}{2}-x+3=0$	1
(ii) $\alpha^2, \beta^2, \gamma^2$. $\left(\sqrt{x}\right)^3+2\left(\sqrt{x}\right)^2-2\left(\sqrt{x}\right)+3=0$ $x\sqrt{x}+2x-2\sqrt{x}+3=0$ $\sqrt{x}(x-2)=-(2x+3)$ $x(x-2)^2=(2x+3)^2$ $x^3-4x^2+4x=4x^2+12x+9$ $x^3-8x^2-8x-9=0$	3 1 this line 1 this line 1 this line
(e)  (i) ∠BDT = ∠BCD (angle between tangent and chord equals angle in alternate segment) ∠BRT = ∠BCD (corresponding angles equal, RQ ll CD) ∴ ∠BDT = ∠BRT. (ii) ∠BDT = ∠BRT from (i). These are a pair of equal angles standing on the same side of BT. Hence B, T, D and R are concyclic points.	2 2

Question 3

(a) $w=-1-3i,\quad \bar{w}=-1+3i$ $\arg\left((2+i)\bar{w}\right)=\arg\left((2+i)(-1+3i)\right)$ $\arg\left((2+i)\bar{w}\right)$ given $=\arg(-2-i+6i-3)$. $=\arg(-5+5i)$ $=\frac{3\pi}{4}$	2 1 for $\frac{-\pi}{4}$ or $\frac{\pi}{4}$.
(b) $x^2-12x+48=(x-6)^2+12$ $=\left(x-6+i\sqrt{12}\right)\left(x-6-i\sqrt{12}\right)$ $=\left(x-6+2\sqrt{3}i\right)\left(x-6-2\sqrt{3}i\right)$	2 $x=\frac{12\pm\sqrt{144-192}}{2}$ $=\frac{12\pm\sqrt{48}i}{2}$ $=6\pm2\sqrt{3}i$
(c) OQ is just OP rotated clockwise through an angle of $\frac{\pi}{2}$. Q is represented by $-i(a+bi)=b-ai$	1
(d)(i) $\arg(z-2)=\arg z+\frac{\pi}{2}$ may be written $\arg(z-2)-\arg z=\frac{\pi}{2}$ which suggests an angle in a semi circle, centre (1, 0) radius 1 in the upper half plane, excluding the points (0, 0) and (2, 0).  (ii) $\|z+3i\|<2\|z\|$ Let $z=x+iy$ $\|x+(3+y)i\|<2\|x+iy\|$ $x^2+(y+3)^2<4\left(x^2+y^2\right)$ $x^2+y^2+6y+9<4x^2+4y^2$ $3x^2+3y^2-6y>9$ $x^2+y^2-2y>3$ $x^2+(y-1)^2>4$  This region is the exterior of the circle centre (0, 1), radius 2	2 1 1 3 1 for dotted circle −1 for wrong centre

Question 3 continued

(e) (i) Let $(r\,\mathrm{cis}\theta)^3=-8$ $r^3\mathrm{cis}3\theta=-8$ $r=2,\ \ \mathrm{cis}3\theta=-1$ $3\theta=\pi,\pi+2\pi,\pi-2\pi$ $\theta=\frac{\pi}{3},\pi,\frac{-\pi}{3}$ The three cube roots of −8 are $2\mathrm{cis}\left(\frac{-\pi}{3}\right),\ \ 2\mathrm{cis}\frac{\pi}{3},\ \ 2\mathrm{cis}\pi$ or $-2\mathrm{cis}\left(\frac{\pm2\pi}{3}\right),\ \ 2\mathrm{cis}\pi$	2
(ii) $2\mathrm{cis}\left(\frac{-\pi}{3}\right)=2\left(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1-\sqrt{3}i$ $2\mathrm{cis}\frac{\pi}{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}i$	1
(iii) Let $w_1=2\mathrm{cis}\left(\frac{-\pi}{3}\right),\ \ w_2=2\mathrm{cis}\frac{\pi}{3}$ $w_1{}^{6n}+w_2{}^{6n}=\left(2\mathrm{cis}\left(\frac{-\pi}{3}\right)\right)^{6n}+\left(2\mathrm{cis}\left(\frac{\pi}{3}\right)\right)^{6n}$ $=2^{6n}\left(\mathrm{cis}(-2n\pi)+\mathrm{cis}(2n\pi)\right)$ $=2^{6n}\left(\cos(-2n\pi)+i\sin(-2n\pi)+\cos(2n\pi)+i\sin(2n\pi)\right)$ $=2^{6n}(1+0+1+0)$ when n is an integer $=2\times2^{6n}$ $=2^{6n+1}$ **OR** $w_1{}^3=-8,\ \ w_2{}^3=-8$ $w_1{}^{6n}+w_2{}^{6n}=(-8)^{2n}+(-8)^{2n}$ $=(-2)^{6n}+(-2)^{6n}$ $=2\times(-2)^{6n}$ $=2\times2^{6n}$ $=2^{6n+1}$	2 1 Induction with everything set up correctly with an error

Question 2

(a)(i) $\int \frac{x}{\sqrt{9-16x^2}}dx = \frac{1}{16}\int \frac{16x}{\sqrt{9-16x^2}}dx = \frac{-1}{16}\sqrt{9-16x^2}+C$ OR $x=\frac{3}{4}\sin\theta,\quad dx=\frac{3}{4}\cos\theta d\theta$ $\int \frac{x}{\sqrt{9-16x^2}}dx = \int \frac{\frac{3}{4}\sin\theta \times \frac{3}{4}\cos\theta d\theta}{\sqrt{9-16\times\frac{9}{16}\sin^2\theta}} = \frac{9}{16}\int \frac{\sin\theta\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}}$ $=\frac{3}{16}\int \sin\theta d\theta = \frac{-3}{16}\cos\theta + C$ $=\frac{-3}{16}\sqrt{1-\frac{4}{9}x^2}+C = \frac{-1}{16}\sqrt{9-4x^2}+C$	2
(ii) $\int \frac{x^2}{x+1}dx = \int \frac{(x+1)(x-1)+1}{x+1}dx$ $=\int\left(x-1+\frac{1}{x+1}\right)dx = \frac{x^2}{2}-x+\ln\lvert x+1\rvert+C$	2
(iii) $\int_0^{\ln 3} xe^x dx = xe^x\Big\rvert_0^{\ln 3} - \int_0^{\ln 3} e^x dx$ $=3\ln 3 - \left[e^x\right]_0^{\ln 3} = 3\ln 3 - 3 + 1 = 3\ln 3 - 2$	3 1 for first line 1 for indefinite integral 1 for correct substit
(b)(i) $\frac{2}{(t+1)(t^2+1)} = \frac{A}{t+1}+\frac{Bt+C}{t^2+1}$ $2 \equiv A(t^2+1)+(Bt+C)(t+1)$ $t=-1 \Rightarrow 2=2A \quad \therefore A=1$ $t=0 \Rightarrow 2=A+C \quad \therefore C=1$ $t=1 \Rightarrow 2=2+(B+1)\times 2 \quad \therefore B=-1$	3 –1 for error carried through

Question 2 continued

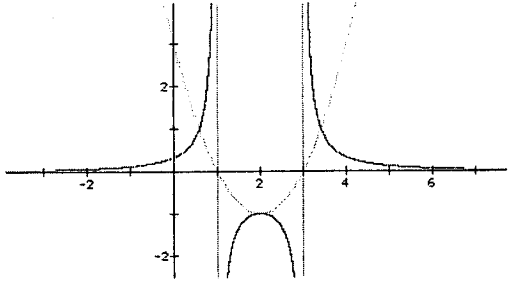
(b)(ii) $\int_0^1 \frac{2}{(t+1)(t^2+1)}dt = \int_0^1\left(\frac{1}{(t+1)}+\frac{1-t}{(t^2+1)}\right)dt$ $=\int_0^1\left(\frac{1}{(t+1)}+\frac{1}{(t^2+1)}-\frac{t}{(t^2+1)}\right)dt$ $=\left[\ln\lvert t+1\rvert+\tan^{-1}t-\frac{1}{2}\ln(t^2+1)\right]_0^1$ $=\left[\tan^{-1}t+\ln\frac{\lvert t+1\rvert}{\sqrt{(t^2+1)}}\right]_0^1$ $=\tan^{-1}1+\ln\frac{2}{\sqrt{2}}-\left(0+\ln\frac{1}{\sqrt{1}}\right)$ $=\frac{\pi}{4}+\frac{1}{2}\ln 2$	3
(iii) $t=\tan\frac{x}{2} \Rightarrow dx=\frac{2dt}{1+t^2}, \quad x=0, t=0 \quad x=\frac{\pi}{2}, t=1$ $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x-\cos x}dx = \int_0^1 \frac{\frac{2t}{1+t^2}\times\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}$ $=\int_0^1 \frac{4t\,dt}{(1+t^2)(1+t^2+2t-1+t^2)}$ $=\int_0^1 \frac{4t\,dt}{(1+t^2)(2t^2+2t)}$ $=\int_0^1 \frac{2\,dt}{(1+t^2)(t+1)}$ from b(ii) $\Rightarrow \quad =\frac{\pi}{4}+\frac{\ln 2}{2}$	2 1 for mostly correct 1 for this integral No mark for last line

~~y = f'(x)~~ MHS Trial 2004 **Mathematics Extension 2**

Question 1

(a) If $f(x)=(x-1)(x-3)$ then sketch

(i) $y=\dfrac{1}{f(x)}$

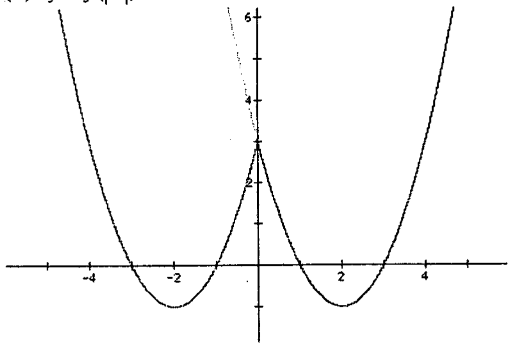


2

1 for both top branches

1 for bottom part with correct local max

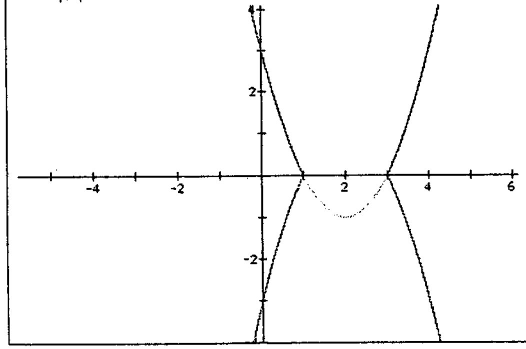
(ii) $y=f(|x|)$



2

1 for one half correct

(iii) $|y|=f(x)$



2

1 for either vertical half or either horizontal half

Question 1 continued

(b) (i) $y=\dfrac{(x+1)^4}{x^4+1}$

$$\frac{dy}{dx}=\frac{4(x+1)^3(x^4+1)-(x+1)^4\times 4x^3}{(x^4+1)^2}$$

$$=\frac{4(x+1)^3(x^4+1-x^4-x^3)}{(x^4+1)^2}=\frac{4(x+1)^3(1-x^3)}{(x^4+1)^2}=0$$

when $x=-1$ or $x=1$.

Stationary points are $(-1,0)$, $(1,8)$

For asymptotes, write $y=\dfrac{(x+1)^4}{x^4+1}=\dfrac{x^4\left(1+\frac{1}{x}\right)^4}{x^4+1}=\dfrac{\left(1+\frac{1}{x}\right)^4}{1+\frac{1}{x^4}}$

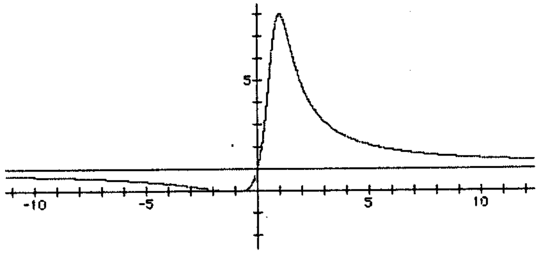
As $x\to\pm\infty$, $y=\dfrac{\left(1+\frac{1}{x}\right)^4}{1+\frac{1}{x^4}}\to 1$, horizontal asymptote is $y=1$.

2

1 for stationary points

1 for asymptotes

(ii)



1 if correct and not made easier from errors above

(iii) $y=\dfrac{(x+1)^4}{x^4+1}$ and $y=k \rightarrow k\left(x^4+1\right)=(x+1)^4$

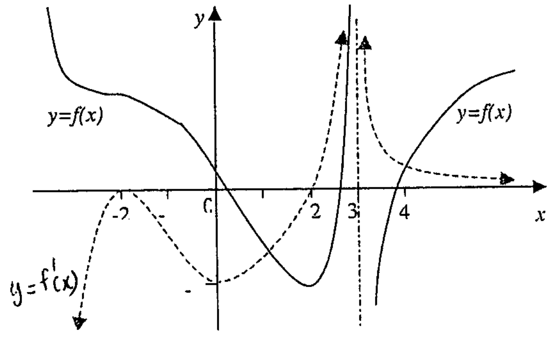
Two roots if $0<k<1$ or $1<k<8$, from the graph.

2

1 for each set of values

1 for $0<k<8$

(c)



4

deduct 1 for each part missing.

horizontal at $x=-2$

slope -1 at $x=0$

minimum at $x=2$

asymptotic to $x=3$

RH branch monotonic increase & flattening