2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (120)

- Attempt Questions 1-8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your name and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) Find $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}dx$ 2

b) i) Use partial fractions to evaluate 3

$$\int_0^1 \frac{5\,dt}{(2t+1)(2-t)}$$

ii) Hence, and by using the substitution $t = \tan\dfrac{\theta}{2}$, evaluate 3

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta}$$

c) By using the table of standard integrals and manipulation, find 2

$$\int_0^1 \frac{dx}{\sqrt{4x^2+36}}$$

d) If $I = \displaystyle\int e^x \sin x\, dx$, find I. 3

e) By completing the square find 2

$$\int \frac{dx}{\sqrt{1-4x-x^2}}$$

End of Question 1

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) Given $z_1 = i\sqrt{2}$ and $z_2 = \dfrac{2}{1-i}$

i) Express z_1 and z_2 in Mod / Arg form. 2

ii) If $z_1 = \omega z_2$, find the complex number ω in Mod / Arg form. 1

iii) α) On the Argand diagram plot the points P and Q representing the complex numbers z_1 and z_2 respectively. 1

β) Show how to construct the point R representing $z_1 + z_2$. 1

iv) α) Find $\arg(z_1 + z_2)$ 1

β) Find the exact value of $\tan\dfrac{3\pi}{8}$ 1

b) Draw a diagram to illustrate the locus of points z in the complex plane such that

i) $Arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ 2

ii) $\text{Re}(z) \leq 1$ and $|z-3+4i| \leq 5$ 2

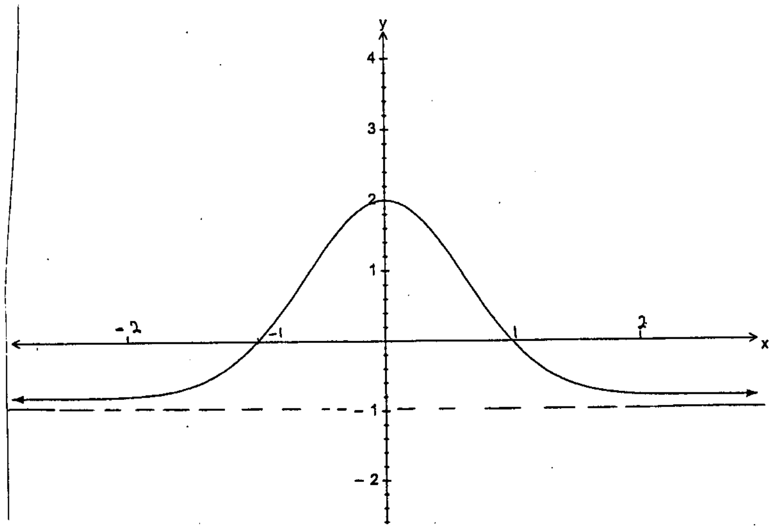
iii) $|z-3| + |z+1| = 6$ 2

c) Given that $(x-2)$ is a factor of $x^3 - 4x^2 + 7x - 6$ reduce $x^3 - 4x^2 + 7x - 6$ to irreducible factors over the complex field. 2

End of Question 2

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) The sketch below is the even function $y = f(x)$.



On separate diagrams sketch each of the following , clearly showing all important features

(Each of the graphs for the questions below to be done on the sheets with the graph of $y = f(x)$ provided)

i) $y = f(x) - 2$ 1

ii) $y = f(x-2)$ 1

iii) $y = |f(x)|$ 1

iv) $y = [f(x)]^2$ 2

v) $y = \dfrac{1}{f(x)}$ 2

vi) $y^2 = f(x)$ 2

$y = \sqrt{f(x)}$

b) Nine people gather to play football by forming two teams of four to play each other with the remaining person to be the referee.

i) In how many ways can the teams be formed 1

ii) If two particular people are not to be in the same team , how many ways are there then to choose the teams 2

c) A particle moving in Simple Harmonic Motion has a speed of $10\sqrt{3}\ ms^{-1}$ at the centre of its motion. Find its speed when it is at half of its amplitude. 3

End of Question 3

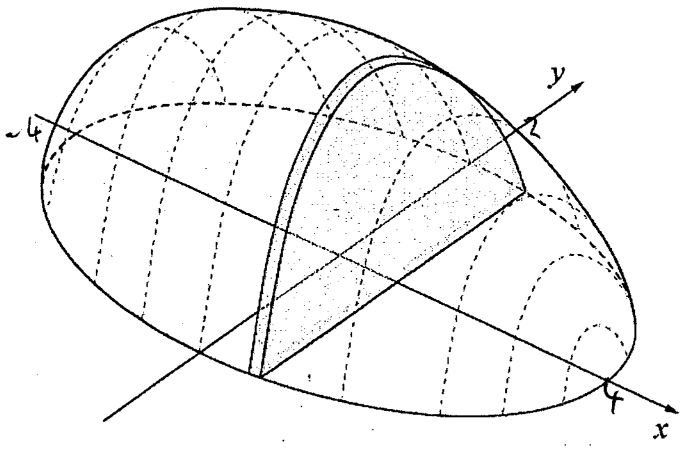
Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) Show that the area enclosed between the parabola $x^2 = 4ay$ and its latus rectum is $\dfrac{8a^2}{3}$ units^2. **3**

(The latus rectum is the focal chord perpendicular to the axis of the parabola)

b) A solid figure has as its base, in the xy plane, the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

Cross-sections perpendicular to the x-axis are parabolas with latus rectums in the xy plane.



i) Show that the area of the cross-section at $x = h$ is $\dfrac{16 - h^2}{6}$ units^2. 4

[Use your answer to part (a)]

ii) Hence, find the volume of this solid. 2

c) i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$, (Let $u = a - x$) 2

ii) Consider $f(x) = \dfrac{1}{1+\tan x}$ where $0 \le x \le \dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right) = 0$ 2

show that $f(x) + f\left(\dfrac{\pi}{2} - x\right) = 1$

iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}dx$ 2

End of Question 4

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) i) On the same diagram sketch the graphs of the ellipses $E_1 : \frac{x^2}{4} + \frac{y^2}{3} = 1$ and $E_2 : \frac{x^2}{16} + \frac{y^2}{12} = 1$, showing clearly the intercepts on the axes. Show the coordinates of the foci and the equations of the directices of the ellipse E_1. 5

ii) $P\left(2\cos p, \sqrt{3}\sin p\right)$ where $0 < p < \frac{\pi}{2}$, is a point on the ellipse E_1. Use differentiation to show that the tangent to the ellipse E_1 at P has equation 3

$$\frac{x\cos p}{2} + \frac{y\sin p}{\sqrt{3}} = 1.$$

iii) The tangent to the ellipse E_1 at P meets the ellipse E_2 at the points $Q\left(4\cos q, 2\sqrt{3}\sin q\right)$ and $R\left(4\cos r, 2\sqrt{3}\sin r\right)$, where $-\pi < q < \pi$ and $-\pi < r < \pi$. Show that q and r differ by $\frac{2\pi}{3}$. 2

b) Let $P(x) = x^4 - 5x + 2$. The equation $P(x) = 0$ has roots α, β, γ *and* δ.

i) Evaluate $P(0)$ & $P(1)$. Hence show that the equation $x^4 - 5x + 2 = 0$ has a real root between $x = 0$ and $x = 1$ 1

ii) Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$ *and* δ^2. Hence or otherwise show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$. 2

iii) Find the number of non-real roots of $x^4 - 5x + 2 = 0$, giving full reasons for your answer. 2

End of Question 5

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) i) Given $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. 2

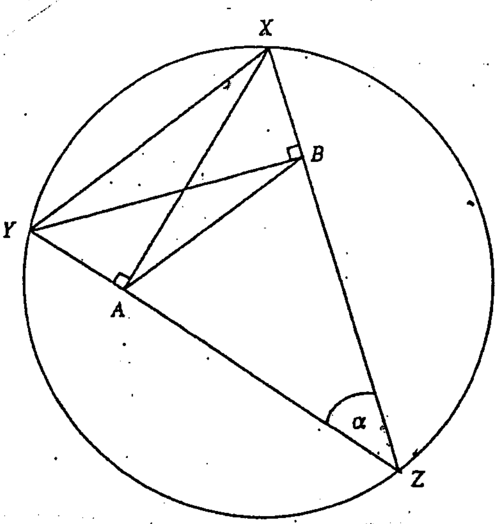
ii) Write down $\left(z + z^{-1}\right)$ in terms of $\cos n\theta$ 1

iii) Show that $\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$ 1

iv) Hence, express $\cos^4\theta$ in terms of $\cos n\theta$. 3

v) Using your answer to part (ii), evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta$. 6 2

b)



XY is a fixed chord of a circle. Z is a point on the major arc XY. The perpendicular from X to the chord YZ meets YZ at A. The perpendicular from Y to the chord XZ meets XZ at B. $\angle XZY = \alpha$

i) Copy the diagram

ii) Show that ABXY is a cyclic quadrilateral 1

similar

iii) Show that $\triangle ABZ \,|||\, \triangle XYZ$. 2

iv) Hence show that $AB = XY\cos\alpha$. 2

v) Deduce that as Z moves on the major arc XY, the length of AB is constant. 1

End of Question 6

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) A mass of m kg is allowed to fall under gravity from a stationary position h metres above the ground. It experiences resistance proportional to the square of its velocity, $v\,\text{ms}^{-1}$.

i) Explain why the equation for this motion is $\ddot{x} = g - kv^2$, (where k is a constant). 1

ii) Show that $\ddot{x} = v\dfrac{dv}{dx}$ 1

iii) Hence show that $v^2 = \dfrac{g}{k}\left(1 - e^{-2kx}\right)$ 4

iv) Find the velocity at which the mass hits the ground in terms of g, h and k. 1

v) Calculate the terminal velocity of the object, given that h is sufficiently large to allow terminal velocity to be reached. 1

b) i) On the same axes sketch the graphs of $y = \sqrt{1-x^2}$ and $y = \sqrt{\dfrac{1}{1-x^2}}$ 2

ii) The region bounded by the curve $y = \dfrac{1}{\sqrt{1-x^2}}$, the coordinate axes and the line $x = \dfrac{1}{2}$ is rotated through one complete revolution about the line $x = 6$. Use the method of cylindrical shells to show that volume V *units*^{3} of the solid of revolution is given by 3

$$V = 2\pi \int_0^{1/2} \frac{6-x}{\sqrt{1-x^2}}\,dx\,.$$

iii) Hence find the value of V in simplest exact form 2

End of Question 7

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

a) i) If $I_n = \int_0^1 \left(x^2-1\right)^n dx$, $n = 0,1,2,.....$ show that 4

$$I_n = \frac{-2n}{2n+1} I_{n-1} \quad , \quad n = 1,2,3,......$$

ii) Hence use the method of Mathematical Induction to show that 4

$$I_n = \frac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!}$$

for all positive integers n

b) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$.

ie $P(x) = \left(x^2 - a^2\right).Q(x) + px + q$ for some polynomial $Q(x)$

i) Show that $p = \frac{1}{2a}\{P(a) - P(-a)\}$ and $q = \frac{1}{2}\{P(a) + P(-a)\}$ 3

ii) Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided

by $x^2 - a^2$ for the cases

α) n even 2

β) n odd 2

End of Examination

1a $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$ let $u = \tan x$, $du = \sec^2 x\,dx$

$= \int e^u\,du$

$= e^u + c$

$= e^{\tan x} + c$ **2**

b (i) $\frac{5}{(2t+1)(2-t)} = \frac{A}{2t+1} + \frac{B}{2-t}$ where $A = 2$, $B = 1$ by Heaviside's rule

$\therefore \int_0^1 \frac{5\,dt}{(2t+1)(2-t)} = \int_0^1 \left(\frac{2}{2t+1} + \frac{1}{2-t}\right)dt$

$= \left[\ln(2t+1) - \ln(2-t)\right]_0^1$

$= \ln 2$ **3**

(ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{6t}{1+t^2} + \frac{4-4t^2}{1+t^2}}$

$t = \tan\frac{\theta}{2}$

$d\theta = \frac{2\,dt}{1+t^2}$

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\theta = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$\theta = 0 \Rightarrow t = \tan 0 = 0$

$= \int_0^1 \frac{2\,dt}{6t+4-4t^2}$

$= \int_0^1 \frac{dt}{(2t+1)(t-2)}$

$= \frac{1}{5}\ln 2$ **3**

c $\int_0^1 \frac{dx}{\sqrt{4x^2+36}} = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{x^2+9}}$

$= \frac{1}{2}\left[\ln\left(x + \sqrt{x^2+9}\right)\right]_0^1$

$= \frac{1}{2}\ln\left(\frac{1+\sqrt{10}}{3}\right)$ **2**

d $I = \int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$

$u = e^x$ $v = -\cos x$
$u' = e^x$ $v' = \sin x$

$u = e^x$ $v = \sin x$
$u' = e^x$ $v' = \cos x$

$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$

$\therefore 2I = e^x(\sin x - \cos x)$

$I = \frac{1}{2}e^x(\sin x - \cos x) + c$ **3**

e $\int \frac{dx}{\sqrt{1-4x-x^2}} = \int \frac{dx}{1+4-(x^2+4x+4)}$

$= \int \frac{dx}{5-(x+2)^2}$

$= \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$ **2**

2a (i) $z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{2}$ 1

$z_2 = \frac{2}{(1-i)} \cdot \frac{(1+i)}{(1+i)}$

$= 1+i$ 1

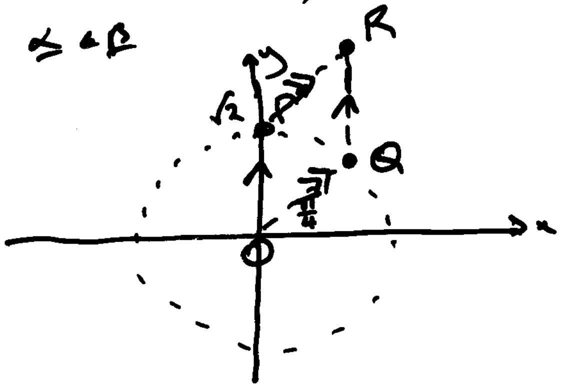
$= \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

(ii) $w = \frac{z_1}{z_2}$

$= \frac{\sqrt{2}\,\text{cis}\,\frac{\pi}{4}}{\sqrt{2}\,\text{cis}\,\frac{\pi}{2}}$ 1

$= \text{cis}(-\frac{\pi}{4})$

(iii) α & β



P & Q 1

R 1

(iv) α OPRQ is a rhombus

$\therefore \arg(z_1+z_2) = \angle xOR$

$= \angle xOQ + \angle QOR$

$= \frac{\pi}{4} + \frac{\pi}{8}$ (diag. of rhombus bisects angles)

$= \frac{3\pi}{8}$ 1

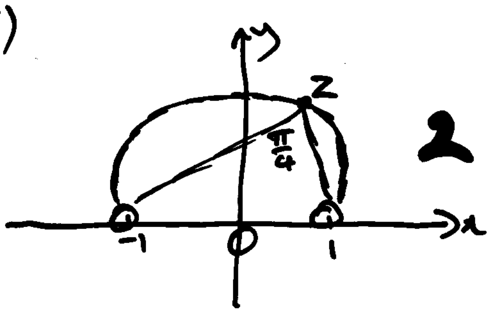
β $z_1 + z_2 = i\sqrt{2} + 1 + i$

$= 1 + (1+\sqrt{2})i$

$\therefore$ ~~[illegible]~~ $\therefore \tan\frac{3\pi}{8} = \frac{1+\sqrt{2}}{1}$

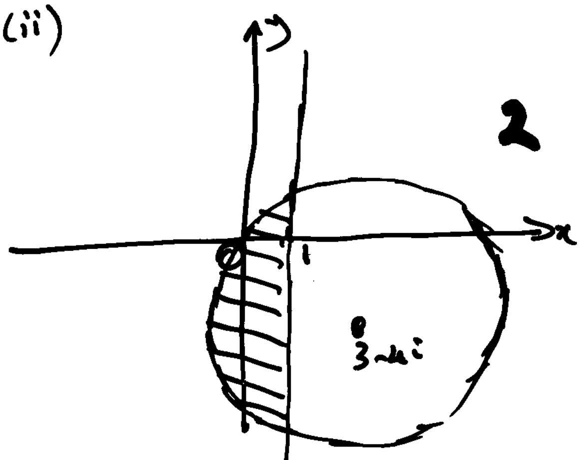
$= 1+\sqrt{2}$ 1

b (i)



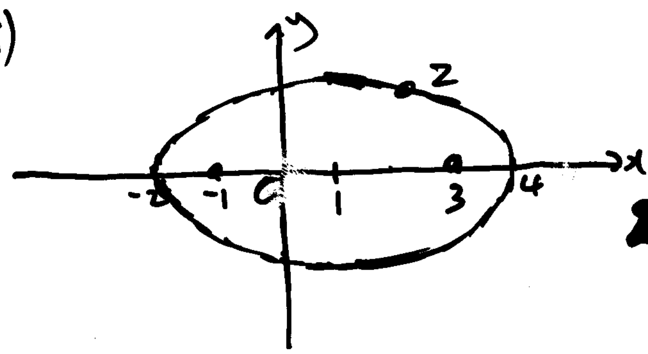
2

(ii)



2

(iii)



2

c Let $P(x) = x^3 - 4x^2 + 7x - 6$

$$\begin{array}{r} x^2 - 2x + 3 \\ x-2 \overline{) x^3 - 4x^2 + 7x - 6} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ -2x^2 + 7x \qquad \\ \underline{-2x^2 + 4x} \qquad \\ 3x - 6 \end{array}$$

$\therefore P(x) = (x-2)(x^2-2x+3)$

$= (x-2)(x^2-2x+1+2)$

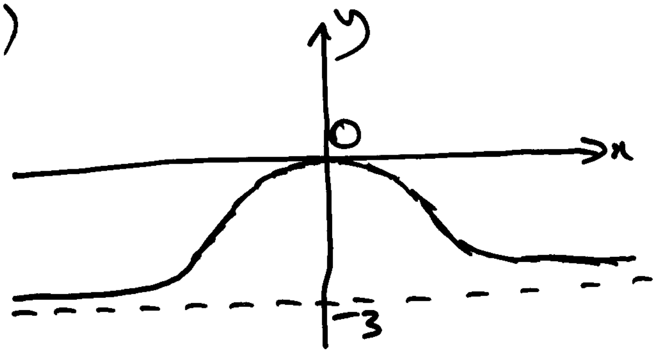
$= (x-2)((x-1)^2+2)$

$= (x-2)\left[(x-1)^2 - (\sqrt{2}i)^2\right]$

$= (x-2)(x-1-\sqrt{2}i)(x-1+\sqrt{2}i)$

2

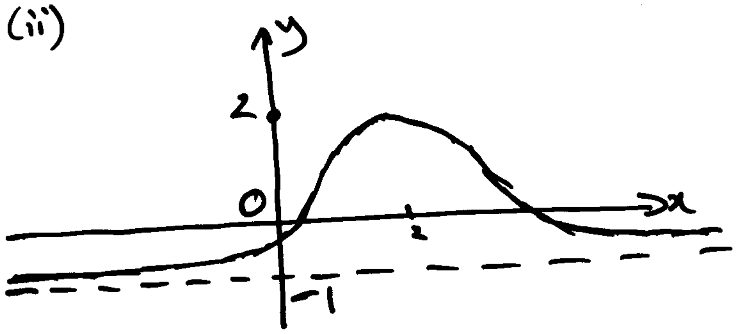
3a (i)



$y = f(x) - 2$

1

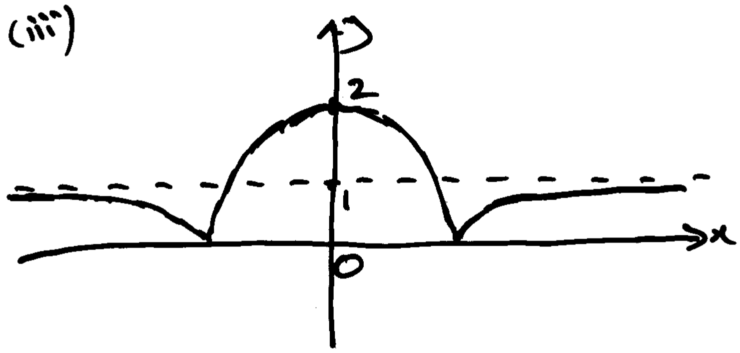
(ii)



$y = f(x-2)$

1

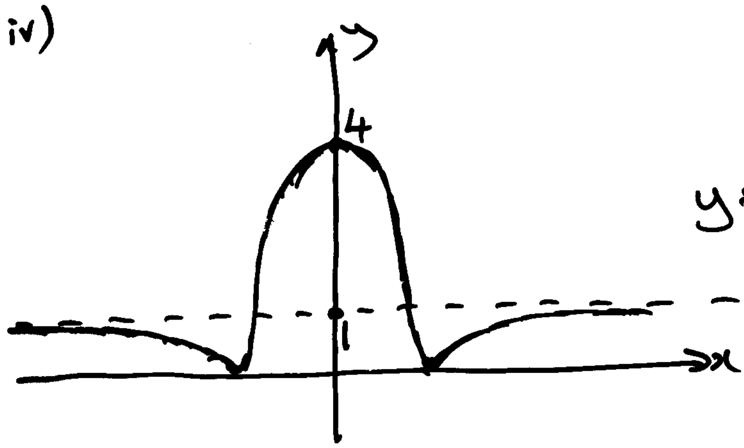
(iii)



$y = |f(x)|$

1

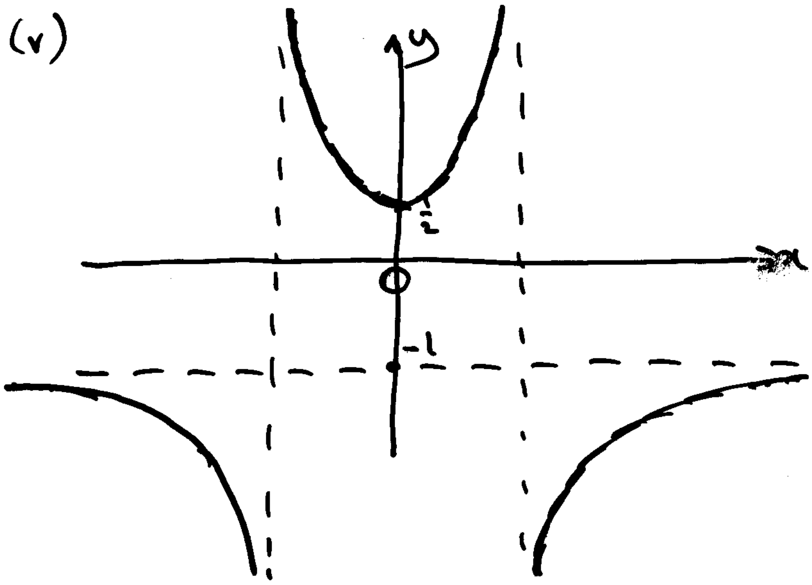
(iv)



$y = [f(x)]^2$

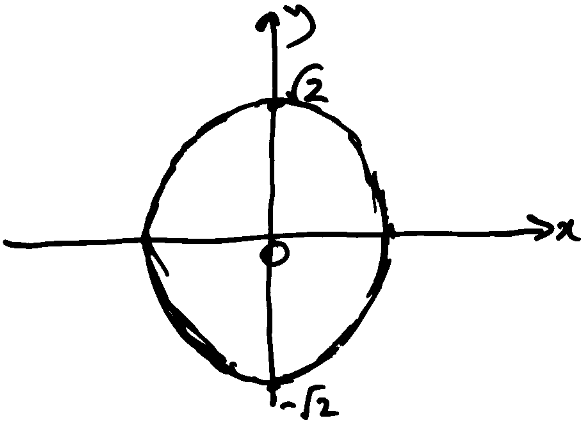
2

(v)



$y = \frac{1}{f(x)}$ **2**

(vi)



$y^2 = f(x)$ **2**

b (i) $\dfrac{\text{Choose referee} \times \text{choose 4 from 8}}{2} = \dfrac{{}^9C_1 \times {}^8C_4}{2}$

$= 315$ **1**

(ii) Let A, B be the two particular people

ways = (# ways with A or B ref) + (# ways with A, B playing)

$= \dfrac{2 \times {}^8C_4}{2} + 7 \times$ (team with A in, B not) $\times$ (team with B in, A not)

$= {}^8C_4 + 7 \times {}^6C_3$

$= 210$ **2**

c Set up S.H.M. with O at centre of motion

$\therefore v^2 = n^2(a^2 - x^2)$

When $x = 0$, $v = 10\sqrt{3}$ $\Longrightarrow 300 = n^2a^2$

When $x = \frac{a}{2}$ $v^2 = n^2\left(a^2 - \frac{a^2}{4}\right)$

$= \frac{3n^2a^2}{4}$

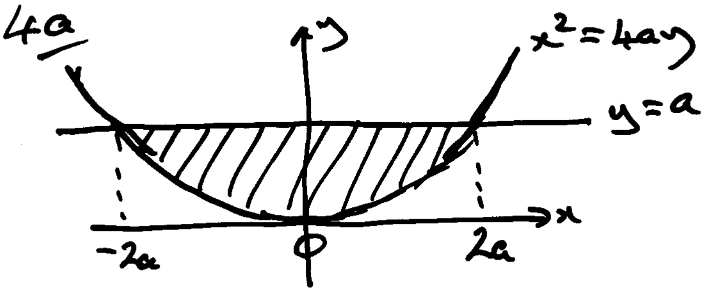
$= \frac{3}{4} \times 300$

$= 225$

$\therefore v = 15 \text{ ms}^{-1}$

3

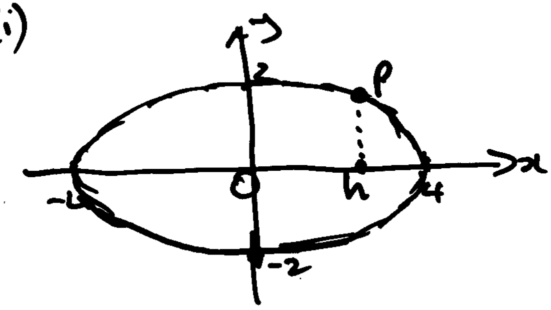
4a



$$A = 2\int_0^{2a}\left(a - \frac{x^2}{4a}\right)dx$$
$$= 2\left[ax - \frac{x^3}{12a}\right]_0^{2a}$$
$$= 2\left[2a^2 - \frac{8a^3}{12a} - (0-0)\right]$$
$$= \frac{8a^2}{3}$$

3

b (i)



Take cross section through point P on ellipse where $x = h$.

$$\therefore \frac{h^2}{16} + \frac{y^2}{4} = 1$$
$$h^2 + 4y^2 = 16$$
$$y^2 = \frac{16-h^2}{4}$$
$$y = \pm\frac{\sqrt{16-h^2}}{2}$$

$\therefore$ length of latus rectum is $2y = \sqrt{16-h^2}$

But length of latus rectum is $4a$ $\quad \therefore a = \frac{\sqrt{16-h^2}}{4}$

From a, area of parabolic cross section is $\frac{8}{3}\left(\frac{\sqrt{16-h^2}}{4}\right)^2$
$$= \frac{8}{3}\left(\frac{16-h^2}{16}\right)$$
$$= \frac{16-h^2}{6}$$

4

(ii) Vol. of slice $\delta V = \frac{16-h^2}{6}\cdot\delta h$

$$\text{Total vol} = 2\times\lim_{\delta h\to 0}\sum_{h=0}^{4}\frac{16-h^2}{6}\cdot\delta h$$
$$= 2\int_0^4 \frac{16-h^2}{6}\,dh$$
$$= \frac{1}{3}\left[16h - \frac{h^3}{3}\right]_0^4$$
$$= \frac{128}{9} \text{ cu. units}$$

2

c (i) $\int_0^a f(x)\,dx$

let $u = a - x$

$du = -dx$

Limits $x = a \Rightarrow u = 0$

$x = 0 \Rightarrow u = a$

$$= \int_a^0 f(a-u) \times -1\, du$$

$$= \int_0^a f(a-u)\, du$$

$$= \int_0^a f(a-x)\, dx$$

2

(ii) $LHS = f(x) + f(\frac{\pi}{2} - x)$

$$= \frac{1}{1+\tan x} + \frac{1}{1+\tan(\frac{\pi}{2}-x)}$$

$$= \frac{1}{1+\tan x} + \frac{1}{1+\cot x} \cdot \frac{\tan x}{\tan x}$$

$$= \frac{1}{1+\tan x} + \frac{\tan x}{\tan x + 1}$$

$$= \frac{1+\tan x}{1+\tan x}$$

$= 1$

$= RHS$

2

(iii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan(\frac{\pi}{2}-x)}$ from (i)

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1+\tan x}\right) dx \quad \text{from (ii)}$$

$$= \int_0^{\frac{\pi}{2}} 1\, dx - \int_0^{\pi/2} \frac{dx}{1+\tan x}$$

$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \int_0^{\pi/2} 1\, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

2

5a (i) $E_1 : \frac{x^2}{4} + \frac{y^2}{3} = 1$

$a = 2$, $b = \sqrt{3}$

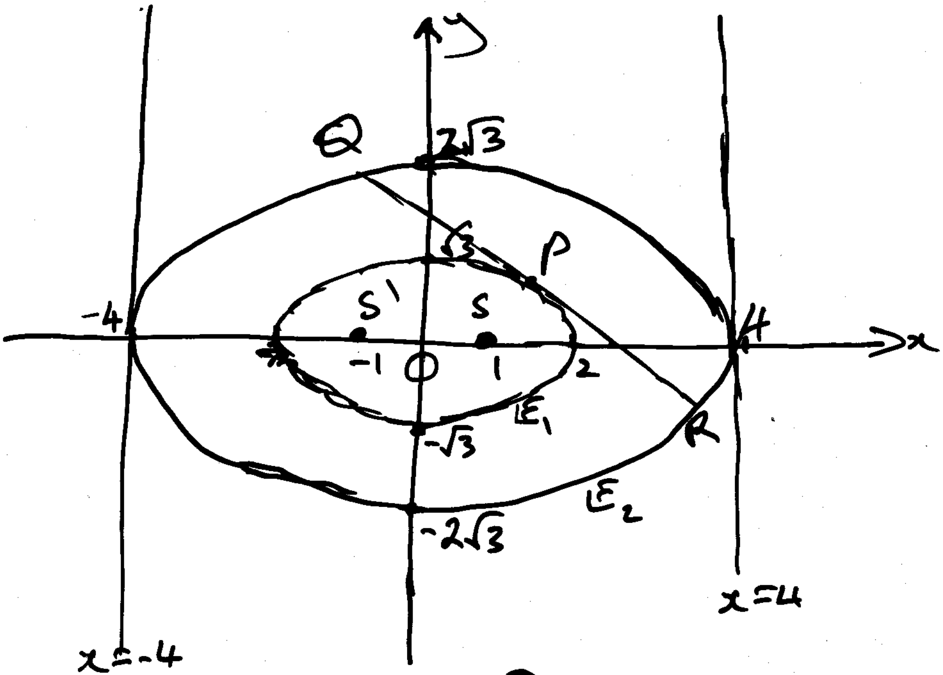
$e = \frac{1}{2}$ — 1

Foci $(\pm 1, 0)$

Directrices $x = \pm 4$

E_1 intercepts — 1
foci — 1
directrices — 1

E_2 intercepts — 1



(ii) At P: $x = 2\cos p$, $y = \sqrt{3}\sin p$ $\Rightarrow$ $\frac{dx}{dp} = -2\sin p$, $\frac{dy}{dp} = \sqrt{3}\cos p$ } 1 { OR implicit diff^n to get $\frac{dy}{dx} = \frac{-3x}{4y}$

At P: $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$\therefore$ grad. tangent at P $= \frac{\sqrt{3}\cos p}{-2\sin p}$ } 1 { At P, grad. of tangent $= \frac{-3 \times 2\cos p}{4 \times \sqrt{3}\sin p} = \frac{-\sqrt{3}\cos p}{2\sin p}$

Eq'n tangent at P is $y - \sqrt{3}\sin p = \frac{\sqrt{3}\cos p}{-2\sin p}(x - 2\cos p)$

etc

to get $\frac{x\cos p}{2} + \frac{y\sin p}{\sqrt{3}} = 1$ } 1

(iii) At Q, $x = 4\cos q$, $y = 2\sqrt{3}\sin q$ } satisfies $\frac{x\cos p}{2} + \frac{y\sin p}{\sqrt{3}} = 1$

$\therefore \frac{4\cos q\cos p}{2} + \frac{2\sqrt{3}\sin q\sin p}{\sqrt{3}} = 1$

$\cos q\cos p + \sin q\sin p = \frac{1}{2}$

$\cos(q - p) = \frac{1}{2}$

Similarly, at R,

$\cos(r - p) = \frac{1}{2}$ } 1

Hence $q - p = \pm\frac{\pi}{3}$ and $r - p = \mp\frac{\pi}{3}$

(q, r are distinct values)

$\therefore (q - p) - (r - p) = \pm\frac{\pi}{3} - \left(\mp\frac{\pi}{3}\right)$

$q - r = \pm\frac{2\pi}{3}$

ie q and r differ by $\frac{2\pi}{3}$ } 1

b (i) $P(0)=2$, $P(1)=-2$ — $P(0)$ and $P(1)$ are on opposite sides of x axis and $P(x)$ is <u>continuous</u>
$\therefore P(x)$ has real root between $x=0$ & $x=1$ — 1

(ii) Eq'n with roots $\alpha^2, \beta^2, \gamma^2, \delta^2$ is $P(\sqrt{x})=0$

$$(\sqrt{x})^4 - 5\sqrt{x} + 2 = 0$$
$$x^2 + 2 = 5\sqrt{x}$$
$$x^4 + 4x^2 + 4 = 25x$$
$$x^4 + 4x^2 - 25x + 4 = 0 \quad - 1$$

Using sum of roots $= -\frac{b}{a}$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0 \quad - 1$$

(iii) For the eq'n $P(x)=0$, with roots $\alpha, \beta, \gamma, \delta$
None of $\alpha, \beta, \gamma, \delta = 0$ as $x=0$ not a solution
and one root, say α, is real (from part (i))
$\therefore \alpha^2 > 0$ & so $\beta^2 + \gamma^2 + \delta^2 < 0$
ie at least one of β, γ, δ is not real — 1

But coefficients of $P(x)$ are real
$\therefore$ non-real roots as conjugate pairs — 1
$\therefore$ 2 non-real roots.

6a (i) $z = \cos\theta + i\sin\theta$

$$\left.\begin{aligned} z^n &= \cos n\theta + i\sin n\theta \\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta) \end{aligned}\right\} \text{by De Moivre} \quad - 1$$

$= \cos n\theta - i\sin n\theta$ as cos even function, sin odd function — 1

$\therefore z^n + z^{-n} = 2\cos n\theta$

(ii) Let $n=1$ $\therefore z + z^{-1} = 2\cos\theta$

(iii) Expand $(z+z^{-1})^4$ to get $z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$ — 1
by Pascal's triangle or repeated expansion

(iv) From (i) with $n=4 \Rightarrow (z^4 + z^{-4}) = 2\cos 4\theta$
with $n=2 \Rightarrow z^2 + z^{-2} = 2\cos 2\theta$

Also $(z+z^{-1})^4 = (2\cos\theta)^4$
$= 16\cos^4\theta$ — 1

Using (iii) $16\cos^4\theta = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$ — 1
$= 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ — 1

(v) $\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta = \frac{1}{8}\int_0^{\frac{\pi}{4}} (\cos 4\theta + 4\cos 2\theta + 3)\, d\theta$
$= \frac{1}{8}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{4}}$ — 1
$=$
$= \frac{8+3\pi}{32}$ — 1

b (ii) ABXY is a cyclic quad. as interval XY subtends equal angles at A and B on same side of it. — 1

OR (converse of angles at circumference)

(iii) Let $\angle BAX = \theta$

$\therefore \angle BAZ = 90 - \theta$ (straight angle)

$\angle BYX = \theta$ (angles at circumference)

$\angle BXY = 90 - \theta$ (angle sum of $\triangle BXY$) } 1

Thus in $\triangle$'s ABZ, XYZ

$\angle Z$ common

$\angle BAZ = \angle YXZ = 90 - \theta$

$\therefore \triangle ABZ \mathbin{|||} \triangle XYZ$ (AAA) — 1

(iv) In $\triangle BYZ$, $\cos\alpha = \frac{BZ}{YZ}$

But $\frac{BZ}{YZ} = \frac{AB}{XY}$ (corr. sides of sim $\triangle$'s) } 1

$\therefore \cos\alpha = \frac{AB}{XY}$

$AB = XY \cos\alpha$ } 1

(v) XY is a fixed chord ie XY is constant

α is constant. 1

$\therefore$ AB is constant

7a (i) As P falls, forces are $\uparrow mkv^2$

$\downarrow mg$

Taking $\downarrow$ as positive & O at point of release, resultant force on P is

$$m\ddot{x} = mg - mkv^2$$ 1

$$\ddot{x} = g - kv^2$$

(ii) $\ddot{x} = \frac{dv}{dt}$

$= \frac{dv}{dx} \cdot \frac{dx}{dt}$ 1

$= v\frac{dv}{dx}$

(iii) $v\frac{dv}{dx} = g - kv^2$

$\frac{dv}{dx} = \frac{g-kv^2}{v}$

$\frac{dx}{dv} = \frac{v}{g-kv^2}$

$x = \int \frac{v}{g-kv^2}\,dv$ 1

$x = -\frac{1}{2k}\ln(g-kv^2) + c$ 1

When $x=0$, $v=0$ $\Rightarrow c = \frac{1}{2k}\ln g$

$\therefore x = -\frac{1}{2k}\ln(g-kv^2) + \frac{1}{2k}\ln g$ 1

$-2kx = \ln\left(\frac{g-kv^2}{g}\right)$

$\frac{g-kv^2}{g} = e^{-2kx}$

$g - kv^2 = ge^{-2kx}$

etc

$v^2 = \frac{g}{k}\left(1-e^{-2kx}\right)$ 1

(iv) Hits ground when $x=h$

$\therefore v^2 = \frac{g}{k}\left(1-e^{-2kh}\right)$

$v = \sqrt{\frac{g}{k}\left(1-e^{-2kh}\right)}$ 1

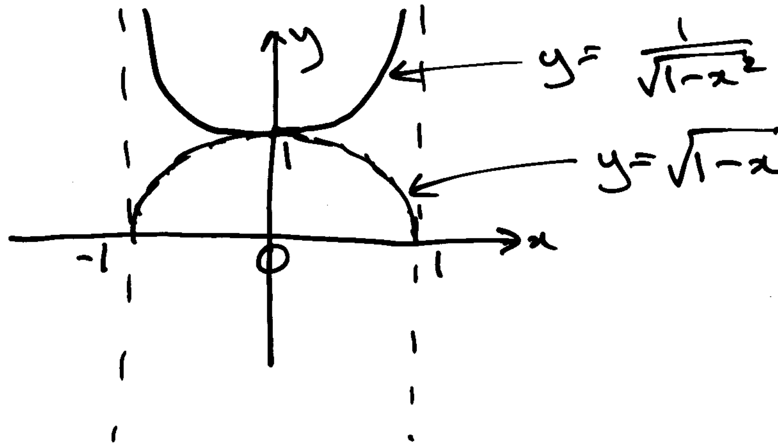
taking pos. square root as $\downarrow$ is pos.

(v) Terminal vel. when $\ddot{x}=0$

$g - kv^2 = 0$

$v = \sqrt{\frac{g}{k}}$ 1

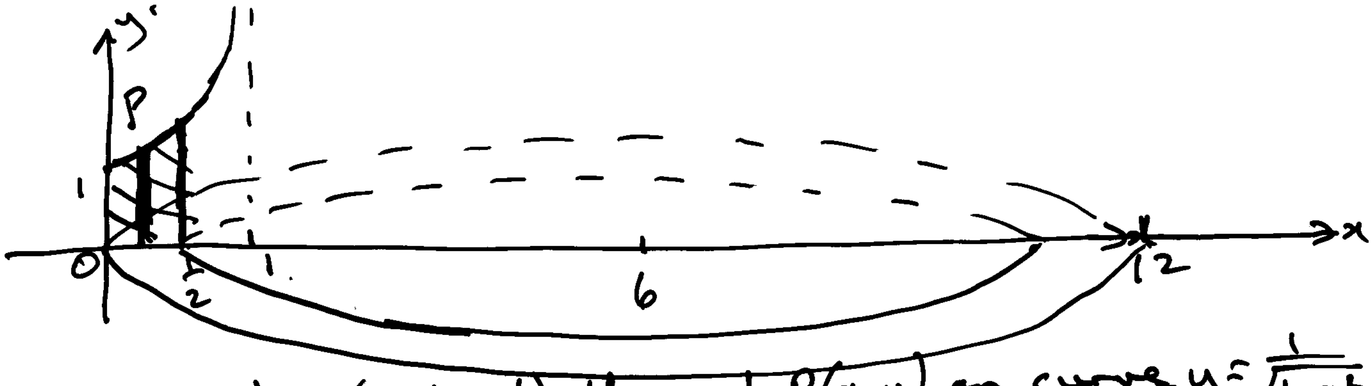
b (i)



1

1

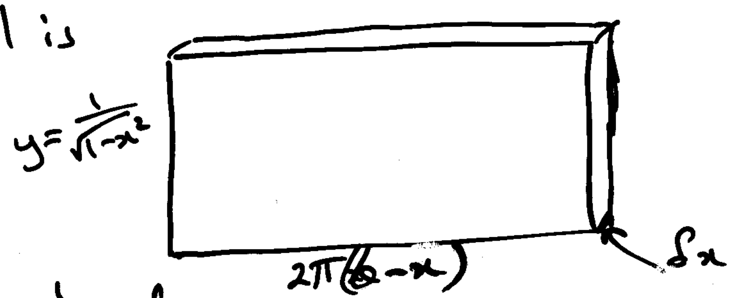
(ii)



Take thin slice (vertical) through $P(x,y)$ on curve, $y = \frac{1}{\sqrt{1-x^2}}$ thickness δx, and rotate it about line $x=6$ to form a thin-walled hollow cylindrical shell of radius $6-x$.

When "opened" shell is

1 for "set up"



Vol. shell $\delta V = 2\pi(6-x) \cdot \frac{1}{\sqrt{1-x^2}} \cdot \delta x$

$$= \frac{2\pi(6-x)}{\sqrt{1-x^2}}\,\delta x \qquad 1$$

Total vol. $V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{1}{2}} \frac{2\pi(6-x)}{\sqrt{1-x^2}}\,\delta x$

$$= 2\pi \int_0^{\frac{1}{2}} \frac{6-x}{\sqrt{1-x^2}}\,dx$$

} 1

(iii) $V = 12\pi \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} + \pi \int_0^{\frac{1}{2}} -2x(1-x^2)^{-\frac{1}{2}}\,dx$ 1

$= 12\pi \left[\sin^{-1} x\right]_0^{\frac{1}{2}} + \pi \left[2(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

since $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1}$

$=$ etc

$= 2\pi^2 + \pi\sqrt{3} - 2\pi$ cu. units. 1

8a (i) $I_n = \int_0^1 (x^2-1)^n \, dx$

$u = (x^2-1)^n \quad v = x$

$u' = 2nx(x^2-1)^{n-1} \quad v' = 1 \Big\} \; 1$

$\therefore I_n = \left[x(x^2-1)^n \right]_0^1 - 2n \int_0^1 x^2 (x^2-1)^{n-1} \, dx \quad 1$

$= 0 - 2n \int_0^1 \left[(x^2-1)(x^2-1)^{n-1} + 1(x^2-1)^{n-1} \right] dx$

$= -2n \int_0^1 (x^2-1)^n + (x^2-1)^{n-1} \, dx \quad 1$

$= -2n \left[I_n + I_{n-1} \right]$

$\therefore (2n+1) I_n = -2n . I_{n-1}$

$I_n = \dfrac{-2n}{(2n+1)} I_{n-1} \quad 1$

(ii) Prove true for $n=1$

By proposition $I_1 = \dfrac{(-1)^1 2^2 (1!)^2}{3!}$

$= -\frac{2}{3}$

But $I_1 = \int_0^1 (x^2-1) \, dx$

$= \left[\frac{x^3}{3} - x \right]_0^1$

$= -\frac{2}{3}$

$\therefore$ proposition true for $n=1$ $\Big\} \; 1$

Assume true for $n=k$ (integer k)

ie assume $I_k = \dfrac{(-1)^k 2^{2k} (k!)^2}{(2k+1)!}$

Prove true for $n=k+1$ if true for $n=k$

ie prove $I_{k+1} = \dfrac{(-1)^{k+1} 2^{2k+2} \left[(k+1)! \right]^2}{(2k+3)!}$ $\Big\} \; 1$

Now $I_{k+1} = \dfrac{-2(k+1)}{(2k+3)} . I_k$ from part (i)

$= \dfrac{-2(k+1)}{(2k+3)} . \dfrac{(-1)^k 2^{2k} (k!)^2}{(2k+1)!}$ | by assumption true for $n=k$

$= \dfrac{(-1)^{k+1} 2^{2k+1} (k+1)(k!)^2}{(2k+3)(2k+1)!} . \dfrac{2(k+1)}{(2k+2)}$

$= \dfrac{(-1)^{k+1} 2^{2k+2} \left[(k+1)! \right]^2}{(2k+3)!} \quad 1$

+ Conclusion

b (i) $P(x) = (x^2 - a^2)Q(x) + px + q$

So $P(a) = 0 . Q(a) + pa + q$

$\therefore pa + q = P(a) \quad \ldots (1)$

Also $P(-a) = 0 . Q(-a) - pa + q$

$\therefore -pa + q = P(-a) \quad \ldots (2)$

1

(1) + (2) $\quad 2q = P(a) + P(-a)$

$q = \frac{1}{2}[P(a) + P(-a)]$ — 1

(1) − (2) $\quad 2pa = P(a) - P(-a)$

$p = \frac{1}{2a}[P(a) - P(-a)]$ — 1

(ii) $P(x) = x^n - a^n$

α/ When n even $\quad P(a) = a^n - a^n = 0$

and $P(-a) = (-a)^n - a^n = a^n - a^n = 0$ as n even

1

$\therefore p = \frac{1}{2a}[0 - 0] = 0$ and $q = \frac{1}{2}[0 + 0] = 0$

$\therefore$ when n even, remainder $px + q$ becomes $0x + 0$

i.e remainder is 0 1

β/ When n odd $\quad P(a) = a^n - a^n = 0$

and $P(-a) = (-a)^n - a^n = -a^n - a^n = -2a^n$ as n odd

1

~~$\therefore$ when n odd, remainder $px+q$ becomes~~

$\therefore p = \frac{1}{2a}[0 - -2a^n] = a^{n-1}$ and $q = \frac{1}{2}[0 + -2a^n] = -a^n$

$\therefore$ when n odd, remainder $px + q$ becomes 1

$a^{n-1}x - a^n$