Total marks – 120
Attempt Questions 1–8

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 \frac{e^x}{1+e^x}\,dx$. **2**

(b) By completing the square, find $\int \frac{dx}{\sqrt{x^2-6x+10}}$. **2**

(c) Use integration by parts to find $\int x \log_e x\,dx$. **3**

(d) Use the substitution $u = 1 - x$ to evaluate $\int_{-1}^{0} \frac{2+x}{\sqrt{1-x}}\,dx$. **4**

(e) (i) Find real numbers a and b such that **2**

$$\frac{4x^2-5x-1}{(x-3)\left(x^2+1\right)} \equiv \frac{a}{x-3} + \frac{bx+1}{x^2+1}.$$

(ii) Hence find $\int \frac{4x^2-5x-1}{(x-3)\left(x^2+1\right)}\,dx$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 1 + i$ and $w = 1 - 2i$. Find, in the form $x + iy$,

(i) zw **1**

(ii) $\frac{1}{\overline{w}}$ **1**

(b) Sketch the region in the Argand diagram where the inequalities **3**

$$|z - 2 + 2i| \leq 2 \text{ and } \frac{-\pi}{4} < \arg z < \frac{-\pi}{6}$$

hold simultaneously

(c) It is given that $3 - i$ is a root of $P(z) = z^3 + rz + 60$, where r is a real number.

(i) State why $3 + i$ is also a root of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **2**

(d) (i) Express $-2 + 2\sqrt{3}i$ in modulus-argument form **2**

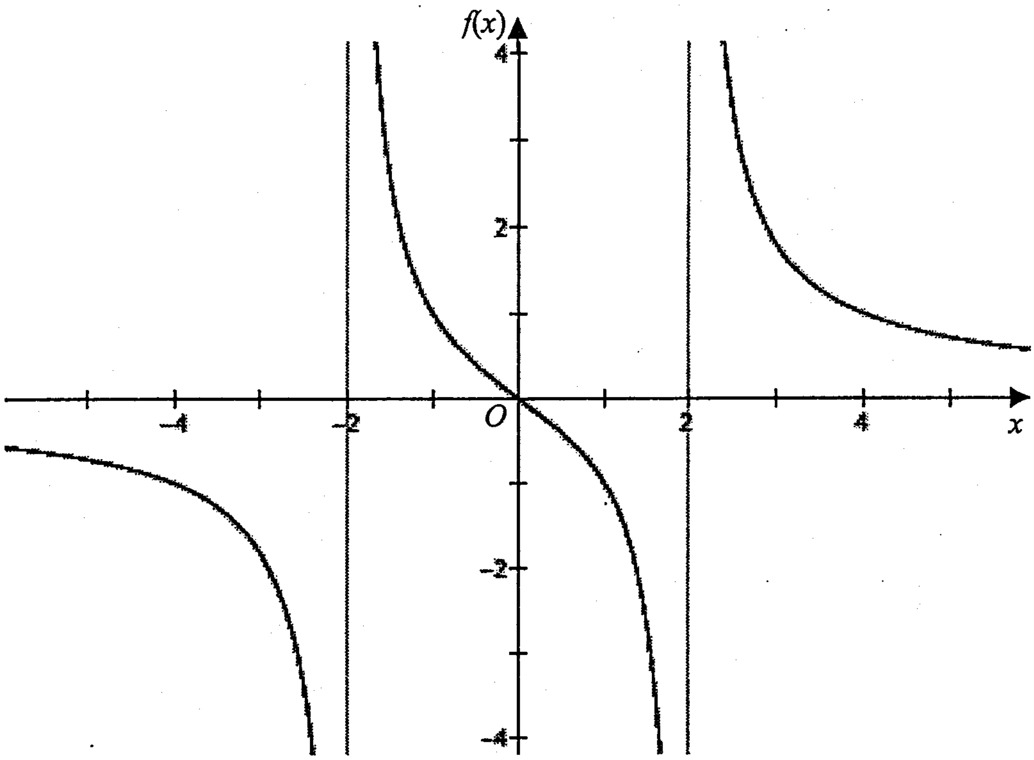
(ii) Hence evaluate $\sqrt[4]{-2 + 2\sqrt{3}i}$ in the form $x + iy$. **2**

(e) By applying de Moivre's theorem and by also expanding $(\cos\theta + i\sin\theta)^4$, obtain expressions for $\cos 4\theta$ in terms of $\cos\theta$. **3**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of $f(x) = \dfrac{3x}{x^2 - 4}$.

(i) Write down the equations of all the asymptotes. **1**

Draw separate one-third page sketches of the graphs of the following:

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $y^2 = f(x)$ **2**

(b) Let α, β and γ be the roots of $x^3 + 3x^2 + 4 = 0$.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$. **2**

Question 3 continues on page 5

Marks

Question 3 continued

(c) (i) If $a > b > 0$, on a sketch of the curve $y = \sqrt{a^2 - x^2}$ shade the region represented by the definite integral $\int_b^a \sqrt{a^2 - x^2}\, dx$. **1**

(ii) By using your diagram, or otherwise, show that **3**

$$\int_b^a \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\cos^{-1}\left(\frac{b}{a}\right) - \frac{b}{2}\sqrt{a^2 - b^2}.$$

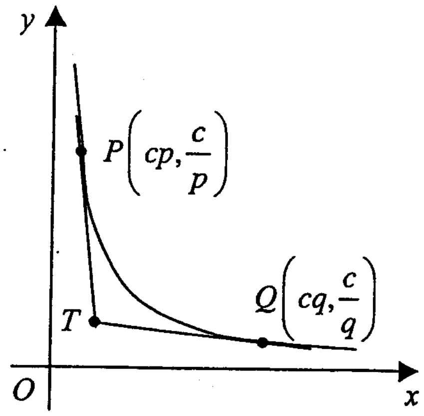
End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Find the equation of the normal to the curve defined by $x^4 + 3xy - y^2 + 9 = 0$ at the point $(-1, 2)$. **3**

(b)



The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, $p \neq q$, lie on the same branch of the hyperbola $xy = c^2$.

The tangents at P and Q meet at the point T.

(i) Show that the equation of the tangent to the hyperbola at Q is $x + q^2 y = 2cq$ **2**

(ii) Show that T has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(iii) If P and Q move so that $pq = k$, a constant, show that the locus of T is a straight line and give its equation in terms of k. **3**
Clearly state any restrictions on the locus.

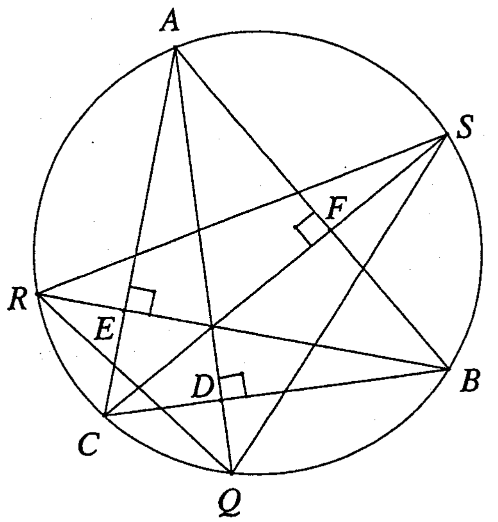
Question 4 continues on page 7

Marks

Question 4 continued

(c) The vertices of an acute-angled triangle ABC lie on a circle. The perpendiculars from A, B and C meet BC, AC and AB at D, E and F respectively.

The perpendiculars AD, BE and CF are produced to meet the circle at Q, R and S respectively.



(i) Using ΔAEB, state the relationship between $\angle EAB$ and $\angle EBA$, giving reasons. **1**

(ii) Prove that $\angle ABE = \angle ACF$. **1**

(iii) Prove that AQ bisects $\angle RQS$. **2**

(iv) Prove that the points E, F, B and C are concyclic. **1**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The region bounded by $0 \le x \le 2$, $0 \le y \le 4x^2 - x^4$ is rotated about the y-axis to form a solid. **3**

Use the method of cylindrical shells to find the volume of the solid.

(b) $z_1 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ and $z_2 = 2i$ are two complex numbers.

(i) On an Argand diagram draw the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ to represent z_1 and z_2 respectively. Also draw the vectors representing $z_1 + z_2$ and $z_2 - z_1$. **2**

(ii) Find the exact values of $\arg(z_1 + z_2)$ and $\arg(z_2 - z_1)$. **2**

(c)(i) Show that $f(x) = x\sqrt{4 - x^2}$ is an odd function. **1**

(ii) Hence evaluate $\int_{-2}^{2} (x-1)\sqrt{4 - x^2}\,dx$ **2**

(d)(i) Use the substitution $u = \frac{\pi}{4} - x$ to show that **3**

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\ dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right) dx$$

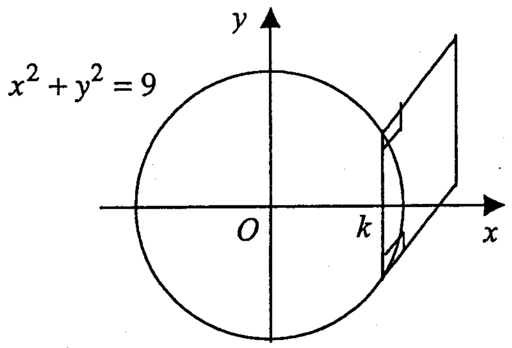
(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\ dx$ **2**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

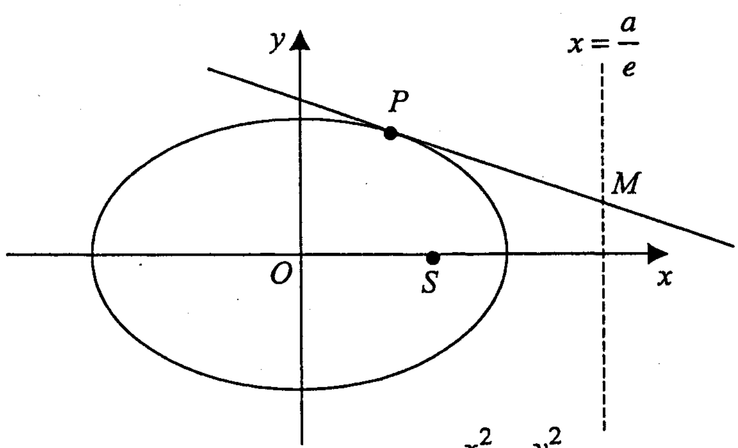
(a) The base of a solid is the region in the xy plane bounded by the circle $x^2 + y^2 = 9$. Each cross-section perpendicular to the x-axis is a square.



(i) Show that the area of the square cross-section at $x = k$, where $-3 < k < 3$, is $4\left(9 - k^2\right)$ **1**

(ii) Hence find the volume of the solid. **2**

(b)



The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

S is the focus $(ae, 0)$.

M the point where the tangent at P cuts the directrix, $x = \dfrac{a}{e}$.

(i) Show that the equation of the tangent to the ellipse at P is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ **2**

(ii) Show that M has coordinates $\left(\dfrac{a}{e}, \dfrac{b(e - \cos\theta)}{e\sin\theta}\right)$. **1**

(iii) Prove that $\angle PSM = 90°$. **3**

Question 6 continues on page 10

Marks

Question 6 continued

(c) (i) Show that the solutions of $z^6 + z^3 + 1 = 0$ are contained in the solutions of $z^9 - 1 = 0$ **1**

(ii) Sketch the nine solutions of $z^9 - 1 = 0$ on an Argand diagram (about one third of a page in size) **2**

(iii) Mark clearly on your diagram the six roots $z_1, z_2, z_3, z_4, z_5, z_6$ of $z^6 + z^3 + 1 = 0$ **1**

(iv) Show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ is given by **2**

$$2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right)$$

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 - 3x^2 + 5x - 2 = 0$ has roots α, β, γ.

(i) Find $\alpha^2 + \beta^2 + \gamma^2$ 1

(ii) Hence find the number of real roots of the equation $x^3 - 3x^2 + 5x - 2 = 0$ 2

(b) A particle of mass m is projected vertically upwards with an initial velocity u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle, that is mkv^2.

Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is

$$\ddot{x} = -\left(g + kv^2\right).$$

Where g ms^{-2} is the acceleration due to gravity.

(i) Assume $k = 10$ and that the acceleration due to gravity is 10 ms^{-2}. Show that $t = \frac{1}{10}\left(\tan^{-1} u - \tan^{-1} v\right)$ 2

(ii) Find, in terms of u, the time taken for the particle to reach its greatest height. 1

(iii) Find an expression for the height, x, in terms of u and v. 2

(iv) Find, in terms of u, the greatest height attained. 1

(c) A sequence of numbers u_n is such that $u_1 = 2$, $u_2 = 16$ and $u_n = 8u_{n-1} - 15u_{n-2}$ for $n \geq 3$.

(i) Use the method of Mathematical Induction to show that $u_n = 5^n - 3^n$ for $n \geq 1$. 4

(ii) Hence show that $u_1 + u_2 + u_3 + \ldots + u_n = \dfrac{5^{n+1} - 2 \times 3^{n+1} + 1}{4}$. 2

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $x^x a^{\ln x} = x$ **2**

(b) Given that $I_n = \int_0^{\frac{\pi}{4}} \sec x \tan^n x dx, \quad n = 1, 2, 3...$

(i) Find I_1 **1**

(ii) Prove that $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$ **3**

(iii) Evaluate I_5 **1**

(c) Two points, X and Y, are undergoing Simple Harmonic Motion on the number plane.

Y moves along the y axis, oscillating between $y = 1\,cm$ and $y = 7\,cm$ with period π seconds.

X moves along the x axis with centre of motion at $x = 3\,cm$ and period 2π seconds.

Initially Y is stationary at $y = 7$ and X is at its central position with velocity of $2cms^{-1}$.

(i) Show that Y moves according to the equation $y = 4 + 3\cos 2t$ and that X moves according to the equation $x = 3 + 2\sin t$
In both cases you should use the displacement-time equation for SHM, $x = b + a\cos(nt + \alpha)$, as your starting point. **5**

(ii) As X and Y move in SHM find the rate at which the area of triangle OXY is changing when $t = \frac{5\pi}{4}$ **3**

End of paper

Question 1

a. $\int_0^1 \frac{e^x}{1+e^x}\,dx = \left[\ln(1+e^x)\right]_0^1$

$= \ln(1+e) - \ln(1+1)$

$= \ln\left(\frac{1+e}{2}\right)$

b. $\int \frac{dx}{\sqrt{x^2-6x+10}} = \int \frac{dx}{\sqrt{1+(x-3)^2}}$

$= \ln\left(x-3+\sqrt{(x-3)^2+1}\right)$

$= \ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

c. $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$

$u=\ln x \qquad v=\frac{x^2}{2}$

$u'=\frac{1}{x} \qquad v'=x$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

d. $\int_{-1}^{0} \frac{2+x}{\sqrt{1-x}}\,dx = -\int_2^1 \frac{3-u}{\sqrt{u}}\,du$

$= \int_1^2 3u^{-\frac{1}{2}} - u^{\frac{1}{2}}\,du$

$u=1-x$

$du=-dx$

$x=1-u$

$= \left[6u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2$

$x=0 \Rightarrow u=1$

$x=-1 \Rightarrow u=2$

$= 6\sqrt{2} - \frac{2}{3}\cdot 2\sqrt{2} - \left(6-\frac{2}{3}\right)$

$= \frac{14\sqrt{2}-16}{3}$

e (i) $a(x^2+1)+(bx+1)(x-3) \equiv 4x^2-5x+1$

$ax^2+a+bx^2-3bx+x-3 \equiv 4x^2-5x+1$

$(a+b)x^2+(1-3b)x+(a-3) \equiv 4x^2-5x+1$

$1-3b=-5 \Rightarrow b=2$

$a-3=-1 \Rightarrow a=2$

(ii) $\int \frac{4x^2-5x-1}{(x-3)(x^2+1)}\,dx = \int \frac{2}{x-3} + \frac{2x+1}{x^2+1}\,dx$

$= \int\left(\frac{2}{x-3} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$= 2\ln|x-3| + \ln(x^2+1) + \tan^{-1}x + c$

Question 2

a (i) $zw = (1+i)(1-2i)$

$= 1 - 2i + i - 2i^2$

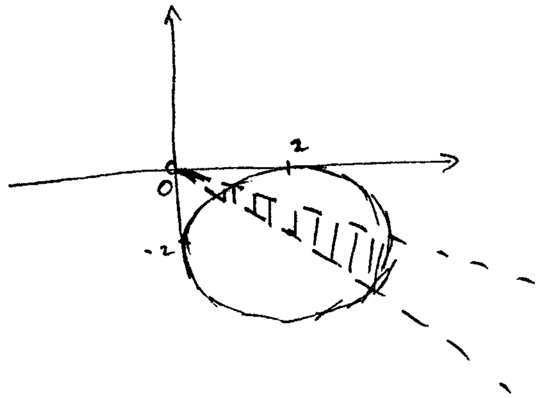
$= 3 - i$

(ii) $\frac{1}{w} = \frac{1}{(1+2i)} \cdot \frac{(1-2i)}{(1-2i)}$

$= \frac{1-2i}{1+4}$

$= \frac{1}{5} - \frac{2}{5}i$

b



c (i) Complex roots of polynomials with real coefficients exist as conjugate pairs.

(ii) Roots are $3+i, 3-i, \alpha$ where α real

Using sum of roots $= -\frac{b}{a}$

$6 + \alpha = 0$

$\alpha = -6$

$\therefore z+6$ is a factor & $(z-3-i)(z-3+i)$ is quadratic factor

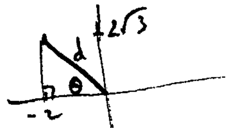
$= (z-3)^2 - i^2$

$= z^2 - 6z + 9 + 1$

$= z^2 - 6z + 10$

$\therefore P(z) = (z+6)(z^2 - 6z + 10)$

d (i) $-2 + 2\sqrt{3}i$



$\therefore -2 + 2\sqrt{3}i = 4 \text{ cis } \frac{2\pi}{3}$

$d^2 = 4 + 12$

$= 16$

$d = 4$

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

(ii) $\sqrt[4]{-2+2\sqrt{3}i} = 4^{\frac{1}{4}} \text{ cis } \frac{2\pi}{12}$

$= \sqrt{2} \text{ cis } \frac{\pi}{6}$

$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ by De-M.

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$
$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$
$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

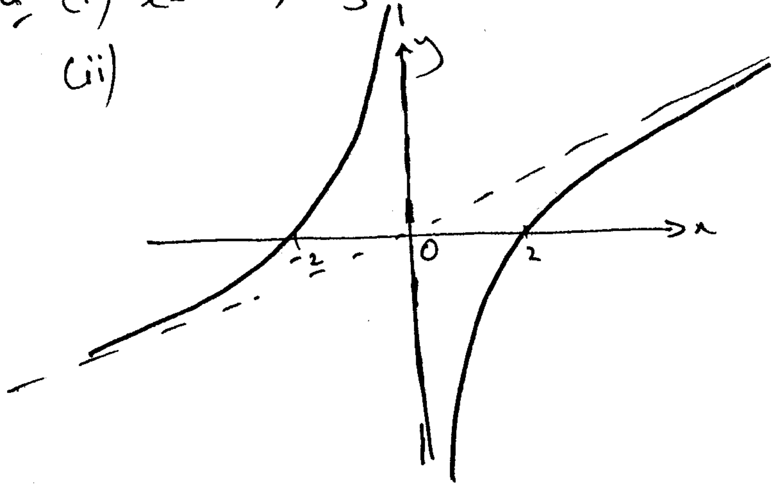
Equating reals

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

Question 3

a (i) $x = \pm 2$, $y = 0$

(ii)



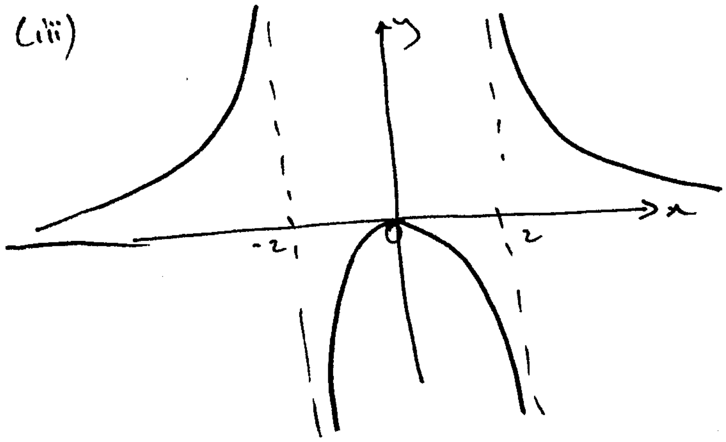
$$\frac{1}{f(x)} = \frac{x^2-4}{3x}$$

$$\lim_{x \to \infty} \frac{x^2-4}{3x}$$

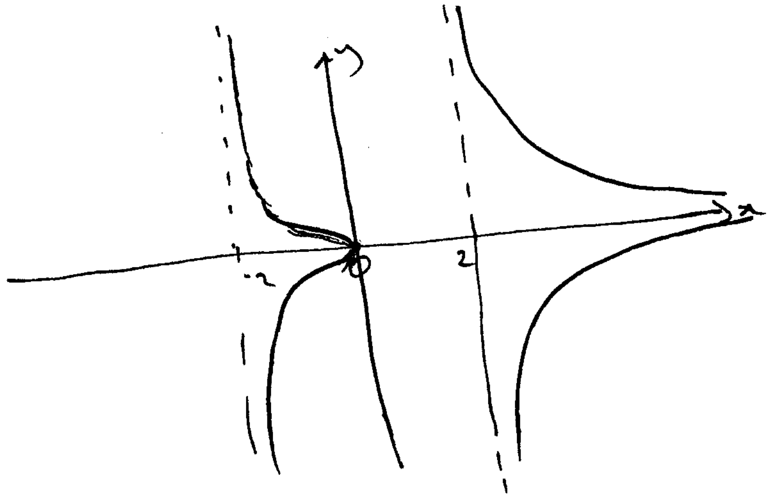
$$= \lim_{x \to \infty} \frac{x}{3} - \frac{4}{3x}$$

$\therefore$ asymptote $y = \frac{x}{3}$

(iii)



(iv)



b (i) $(\sqrt{x})^3 + 3(\sqrt{x})^2 + 4 = 0$

$x\sqrt{x} = -(3x+4)$

$x^3 = (3x+4)^2$

$= 9x^2 + 24x + 16$

$x^3 - 9x^2 - 24x - 16 = 0$

(ii) From (i) $\alpha^2 + \beta^2 + \gamma^2 = 9$

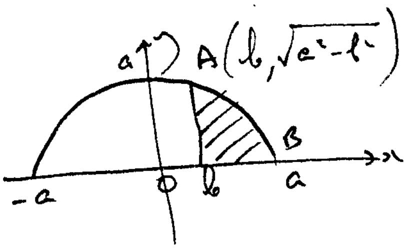
But $\alpha\beta\gamma = -4$

$\therefore \alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3 = \alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)$

$= -4 \times 9$

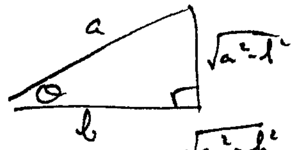
$= -36$

).



(ii) $\int_b^a \sqrt{a^2-x^2}\,dx$ = area sector OAB − area ΔOAB

$\cos\theta = \frac{b}{a}$



Now ∠AOB = θ such that

and $\sin\theta = \frac{\sqrt{a^2-b^2}}{a}$

$\therefore \int_b^a \sqrt{a^2-x^2}\,dx = \frac{1}{2}a^2\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}\cdot b\cdot\sqrt{a^2-b^2}$

Question 4

a $x^4 + 3xy - y^2 + 9 = 0$

Diff. w.r.t. x

$$4x^3 + 3x\frac{dy}{dx} + 3y - 2y\frac{dy}{dx} = 0$$

$$4x^3 + 3y = (2y - 3x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3 + 3y}{2y - 3x}.$$

For $(-1, 2)$ grad. tangent $= \frac{-4+6}{4+3} = \frac{2}{7}$

$\therefore$ grad normal $= -\frac{7}{2}$

$\therefore$ eq'n normal is $y - 2 = -\frac{7}{2}(x+1)$

$$2y - 4 = -7x - 7$$

$$7x + 2y + 3 = 0$$

b (i) $xy = c^2$

$$y = c^2x^{-1}$$

$$y' = -c^2x^{-2}$$

At Q, grad. tangent $= \frac{-c^2}{c^2q^2} = -\frac{1}{q^2}$

$\therefore$ eq'n tangent is $y - \frac{c}{q} = -\frac{1}{q^2}(x - cq)$

$$q^2y - cq = -x + cq$$

$$x + q^2y = 2cq$$

(ii) $x + p^2y = 2cp$ .. ①

$x + q^2y = 2cq$ .. ②

① - ② $(p^2 - q^2)y = 2c(p - q)$

$$y = \frac{2c}{p+q}$$

Sub into ① $x + \frac{2cp^2}{p+q} = 2cp$

$$(p+q)x + 2cp^2 = 2cp^2 + 2cpq$$

$$x = \frac{2cpq}{p+q}$$

$\therefore$ T is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

(ii) At T, with $pq = k$

$$x = \frac{2ck}{p+q} \quad \& \quad y = \frac{2c}{p+q}$$

$\therefore x = ky$

$y = \frac{x}{k}$ which is a st. line.

But T is restricted to being "between" the branches of the hyp, and not including O.

For intersection of $y = \frac{x}{k}$ and $xy = c^2$

get $\frac{x^2}{k} = c^2$

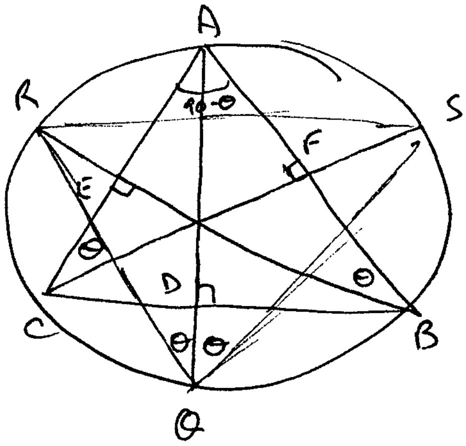
$x^2 = c^2 k$

$x = \pm c\sqrt{k}$

So, locus is that part of line $y = \frac{x}{k}$ with domain

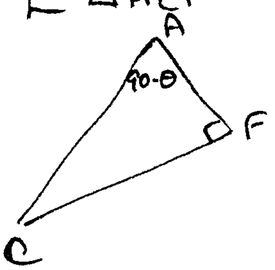
$-c\sqrt{k} < x < 0$, $0 < x < c\sqrt{k}$.

c



(i) $\angle EAB = 90 - \angle EBA$ ($\angle AEB = 90°$ as $BE \perp AC$ data) angle sum of Δ

(ii) In ΔACF



$\therefore \angle ACF = \theta$ (angle sum of Δ)

$\therefore \angle ACF = \angle ABE$

(iii) $\angle AQS = \angle ACF = \theta$ (angles on same arc)

$\angle RQA = \angle RAB = \theta$ (– – –)

$\therefore \angle RQA = \angle AQS$

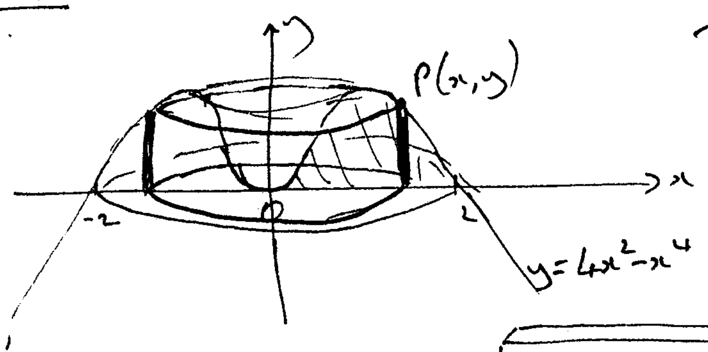
$\therefore$ AQ bisects $\angle RQS$

(iv) $\angle BEC = \angle BFC = 90°$ (data)

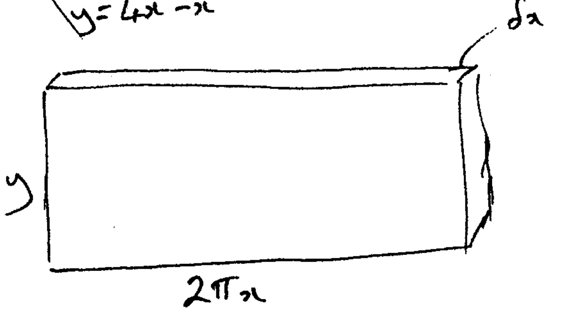
$\therefore$ EFBC concyclic (BC subtends equal angles at E, F on same side of BC)

Question 5

a



Take thin walled hollow cylindrical shell through P(x,y) on curve, thickness δx



Vol shell $\delta V = 2\pi x y \,\delta x$ — 1

Total vol $= \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x y \,\delta x$

$= 2\pi \int_0^2 xy \, dx$

$= 2\pi \int_0^2 4x^3 - x^5 \, dx$ — 1

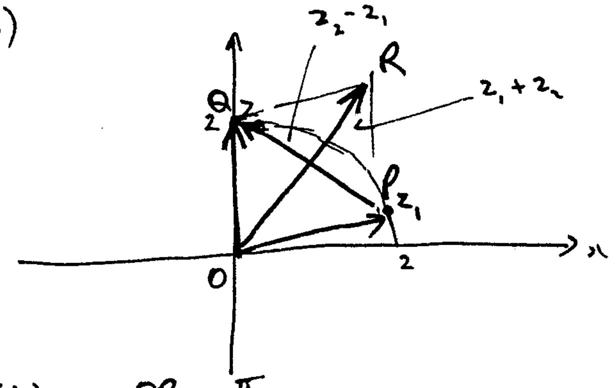
$= 2\pi \left[x^4 - \frac{x^6}{6} \right]_0^2$

$= 2\pi \left[16 - \frac{64}{6} - (0-0) \right]$

$= 2\pi \cdot \frac{16}{3}$

$= \frac{32\pi}{3}$ cu. units — 1

b (i)



$\overrightarrow{OR}$ represents $z_1 + z_2$ — 1

$\overrightarrow{PQ}$ represents $z_2 - z_1$ — 1

(ii) $\angle xOP = \frac{\pi}{12}$

$\angle POQ = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

Now OPRQ is a rhombus (parallelogram with OP = OQ)

$\therefore$ OR bisects $\angle POQ$

$\therefore \angle POR = \frac{5\pi}{24}$

$\therefore \angle xOR = \frac{\pi}{12} + \frac{5\pi}{24} = \frac{7\pi}{24}$

$\therefore \arg(z_1 + z_2) = \frac{7\pi}{24}$ — 1

(iii) diag. of rhombus are perp. $\therefore \arg(z_2 - z_1) = \frac{7\pi}{24} + \frac{\pi}{2} = \frac{19\pi}{24}$ — 1

(i) $f(x) = x\sqrt{4-x^2}$

$f(-x) = -x\sqrt{4-(-x)^2}$

$= -x\sqrt{4-x^2}$

$= -f(x)$

$f(x) = x\sqrt{4-x^2}$ is an odd function

(ii) $\int_{-2}^{2} (x-1)\sqrt{4-x^2}\,dx = \int_{-2}^{2} x\sqrt{4-x^2}\,dx - \int_{-2}^{2}\sqrt{4-x^2}\,dx$

$= 0$ − area of semi-circle of radius 2

$= -\frac{\pi}{2}\cdot 2^2$

$= -2\pi$

d (i) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = -\int_{\frac{\pi}{4}}^{0} \ln(1+\tan(\frac{\pi}{4}-u))\,du$

$u = \frac{\pi}{4} - x \quad \therefore x = \frac{\pi}{4} - u$

$dx = -du$

Limits $x = \frac{\pi}{4} \Rightarrow u = 0$

$x = 0 \Rightarrow u = \frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{\tan\frac{\pi}{4} - \tan u}{1+\tan\frac{\pi}{4}\tan u}\right]du$

$= \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{1-\tan u}{1+\tan u}\right]du$

$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan u+1-\tan u}{1+\tan u}\right]du$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right)du$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

(ii) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2 - \ln(1+\tan x)\;dx$

$= \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$\therefore 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \left[\ln 2 \times x\right]_0^{\frac{\pi}{4}}$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\cdot\frac{\pi}{4}\ln 2$

$= \frac{\pi}{8}\ln 2$

Question 6

a (i) For X-section at $x=k$, base is $2y = 2\sqrt{9-k^2}$

$\therefore$ area of X-section $= 4(9-k^2)$ — 1

(ii) Vol. X-sectional slice $\delta V = 4(9-x^2)\delta x$

Total vol $V = \lim_{\delta x \to 0} \sum_{x=-3}^{3} 4(9-x^2)\delta x$

$= 8\int_0^3 (9-x^2)\,dx$ — 1

$= 8\left[9x - \frac{x^3}{3}\right]_0^3$

$= 8\left[27 - 9 - (0-0)\right]$ — 1

$= 144$ sq. units

b (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Diff. w.r. to x $\quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{a^2}\cdot\frac{b^2}{2y} = -\frac{b^2 x}{a^2 y}$

At P, grad. of tangent $= \frac{-b^2 a\cos\theta}{a^2 b\sin\theta} = -\frac{b\cos\theta}{a\sin\theta}$ — 1

Grad. of tangent is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$a\sin\theta . y - ab\sin^2\theta = -b\cos\theta . x + ab\cos^2\theta$

$b\cos\theta . x + a\sin\theta . y = ab(\cos^2\theta + \sin^2\theta)$

$b\cos\theta x + a\sin\theta . y = ab$ — 1

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

(ii) At M, $x = \frac{a}{e}$ and $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\therefore \frac{\cancel{a}\cos\theta}{\cancel{a}e} + \frac{y\sin\theta}{b} = 1$

$\frac{y\sin\theta}{b} = 1 - \frac{\cos\theta}{e} = \frac{e-\cos\theta}{e}$

$y = \frac{b(e-\cos\theta)}{e\sin\theta}$

$\therefore$ M is $\left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$ — 1

i) grad PS $= \dfrac{b\sin\theta}{a\cos\theta - ae}$ — 1

grad. SM $= \dfrac{b(e-\cos\theta)}{\dfrac{e\sin\theta}{\frac{a}{e} - ae}}$ — 1

grad PS . grad SM =

$= \dfrac{b\sin\theta}{a(\cos\theta - e)} \cdot \dfrac{b(e-\cos\theta)}{e\sin\theta} \cdot \dfrac{e}{a(1-e^2)}$

$= \dfrac{b^2}{-a^2(1-e^2)}$

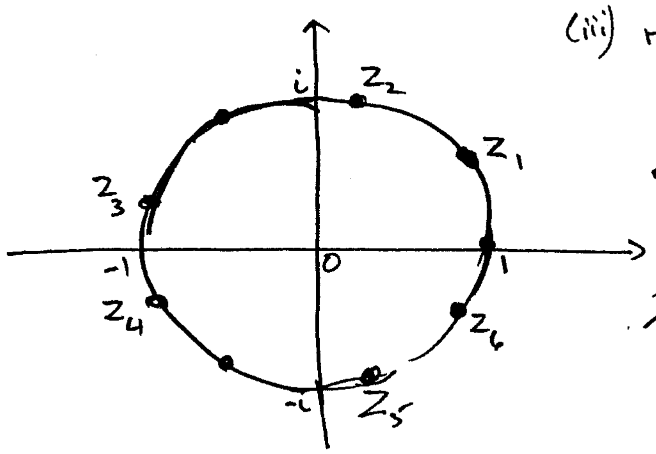
$= \dfrac{b^2}{-b^2}$

$= -1$ — 1

$\therefore PS \perp SM$

c (i) $z^9 - 1 = (z^3)^3 - 1$
$= (z^3-1)(z^6+z^3+1)$ — 1

(ii)



(iii) roots of $z^3 - 1 = 0$
are 1, cis $\frac{2\pi}{3}$, cis $-\frac{2\pi}{3}$
other roots are for 1
$z^6 + z^3 + 1 = 0$

2 { 1 for r = 1
1 for $\frac{2\pi}{9}$ difference

(iv) $z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = \text{cis}\frac{2\pi}{9} + \text{cis}\frac{4\pi}{9} + \text{cis}\frac{8\pi}{9}$ 1
$+ \text{cis}\left(-\frac{2\pi}{9}\right) + \text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\left(-\frac{8\pi}{9}\right)$
$= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right)$ 1

Question 7

a $z^3 - 3z^2 + 5z - 2 = 0$

(i) $\alpha + \beta + \gamma = 3$

$\alpha\beta + \alpha\gamma + \beta\gamma = 5$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 9 - 10$ — 1

$= -1$

(ii) Since $\Sigma\alpha^2$ is negative, at least one of $\alpha^2, \beta^2, \gamma^2$ is negative

ie at least one of α, β, γ is complex. — 1

But coeff of eqn are real

$\therefore$ complex roots exist as conjugate pairs. — 1

Hence 2 complex roots

ie 1 real root.

b (i) $\ddot{x} = -(10 + 10v^2)$

$\frac{dv}{dt} = -10(1+v^2)$

$\frac{dt}{dv} = -\frac{1}{10(1+v^2)}$

$t = -\frac{1}{10}\int \frac{dv}{1+v^2}$ — 1

$t = -\frac{1}{10}\tan^{-1}v + c$

When $t=0$, $v=u$ $\therefore$ $c = \frac{1}{10}\tan^{-1}u$

$\therefore t = \frac{1}{10}(\tan^{-1}u - \tan^{-1}v)$ — 1

(ii) Greatest height when $v=0$ — 1

$\therefore t = \frac{1}{10}\tan^{-1}u$

(iii)

$v\frac{dv}{dx} = -10(1+v^2)$

$\frac{dv}{dx} = -\frac{10(1+v^2)}{v}$

$\frac{dx}{dv} = -\frac{v}{10(1+v^2)}$

$x = -\frac{1}{10}\int \frac{v\,dv}{1+v^2}$ — 1

$x = -\frac{1}{10}\cdot\frac{1}{2}\ln(1+v^2) + c$

When $x=0$, $v=u$ $\therefore$ $c = \frac{1}{20}\ln(1+u^2)$

$x = \frac{1}{20}\ln\left(\frac{1+u^2}{1+v^2}\right)$ — 1

(iv) Greatest height when $v=0$ $\therefore x = \frac{1}{20}\ln(1+u^2)$ — 1

(i) Prove true for $n=1$ & $n=2$

$u_1 = 5^1 - 3^1 = 2$ True $\qquad u_2 = 5^2 - 3^2 = 16$ True $\qquad$ } 1

$\therefore$ true for $n=1$ & $n=2$

Assume true for $n = 1, 2, \ldots k$

ie assume $u_1 = 5^1 - 3^1$

$u_2 = 5^2 - 3^2$

$\vdots$

$u_k = 5^k - 3^k$ $\qquad$ } 1

Prove true for $n=k+1$ if true for $n=1,2,\ldots k$

ie prove $u_{k+1} = 5^{k+1} - 3^{k+1}$

Now $u_{k+1} = 8u_k - 15u_{k-1}$

$= 8(5^k - 3^k) - 15(5^{k-1} - 3^{k-1})$ $\qquad$ – 1

$= 8\times5^k - 3\times5\times5^{k-1} + 5\times3\times3^{k-1} - 8\times3^k$

$= 8\times5^k - 3\times5^k + 5\times3^k - 8\times3^k$

$= 5\times5^k - 3\times3^k$

$= 5^{k+1} - 3^{k+1}$ $\qquad$ – 1

Conclusion

True for $n=k+1$ if true for $n=1,2,\ldots k$

But true for $n=1,2$

$\therefore$ true for all integer $n \geq 1$

(ii) $u_1 + u_2 + \ldots + u_n = 5^1 - 3^1 + 5^2 - 3^2 + \ldots + 5^n - 3^n$

$= (5^1 + 5^2 + \ldots + 5^n) - (3^1 + 3^2 + \ldots + 3^n)$ $\qquad$ – 1

$=$ GP $a=5$, $r=5$ $\qquad$ GP $a=3$, $r=3$

$= \dfrac{5(5^n - 1)}{4} - \dfrac{3(3^n - 1)}{2}$

$= \dfrac{5^{n+1} - 5}{4} - \dfrac{3^{n+1} - 3}{2}$

$= \dfrac{5^{n+1} - 5 - 2\times3^{n+1} + 6}{4}$

$= \dfrac{5^{n+1} - 2\times3^{n+1} + 1}{4}$ $\qquad$ – 1

Question 8

a $x^x a^{\ln x} = x$

$\therefore \ln x^x a^{\ln x} = \ln x$

$\ln x^x + \ln a^{\ln x} = \ln x$

$x \ln x + \ln x . \ln a = \ln x$ — 1

$\ln x(x + \ln a - 1) = 0$

$\ln x = 0$ or $x + \ln a - 1 = 0$

$x = 1$ or $x = 1 - \ln a$ — 1

b $I_n = \int_0^{\frac{\pi}{4}} \sec x \tan^n x \, dx$

(i) $I_1 = \int_0^{\frac{\pi}{4}} \sec x \tan x \, dx$

$= \left[\sec x\right]_0^{\frac{\pi}{4}}$

$= \sec \frac{\pi}{4} - \sec 0$

$= \sqrt{2} - 1$ — 1

(ii) $I_n = \int_0^{\frac{\pi}{4}} \sec x \tan^n x \, dx$ $= \left[\tan^{n-1} x \sec x\right]_0^{\frac{\pi}{4}}$ — 1

$- \int_0^{\frac{\pi}{4}} (n-1) \tan^{n-2} x \sec^2 x \sec x \, dx$

$u = \tan^{n-1} x$ $\quad v = \sec x$

$u' = (n-1)\tan^{n-2} x \sec^2 x$ $\quad v' = \sec x \tan x$

$= \left[\left(\tan \frac{\pi}{4}\right)^{n-1} . \sec \frac{\pi}{4} - 0\right]$

$-(n-1)\int_0^{\frac{\pi}{4}} \sec x \tan^{n-2} x (1 + \tan^2 x) \, dx$

$= \sqrt{2} - (n-1)\left[\int_0^{\frac{\pi}{4}} \sec x \tan^{n-2} x \, dx - \int_0 \sec x \tan^n x \, dx\right]$

$I_n = \sqrt{2} - (n-1)\left[I_{n-2} - I_n\right]$ — 1

$= \sqrt{2} - (n-1) I_{n-2} - (n-1) I_n$

$\therefore n I_n = \sqrt{2} - (n-1) I_{n-2}$

$I_n = \frac{\sqrt{2}}{n} - \frac{(n-1)}{n} I_{n-2}$ — 1

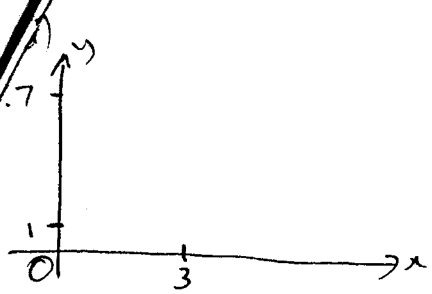
(iii) $I_5 = \frac{\sqrt{2}}{5} - \frac{4}{5} I_3$

$= \frac{\sqrt{2}}{5} - \frac{4}{5}\left[\frac{\sqrt{2}}{3} - \frac{2}{3} I_1\right]$

$= \frac{\sqrt{2}}{5} - \frac{4\sqrt{2}}{15} + \frac{8}{15}(\sqrt{2} - 1)$

$= \frac{3\sqrt{2}}{15} - \frac{4\sqrt{2}}{15} + \frac{8\sqrt{2}}{15} - \frac{8}{15}$

$= \frac{7\sqrt{2} - 8}{15}$ — 1



Y: centre of motion $y = 4$ $\therefore b = 4$
amplitude $a = 3$ (1)
period $\frac{2\pi}{n} = \pi$ $\therefore n = 2$
Initially at $y = 7$ $\therefore$ $\alpha = 0$ (1)

$\therefore y = 4 + 3\cos 2t$

X: centre of motion $x = 3$ $\therefore b = 3$ (1)
period $\frac{2\pi}{n} = 2\pi$ $\therefore n = 1$ (1)
At centre $v^2 = n^2 a^2$ $\therefore a = 2$
Initially at centre, moving → $\therefore \alpha = -\frac{\pi}{2}$ (1)

$\therefore x = 3 + 2\cos\left(t - \frac{\pi}{2}\right)$
$= 3 + 2\cos\left(\frac{\pi}{2} - t\right)$ (1)
$= 3 + 2\sin t$

(ii) Area $\triangle OXY = \frac{1}{2}(4 + 3\cos 2t)(3 + 2\sin t)$ (1)
$\frac{dA}{dt} = \frac{1}{2}\left[(4 + 3\cos 2t) \times 2\cos t + (3 + 2\sin t) \times -6\sin 2t\right]$

When $t = \frac{5\pi}{4}$, $\frac{dA}{dt} = \frac{1}{2}\left[\left(4 + 3\cos\frac{5\pi}{2}\right) \times 2\cos\frac{5\pi}{4} + \left(3 + 2\sin\frac{5\pi}{4}\right) \times -6\sin\frac{5\pi}{2}\right]$ (1)
$= \frac{1}{2}\left[(4 + 0) \times -\frac{2}{\sqrt{2}} + \left(3 - \frac{2}{\sqrt{2}}\right) \times -6\right]$
$= \frac{1}{2}\left[-\frac{8}{\sqrt{2}} - 18 + \frac{12}{\sqrt{2}}\right]$
$= \frac{1}{2}\left[\frac{4}{\sqrt{2}} - 18\right]$
$= \frac{2}{\sqrt{2}} - 9$
$= \sqrt{2} - 9$

$\therefore$ area decreasing at $(9 - \sqrt{2})$ cm^2 s^{-1}
or area changing at $(\sqrt{2} - 9)$ cm^2 s^{-1}