START A NEW BOOKLET FOR QUESTIONS 1 - 4

Question 1 (15 marks) Start a NEW page. **Marks**

(a) Let $z = 1 + 2i$ and $w = 3 + i$. Find $\dfrac{1}{zw}$ in the form $x + iy$. 2

(b) (i) Express $\dfrac{1}{2}\left(-1+i\sqrt{3}\right)$ in modulus-argument form. 2

(ii) Hence express $\dfrac{1}{16}\left(-1+i\sqrt{3}\right)^4$ in the form $x + iy$. 2

(c) Sketch the region in the Argand plane where the inequalities

$\dfrac{\pi}{4} \le \arg(z-i) \le \dfrac{3\pi}{4}$ and $|z-i| \le 2$ 3

both hold simultaneously.

(d) The origin O and the points A, B and C representing the complex numbers z, $\dfrac{1}{z}$ and $z+\dfrac{1}{z}$ respectively are joined to form a quadrilateral. Write down the condition or conditions for z so that the quadrilateral $OABC$ will be

(i) a rhombus, 1

(ii) a square. 1

(e) (i) Write down the six complex sixth roots of unity in modulus-argument form. Sketch the roots on an Argand diagram and explain why they form a regular hexagon. 2

(ii) Factorise $z^6 - 1$ completely into real factors. 2

Question 2 (15 marks) Start a NEW page. **Marks**

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \cos\theta \sin^3\theta \, d\theta.$ 2

(ii) $\displaystyle\int_0^3 \frac{\sqrt{x}}{1+x} dx.$ $\left(\text{Let } u^2 = x\right).$ 3

(iii) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x+2\sin x} dx.$ 4

(b) Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$.

(i) Prove that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$. 4

(ii) Evaluate $\int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$. 2

Question 3 (15 marks) Start a NEW page. **Marks**

(a) Given the equation $x^2 + xy + y^2 = 12$

(i) Show that $\frac{dy}{dx} = \frac{-(y+2x)}{x+2y}$ 2

(ii) Deduce that vertical tangents exist at $(4,-2)$ and $(-4,2)$ and horizontal tangents exist at $(2,-4)$ and $(-2,4)$. 2

(iii) Show that the curve is symmetrical about $y = x$ 1

(iv) Sketch the curve showing these tangents and the intercepts on the coordinate axes. 2

(b) Give a sketch of the curve $y = \frac{1}{1+t}$, for $t > -1$. Indicate on your diagram areas which represent $\log(1+x)$ 1

(i) for $x \geq 0$ 1

(ii) for $-1 < x \leq 0$ 1

and hence show that if $x > -1$,

$$\frac{x}{1+x} < \log(1+x) < x.$$ 2

Deduce that in n is a positive integer,

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}$$ 2

(c) Find $\int xe^{x^2} dx$ 1

Question 4 (15 marks) Start a NEW page. **Marks**

(a) Find $\int \frac{\cos^3 x}{\sin^2 x} dx$. 3

(b) On a certain day high water for a harbour occurs at 5a.m. and low water at 11.20a.m., the corresponding depths being 30 metres and 10 metres. If the tidal motion is assumed to be simple harmonic prove that, to the nearest minute, the latest time before noon that a ship, drawing 25 metres, can enter the harbour is 7.06a.m. 7

(c) A sequence of numbers $T_n,\ n = 1, 2, 3, \ldots$ is defined by $T_1 = 2, T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$ for $n = 3, 4, 5, \ldots$. Use Mathematical Induction to show that $T_n = \left(\sqrt{2}\right)^{n+2} \cos\frac{n\pi}{4}, n = 1, 2, 3, \ldots$. 5

START A NEW BOOKLET FOR QUESTIONS 5 - 8

Question 5 (15 marks) Start a NEW page. **Marks**

(a) A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y axis through 360° about the line $x = 2$.

(i) By slicing perpendicular to the axis of rotation, find the exact volume of S. 4

(ii) (α) Use the method of cylindrical shells to show that the volume of S is also given by $\int_0^1 16\pi(2-x)\sqrt{1-x}\, dx$. 2

(β) Confirm your answer to part (i) by calculating this definite integral using the substitution $u = 1 - x$. 3

(b) The region $(x-2R)^2 + y^2 \le R^2$ is rotated about the y-axis forming a solid of revolution called a torus. 6
By summing volumes of cylindrical shells, show that the volume of the torus is $4\pi^2 R^3$ units^3.

Question 6 (15 marks) Start a NEW page. **Marks**

(a) If α, β, γ are the roots of the cubic equation $x^3 - px + q = 0$ find in terms of p and q the value of:

(i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$ 3

(ii) $\alpha^3+\beta^3+\gamma^3$ 3

(b) If α is a non-real double root of $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$ factorise $P(x)$ completely into linear factors. 4

(c) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$. Show that:

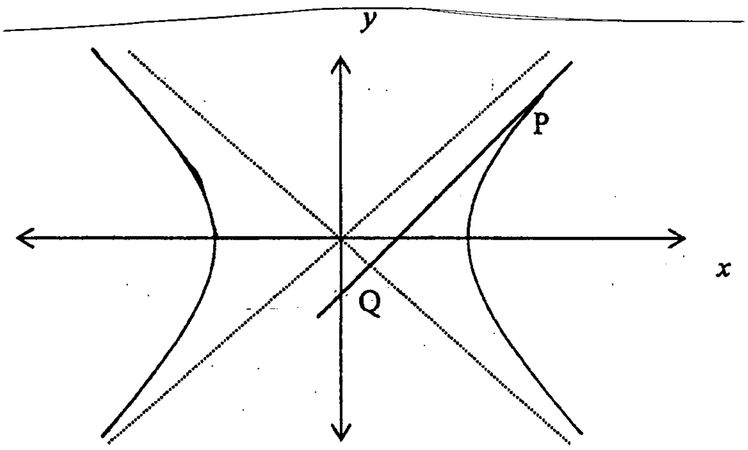
$$p = \frac{1}{2a}\{P(a) - P(-a)\} \qquad \text{and} \qquad q = \frac{1}{2}\{P(a) + P(-a)\}$$ 3

Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for the cases:

(i) n even 1

(ii) n odd 1

Question 7 (15 marks) Start a NEW page. **Marks**



Let P be a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. Let Q be the point of intersection of the tangent at P with an asymptote of the hyperbola. From Q perpendiculars QM and QN are drawn to the co-ordinate axes. Prove that MN passes through P. 4

(a) If Z represents the complex number $x+iy$, sketch on the complex plane $\operatorname{Re}\left(Z^2\right)>0$. 4

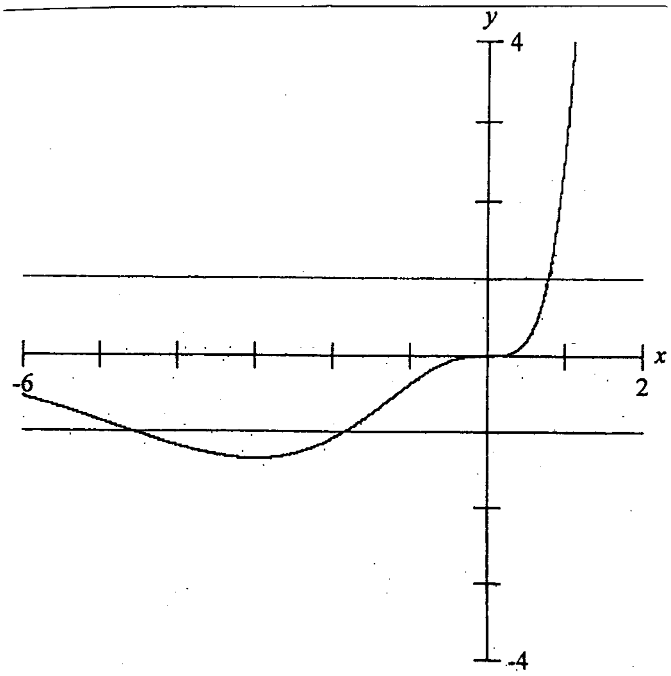
(b) If $0<x<y<\frac{1}{2}$ prove that: $\sqrt{xy}<x+y<\sqrt{x+y}$ 4

(c) Prove that if α, β are the roots of the equation $t^2-2t+2=0$ then:

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{\sin n\theta}{\sin^n\theta} \quad \text{where } \cot\theta=x+1$$ 4

Question 8 (15 marks) Start a NEW page. **Marks**

(a) The graphs of $y=f(x)$ and $y=\pm 1$ are shown.

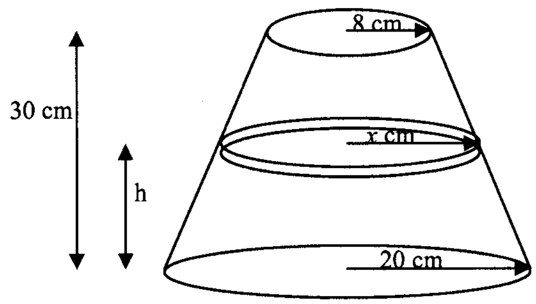


Draw a neat sketch of

(i) $y^2=f(x)$ 1

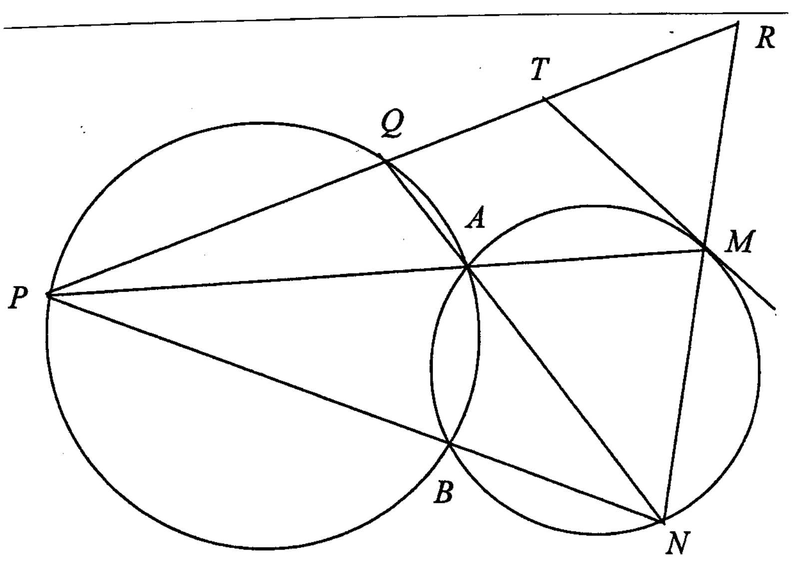
(ii) $y=\frac{1}{f(x)}$ 1

(b)



Calculate the volume in terms of π of the frustum of a cone, with radii of the top and bottom circles being 8 cm and 20 cm respectively. The height of the frustum is 30 cm. 4

(c)



In the diagram, the two circles intersect at A and B. P is a point on one circle. PA and PB produced meet the other circle at M and N respectively. NA produced meets the first circle at Q. PQ and NM produced meet at R. The tangent at M to the second circle meets PR at T.

(i) Copy the diagram. Show that $QAMR$ is a cyclic quadrilateral. 2

(ii) Show that $TM = TR$. 4

(d) ΔABC has sides of length a, b, c. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that ΔABC is equilateral. 3

END OF PAPER

Ext 2 trial 2008 solutions

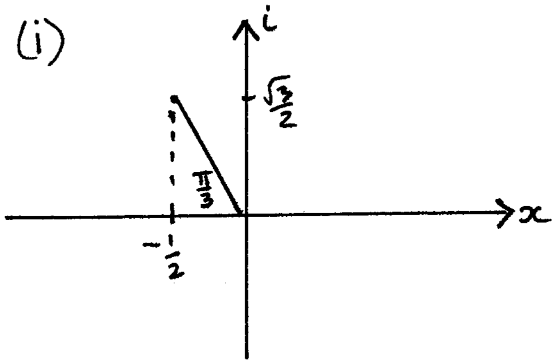
Question 1

a.) $\dfrac{1}{zw} = \dfrac{1}{(1+2i)(3+i)}$

$= \dfrac{1}{1+7i} \times \dfrac{1-7i}{1-7i}$

$= \dfrac{1-7i}{50}$ [2]

b.) (i)



$\frac{1}{2}\left(-1+i\sqrt{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

[2]

(ii) $\frac{1}{16}\left(-1+i\sqrt{3}\right)^4$

$= \left[\frac{1}{2}\left(-1+i\sqrt{3}\right)\right]^4$

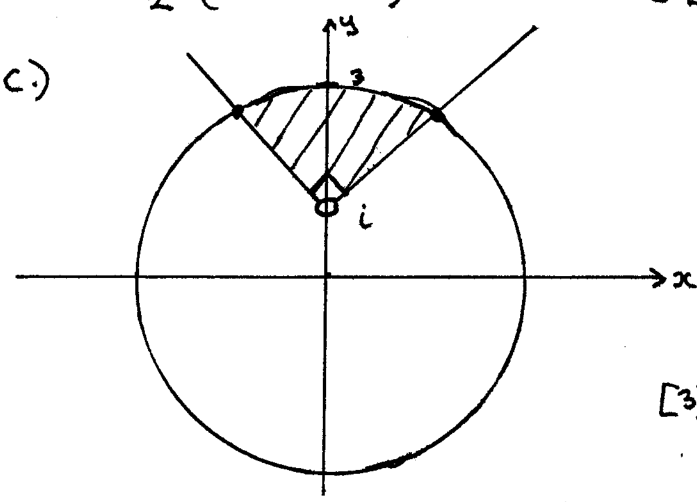
$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$

$= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \frac{1}{2}\left(-1+i\sqrt{3}\right)$ [2]

c.)



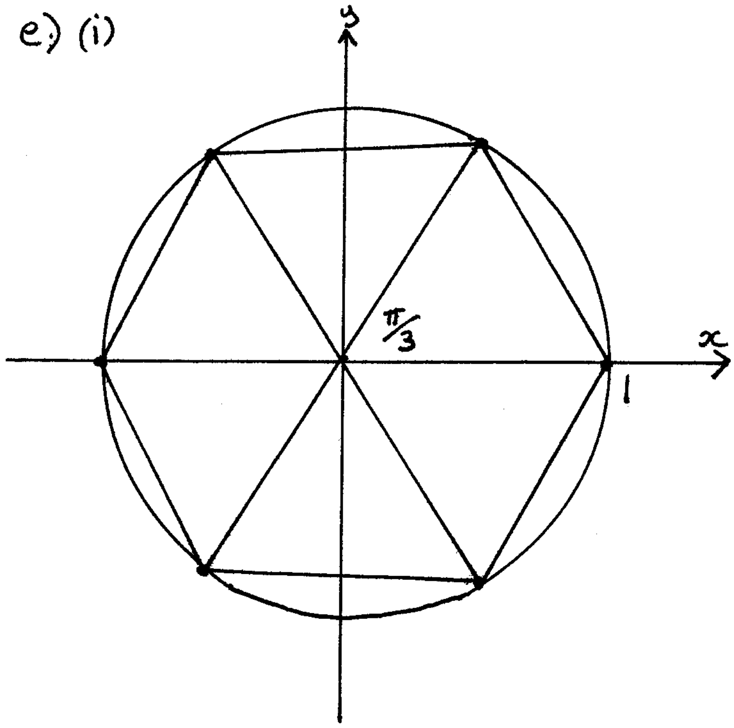
[3]

d.) (i) $|z| = 1$ [1]

(ii) $|z| = 1$ and [1]

$\arg z = \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$

e.) (i)



Roots are ± 1, $\text{cis}\frac{\pi}{3}$, $\text{cis}\frac{2\pi}{3}$, $\text{cis}\frac{4\pi}{3}$, $\text{cis}\frac{5\pi}{3}$

or $\text{cis}\frac{k\pi}{3}$ for $k = 0, 1, \cdots, 5$

since their moduli equal 1, their arguments differ by $\frac{\pi}{3}$ they form the vertices of a regular hexagon on the unit circle.

[2]

(ii) $z^6 - 1 = (z^3)^2 - 1$

$= (z^3+1)(z^3-1)$

$= (z+1)(z^2-z+1)(z-1)(z^2+z+1)$

[2]

Question 2.

a.) (i) $\left[\frac{1}{4}\sin^4\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{4}\left(\sin^4\frac{\pi}{6} - \sin^4 0\right)$

$= \frac{1}{4}\left(\left(\frac{1}{2}\right)^4 - 0\right)$

$= \frac{1}{64}$ [2]

(ii) $u^2 = x$ $\quad x=0, u=0$

$2u\,du = dx$ $\quad x=3, u=\sqrt{3}$

$$\int_0^3 \frac{\sqrt{x}}{1+x}\,dx = \int_0^{\sqrt{3}} \frac{u}{1+u^2}\cdot 2u\,du$$

$$= 2\int_0^{\sqrt{3}} \frac{u^2}{1+u^2}\,du$$

$$= 2\int_0^{\sqrt{3}} 1 - \frac{1}{1+u^2}\,du$$

$$= 2\left[u - \tan^{-1}u\right]_0^{\sqrt{3}}$$

$$= 2\left[(\sqrt{3} - \tan^{-1}\sqrt{3}) - (0 - \tan^{-1}0)\right]$$

$$= 2\left(\sqrt{3} - \frac{\pi}{6}\right)$$ [3]

(iii) $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$= \frac{1}{2}(1+t^2)\,dx$

$dx = \frac{2}{1+t^2}\,dt$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$2 - \cos x + 2\sin x$

$= \frac{2(1+t^2) - (1-t^2) + 4t}{1+t^2}$

$= \frac{3t^2+4t+1}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x + 2\sin x}\,dx = \int_0^1 \frac{1+t^2}{(3t+1)(t+1)}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \left\{\frac{3}{(3t+1)} - \frac{1}{(t+1)}\right\}dt$$

$$= \left[\ln(3t+1) - \ln(t+1)\right]_0^1$$

$$= (\ln 4 - \ln 1) - (\ln 2 - \ln 1)$$

$$= 2\ln 2 - \ln 2$$

$$= \ln 2$$

[4]

b.) (i) $I_n = \int_0^{\frac{\pi}{2}} x^n \cdot \frac{d}{dx}(-\cos x)\,dx$

$$= \left[-x^n\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x \cdot n x^{n-1}\,dx$$

$$= 0 + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx$$

$$= n\int_0^{\frac{\pi}{2}} x^{n-1}\frac{d}{dx}(\sin x)\,dx$$

$$= n\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} \sin x\cdot(n-1)x^{n-2}\,dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\,dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ [4]

(ii) $\int_0^{\frac{\pi}{2}} x^4\sin x\,dx = I_4$

$I_4 = -4(3)I_2 + 4\left(\frac{\pi}{2}\right)^3$

$I_2 = -2(1)I_0 + 2\left(\frac{\pi}{2}\right)^1$

$I_0 = \int_0^{\frac{\pi}{2}} \sin x\,dx$

$= 1$

$\therefore I_2 = -2(1) + \pi$

$I_4 = -12(-2+\pi) + \frac{\pi}{2}^3$

$= 24 - 12\pi + \frac{\pi}{2}^3$ [2]

Question 3

a) (i) $x^2 + xy + y^2 = 12$

$2x + x\frac{dy}{dx} + y.1 + 2y\frac{dy}{dx} = 0$

$(x+2y)\frac{dy}{dx} = -(2x+y)$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$ [2]

(ii) Vertical tangents exist when denominator of $\frac{dy}{dx} = 0$ ie $x+2y=0$ or $y = -\frac{1}{2}x$

Solve $x^2+xy+y^2=12$ and $y = -\frac{1}{2}x$

$x^2 + x(-\frac{1}{2}x) + (-\frac{1}{2}x)^2 = 12$

$x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 12$

$4x^2 - 2x^2 + x^2 = 48$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

If $x=4$, $y=-2$

and if $x=-4$, $y=2$

$\therefore$ Vertical tangents exist at $(4,-2)$ and $(-4,2)$.

Horizontal tangents exist when the denominator of $\frac{dy}{dx} = 0$ $\therefore 2x+y=0$

$y = -2x$

Solve $x^2+xy+y^2=12$ and $y=-2x$

$\therefore x^2 + x(-2x) + (-2x)^2 = 12$

$x^2 - 2x^2 + 4x^2 = 12$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

If $x=2$, $y=-4$ and if $x=-2$, $y=4$

$\therefore$ horizontal tangents exist at $(2,-4)$ and $(-2,4)$. [2]

(iii) $x^2 + xy$

$x^2+xy+y^2=12$

Interchange x and y

$\therefore y^2+yx+x^2=12$

i.e we obtain the same curve $\therefore$ the curve is symmetrical about $y=x$. [1]

(iv) Solve $x^2+xy+y^2=12$ and $y=x$

$x^2+x^2+x^2=12$

$3x^2=12$

$x^2=4$

$x=\pm 2$

$\therefore y=x$ & $x^2+xy+y^2=12$ intersect at $(2,2)$ and $(-2,-2)$

$x^2+xy+y^2=12$ cuts x-axis at $y=0$

$\therefore x^2=12$

$x = \pm 2\sqrt{3}$

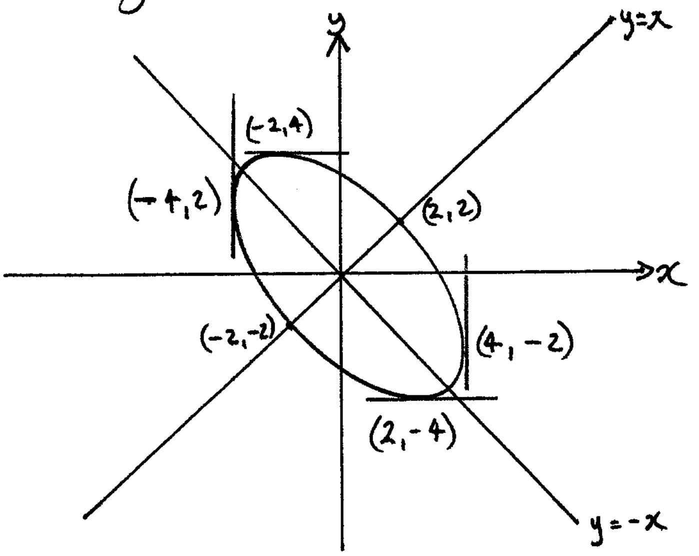
Question 3 continued.

(iv) cont.

The curve cuts the y-axis at $x=0$

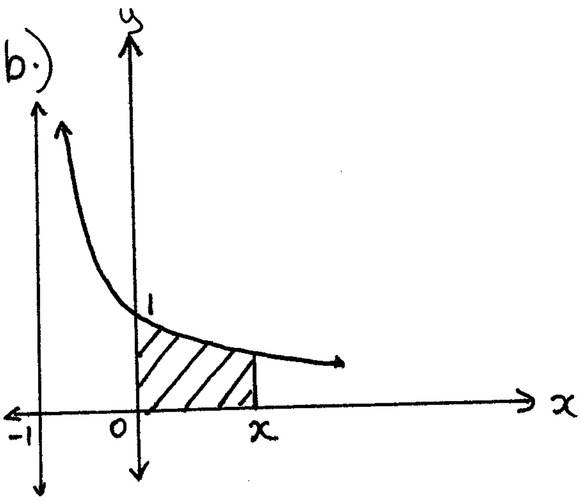
$\therefore\ y^2 = 12$

$y = \pm 2\sqrt{3}$



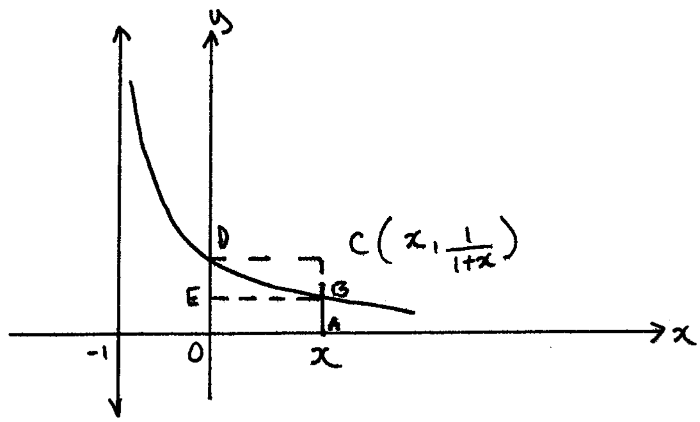
[2]

b.)



[1]

$$\int_0^x \frac{1}{1+t}\,dt = \Big[\ln(1+t)\Big]_0^x$$
$$= \ln(1+x) - \ln 1$$
$$= \ln(1+x)$$



Area OABE $< \int_0^x \frac{1}{1+t}\,dt <$ Area OACD

$$x \cdot \frac{1}{1+x} < \int_0^x \frac{1}{1+t}\,dt < x \cdot 1$$

$$\therefore\ \frac{x}{1+x} < \int_0^x \frac{1}{1+t}\,dt < x$$

i.e $\frac{x}{1+x} < \ln(1+x) < x$

Let $x = \frac{1}{n}$

$$\therefore\ \frac{\frac{1}{n}}{1+\frac{1}{n}} < \log\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

$$\frac{1}{n+1} < \log\left(\frac{n+1}{n}\right) < \frac{1}{n}$$ [6]

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}$$

c.) $\int x e^{x^2}\,dx$

$= \frac{1}{2}\int 2x e^{x^2}\,dx$

$= \frac{1}{2} e^{x^2} + C$ [1]

Question 4

a) $\int \frac{\cos^3 x}{\sin^2 x}\,dx = \int \frac{(1-\sin^2 x)\cos x\,dx}{\sin^2 x}$

Let $u = \sin x$
$du = \cos x\,dx$

$= \int (u^{-2} - 1)\,du$

$= -\frac{1}{u} - u + c$

$= -\operatorname{cosec} x - \sin x + c$

[3]

b.) The time between high water at 5a.m. and low water at 11·20 a.m. is 6h 20min, and this is the half period of the SHM.

The period of motion is 12h 40min, ie $\frac{38}{3}$ h.

Since the period T is $\frac{2\pi}{n}$,

then $\frac{2\pi}{n} = \frac{38}{3}$

$n = \frac{3\pi}{19}$.

Take the origin at the centre of motion,

$\left[10 + \frac{1}{2}(30-10)\right] = 20$m above the bottom of the harbour,

then $x = +10$ at high water

and $x = -10$ at low water.

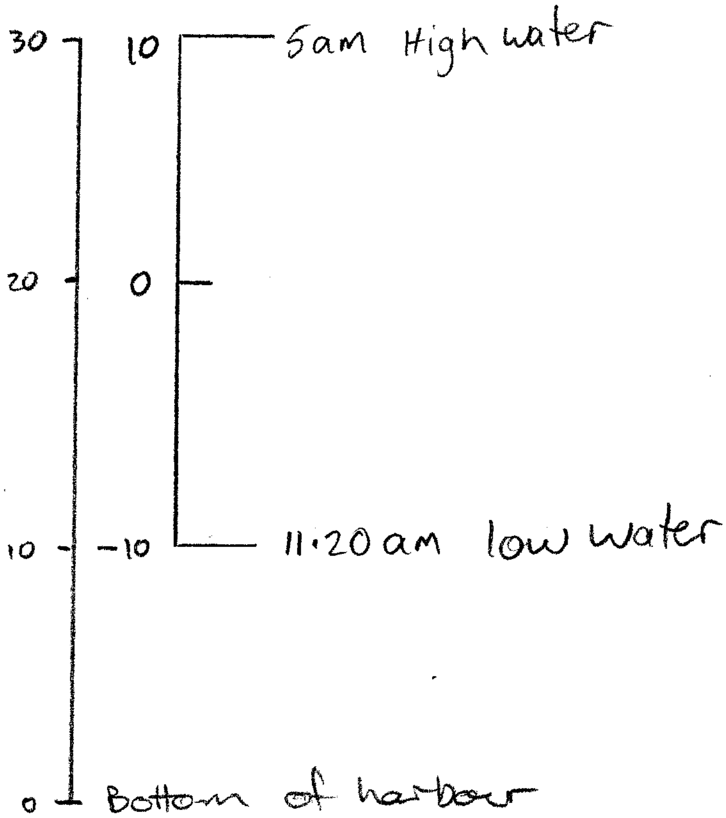
Take initial time $t=0$ at high water, (ie at 5a.m.)

since when $t=0$ $x=10$, then the amplitude a of the motion is 10.

From SHM $x = a\cos nt$

∴ in this case

$x = 10\cos\frac{3\pi t}{19}$



For a ship, drawing 25m, the depth of water in the harbour must be at least 25m, ie $x \geq 5$.

sub $x=5$ into $x = 10\cos\frac{3\pi}{19}t$

$t = \frac{\pi}{3} \cdot \frac{19}{3\pi} = \frac{19}{9}h = 2\frac{1}{9}h$.

The water in the harbour is at a depth of 30m (high H_2O) at 5a.m. (at $t=0$), and falls to a depth of 25m (min. safe level for the ship) for the first time when $t = 2\frac{1}{9}$ h

Question 4 continued.

Thus, the ship can enter the harbour at any time between 5am and $7\frac{1}{9}$ h a.m. ie between 5 a.m. and 7.06 a.m., (at 7.07 a.m. the depth of water is below 25m and it would be unsafe.)

The <u>next</u> time the depth of water in the harbour is 25m ($x=5$) is when

$$t = \frac{5\pi}{3n} = \frac{95}{9} h = 10\frac{5}{9} h$$

ie at 5am + $10\frac{5}{9}$ h = $3\frac{5}{9}$ h <u>after</u> noon. The water is <u>rising</u> then. [7]

Hence the latest time <u>before</u> noon will be 7.06 a.m.

c) Define the sequence of statements $S(n)$, $n = 1, 2, 3, \ldots$ by $S(n)$: $T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}$

Consider $S(1)$, $S(2)$:

$$(\sqrt{2})^{1+2} \cos \frac{1 \cdot \pi}{4} = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2 = T_1$$

$\therefore$ $S(1)$ is true

$$(\sqrt{2})^{2+2} \cos \frac{2\pi}{4} = 4 \times 0 = 0 = T_2$$

$\therefore$ $S(2)$ is true

Assume true for $n=k$:

ie $S(n)$ is true, $n \le k$ where

$$T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}, \quad n = 1, 2, 3, \ldots, k$$

Consider $S(k+1)$, $k \geq 2$:

$$T_{k+1} = 2T_k - 2T_{k-1} \quad (\text{since } k+1 \geq 3)$$

$$= 2(\sqrt{2})^{k+2} \cos \frac{k\pi}{4} - 2(\sqrt{2})^{(k-1)+2} \cos \frac{(k-1)\pi}{4},$$

if $S(n)$ is true, $n \le k$

$$= (\sqrt{2})^{k+3} \left[\sqrt{2} \cos \frac{k\pi}{4} - \cos\left(\frac{k\pi}{4} - \frac{\pi}{4}\right) \right]$$

$$= (\sqrt{2})^{k+3} \left[2 \frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \left(\cos \frac{k\pi}{4} \cos \frac{\pi}{4} + \sin \frac{k\pi}{4} \sin \frac{\pi}{4} \right) \right]$$

$$= (\sqrt{2})^{k+3} \left[2 \cdot \frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \sin \frac{k\pi}{4} \right]$$

$$= (\sqrt{2})^{k+3} \left[\frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \sin \frac{k\pi}{4} \right]$$

$$= (\sqrt{2})^{k+3} \left[\cos \frac{k\pi}{4} \cos \frac{\pi}{4} - \sin \frac{k\pi}{4} \sin \frac{\pi}{4} \right]$$

$$= (\sqrt{2})^{k+3} \cos\left(\frac{k\pi}{4} + \frac{\pi}{4} \right)$$

$$= (\sqrt{2})^{(k+1)+2} \cos \frac{(k+1)\pi}{4}$$

$\therefore$ If $k \geq 2$ and $S(n)$ is true for $n \le k$, then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, and hence $S(3)$ is true, and then $S(4)$ is true etc. Hence by

Question 4 continued

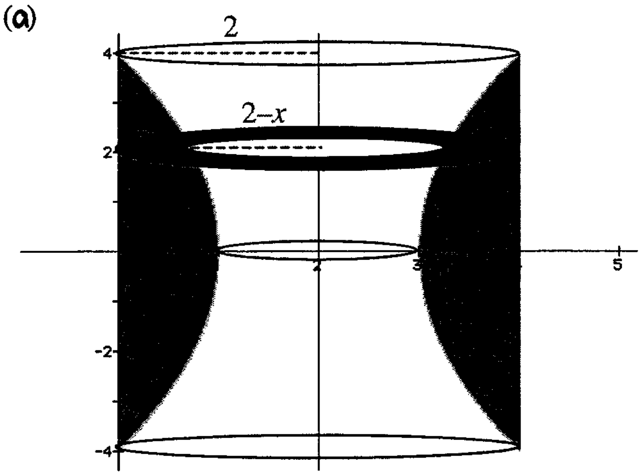
mathematical induction, $s(n)$ is true for all positive integers n.

$\therefore T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}, \; n = 1, 2, 3, \ldots$

[5]

Question 5

(a)



(i) $\delta V = \pi\left(2^2 - (2-x)^2\right)\delta y$

$\delta V = \pi(2-2+x)(2+2-x)\delta y$

$\delta V = \pi x(4-x)\delta y$

$V = \int_{-4}^{4} \pi x(4-x)\,dy$ but $x = 1 - \dfrac{y^2}{16}$ or $x = \dfrac{16-y^2}{16}$

$= \pi \int_{-4}^{4} \left(1 - \dfrac{y^2}{16}\right)\left(4 - 1 + \dfrac{y^2}{16}\right) dy$

$= \pi \int_{-4}^{4} \left(3 - \dfrac{y^2}{8} - \dfrac{y^4}{256}\right) dy$

$= 2\pi \left[3y - \dfrac{y^3}{24} - \dfrac{y^5}{1280}\right]_0^4 = 2\pi\left(12 - \dfrac{64}{24} - \dfrac{4^4 \times 4}{256 \times 5} - 0\right) = \dfrac{256\pi}{15}$ units^3

4

1 for δV correct

1 evidence of the use of a correct substitution for x

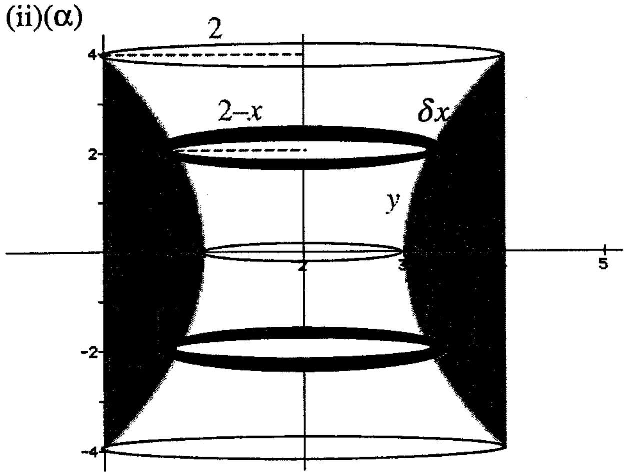
1 correct primitive

1 correct substitution

3 for $\dfrac{416\pi}{15}$ with working

Question 5 continued

(ii)(α) **2**



$$\delta V = 2\pi(2-x)\times 2y \times \delta x$$

$$V = 4\pi\int_0^1 (2-x)y\,dx$$

but $y^2 = 16(1-x)$

Use $y = 4\sqrt{1-x}$ as upper branch

$$V = 4\pi\int_0^1 (2-x)4\sqrt{1-x}\,dx$$

$$= 16\pi\int_0^1 (2-x)\sqrt{1-x}\,dx$$

OR

$$R - r = 2 - x + \delta x - 2 + x$$
$$= \delta x$$
$$R + r = 2 - x + \delta x + 2 - x$$
$$= 4 - 2x + \delta x$$
$$(R-r)(R+r)$$
$$= \delta x(4 - 2x + \delta x)$$
$$\approx 2(2-x)\delta x$$
$$\delta V = \pi\left(R^2 - r^2\right)\times 2y$$
$$= \pi \times 2(2-x)\delta x \times 2y$$
$$= 4\pi(2-x)y\delta x$$

Evidence of $y = 4\sqrt{1-x}$ is needed for **1**

(β) **3**

$$\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx \qquad \text{Let } u = 1 - x, \quad du = -dx$$

$$= \int_1^0 -16\pi(1+u)\sqrt{u}\,du$$

1 evidence of correct substitution

$$= 16\pi\int_0^1 \left(\sqrt{u} + u^{\frac{3}{2}}\right)du$$

1 correct primitive

$$= 16\pi\left[\frac{2}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$

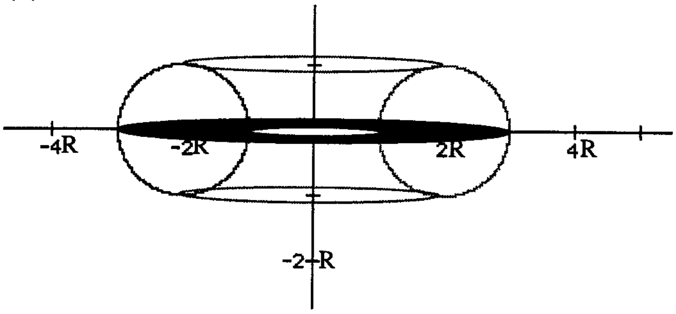
1 correct substitution

$$= 16\pi\left(\frac{2}{3} + \frac{2}{5} - 0\right) = \frac{256\pi}{15}$$

2 for $\dfrac{-256\pi}{15}$ with working

Question 5 continued

(b)	**6**



$(x-2R)^2+y^2=R^2$ $y^2=R^2-(x-2R)^2$ $y=\pm\sqrt{R^2-(x-2R)^2}$ $y=\sqrt{R^2-(x-2R)^2}$ is upper boundary.	**1** equ'n of upper boundary
$\delta V=2\pi x\times 2y\times\delta x$ $V=\int_R^{3R}4\pi xy\,dx$	**1** correct definite integral for V
$=\int_R^{3R}4\pi x\sqrt{R^2-(x-2R)^2}\,dx$	**1** correct substitution for y
Let $x-2R=R\sin\theta$ $x=R,\ \theta=\frac{-\pi}{2}$ $x=3R,\ \theta=\frac{\pi}{2}$ $dx=R\cos\theta d\theta$	**1** set up correct substitution with new limits
$V=4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(2R+R\sin\theta)\sqrt{R^2-R^2\sin^2\theta}\,R\cos\theta d\theta$	
$=4\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(2R+R\sin\theta)R^2\cos^2\theta d\theta$	**1** simplified integrand
$=4\pi R^3\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(2\cos^2\theta+\sin\theta\cos^2\theta\right)d\theta$	
$=4\pi R^3\left[\theta+\sin 2\theta-\frac{\cos^3\theta}{3}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$	**1** correct primitive with correct limits
$=4\pi R^3\left(\frac{\pi}{2}+0-0-\left(\frac{-\pi}{2}+0-0\right)\right)$ $=4\pi^2R^3$ units^3	

Question 6

a) $x^3 - px + q = 0$

i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$= \frac{\gamma^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$

$= \frac{p^2}{q^2}$ [3]

ii) If α, β, γ are roots then:

$\alpha^3 - p\alpha + q = 0$ --- (1)

$\beta^3 - p\beta + q = 0$ --- (2)

$\gamma^3 - p\gamma + q = 0$ --- (3)

(1)+(2)+(3) $\alpha^3 + \beta^3 + \gamma^3 - p(\alpha+\beta+\gamma) + 3q = 0$

$\alpha^3 + \beta^3 + \gamma^3 = p(\alpha+\beta+\gamma) - 3q$

$= -3q$ [3]

b) $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$

If α is a double root then $\bar{\alpha}$ is also a double root.

$\therefore \alpha + \alpha + \bar{\alpha} + \bar{\alpha} = -b/a$ and $\alpha\alpha\bar{\alpha}\bar{\alpha} = e/a$

$= 4$ $\quad (\alpha\bar{\alpha})^2 = 25$

$\alpha\bar{\alpha} = 5$

Now if $\alpha = a + ib$ and $a^2 + b^2 = 5$

then $4a = 4$ ie $1 + b^2 = 5$

$a = 1$ $\quad b = \pm 2$

$\therefore P(x) = (x + 1 + 2i)^2 (x + 1 - 2i)^2$ [4]

c) $P(x) = (x^2 - a^2)\,Q(x) + px + q$

$= (x-a)(x+a)\,Q(x) + px + q$

$\therefore P(a) = pa + q$ --- (1)

$P(-a) = -pa + q$ --- (2)

(1)+(2) $2q = P(a) + P(-a)$ $\quad$ (1)−(2): $2ap = P(a) - P(-a)$

$q = \frac{1}{2}(P(a) + P(-a))$ $\quad p = \frac{1}{2a}(P(a) - P(-a))$ [3]

When $P(x) = x^n - a^n$ then

i) When n is even, $P(a) = 0$ and $P(-a) = 0$

$\therefore$ the remainder $= 0$. [1]

ii) When n is odd, $P(a) = 0$ and $P(-a) = -2a^n$

$\therefore pa + q = 0$ --- (1) $\quad$ (1)−(2) $2ap = 2a^n$

$-pa + q = -2a^n$ --- (2) $\quad ap = a^n$

(1)+(2) $2q = -2a^n$ $\quad p = a^{n-1}$

$q = -a^n$

$\therefore$ the remainder is $a^{n-1}x - a^n$. [1]

Question 7

Ques 7.

a) Let P be the point $(a\sec\theta, b\tan\theta)$

$\therefore$ the equation of P is: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ ——①

the equation of the asymptote: $y = \frac{-bx}{a}$ ···(2)

Subst. (2) into (1):

$$\frac{x\sec\theta}{a} + \frac{\frac{bx}{a}\tan\theta}{b} = 1$$

$$\frac{x\sec\theta}{a} + \frac{x\tan\theta}{a} = 1$$

$$x(\sec\theta + \tan\theta) = a$$

$$x = \frac{a}{\sec\theta + \tan\theta}$$

Subst into (2): $y = \frac{-b}{a}\left(\frac{a}{\sec\theta + \tan\theta}\right)$

$$= \frac{-b}{\sec\theta + \tan\theta}$$

$\therefore$ Q is the point $\left(\frac{a}{\sec\theta+\tan\theta}, \frac{-b}{\sec\theta+\tan\theta}\right)$

$\therefore$ M is the point $\left(\frac{a}{\sec\theta+\tan\theta}, 0\right)$

and N is the point $\left(a, \frac{-b}{\sec\theta+\tan\theta}\right)$

Now gradient of MN $= \dfrac{\frac{b}{\sec\theta+\tan\theta}}{\frac{a}{\sec\theta+\tan\theta}}$

$$= \frac{b}{a}$$

$\therefore$ the equation of MN: $y - 0 = \frac{b}{a}\left(x - \frac{a}{\sec\theta+\tan\theta}\right)$

$$y = \frac{bx}{a} - \frac{b}{\sec\theta + \tan\theta}$$

If P lies on the line MN it must satisfy the equation.

ie $b\tan\theta = \frac{ba\sec\theta}{a} - \frac{b}{\sec\theta+\tan\theta}$

$$= b\sec\theta - \frac{b}{\sec\theta + \tan\theta}$$

$$= b\left(\sec\theta - \frac{1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\sec\theta(\sec\theta+\tan\theta) - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\sec^2\theta + \sec\theta\tan\theta - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{1+\tan^2\theta + \sec\theta\tan\theta - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\tan^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\right)$$

$$= b\tan\theta\left(\frac{\tan\theta + \sec\theta}{\sec\theta+\tan\theta}\right)$$

$$= b\tan\theta.$$

$\therefore$ P lies on the line. [4]

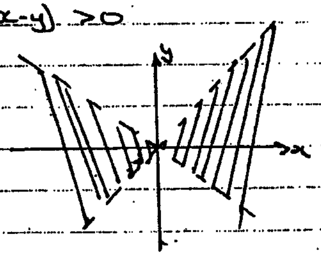
b) $\mathrm{Re}(z^2) > 0$

$\mathrm{Re}((x+iy)^2) > 0$

$\mathrm{Re}(x^2 - y^2 + 2ixy) > 0$

$x^2 - y^2 > 0$

$(x+y)(x-y) > 0$



[4]

Question 7 continued.

c) $(\sqrt{x}-\sqrt{y})^2 \geq 0$

$x+y-2\sqrt{xy} \geq 0$

$x+y \geq 2\sqrt{xy}$

$\therefore \quad x+y \geq \sqrt{xy}$ ------ (1)

If $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ then

$x+y \leq 1$

$\therefore \sqrt{x+y} \geq x+y$ ..(2) as the square root of a number between 0 and 1 is greater than the number.

$\therefore$ from (1) and (2)

$\sqrt{xy} \leq x+y \leq \sqrt{x+y}$ [4]

d) $t^2-2t+2=0$

$t = \frac{2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

let $\alpha = 1+i$, $\beta = 1-i$

$\therefore x+\alpha = (\cot\theta - 1) + 1 + i$ also $x+\beta = (\cot\theta - 1) + (1-i)$

$= \cot\theta + i$ $\qquad$ $= \cot\theta - i$

$= \frac{\cos\theta + i\sin\theta}{\sin\theta}$ $\qquad$ $= \frac{\cos\theta - i\sin\theta}{\sin\theta}$

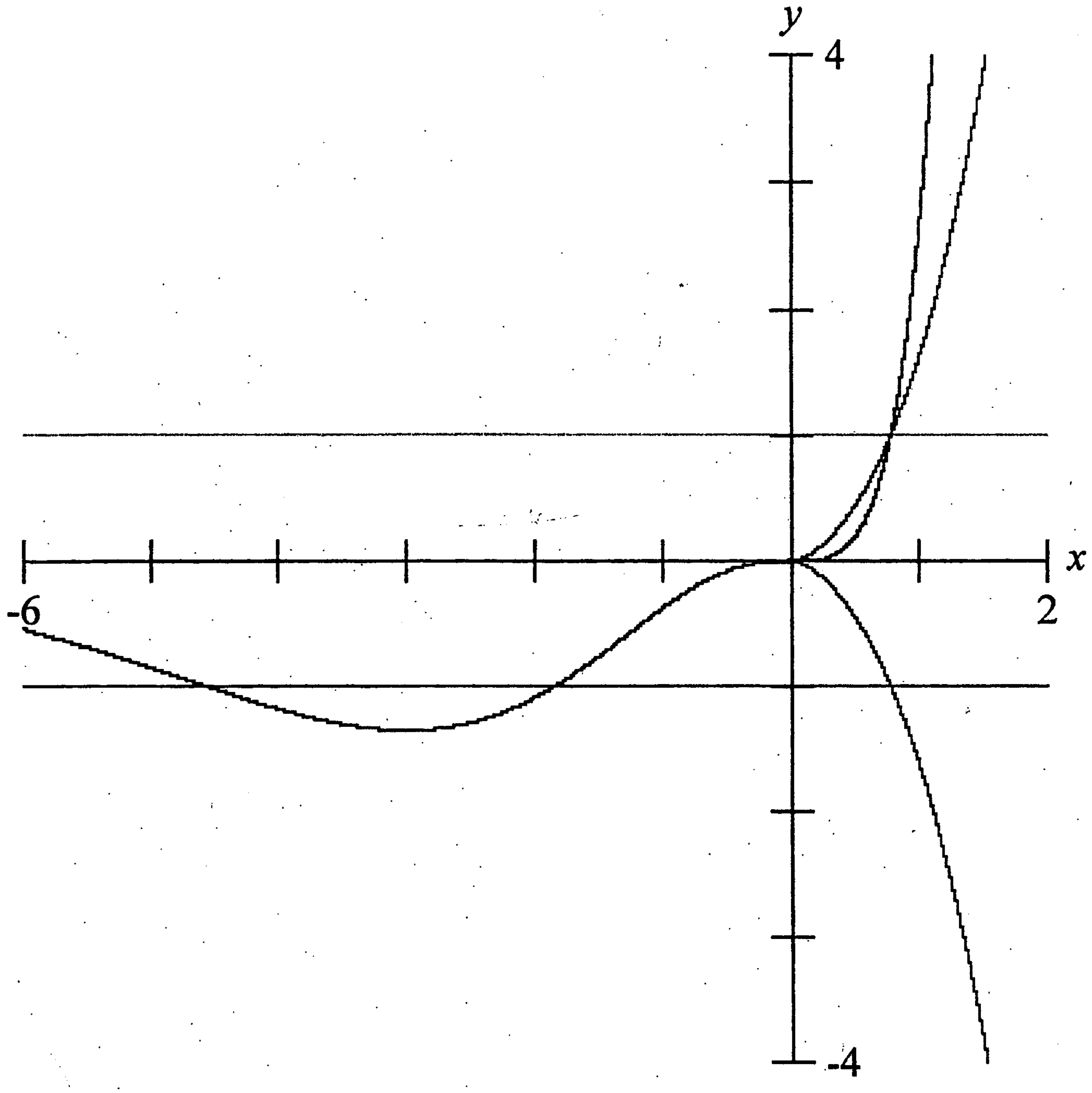
$\therefore \frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta}$

$= \frac{\frac{(\cos\theta + i\sin\theta)^n}{\sin^n\theta} - \frac{(\cos\theta - i\sin\theta)^n}{\sin^n\theta}}{2i}$

$= \frac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{2i\sin^n\theta}$

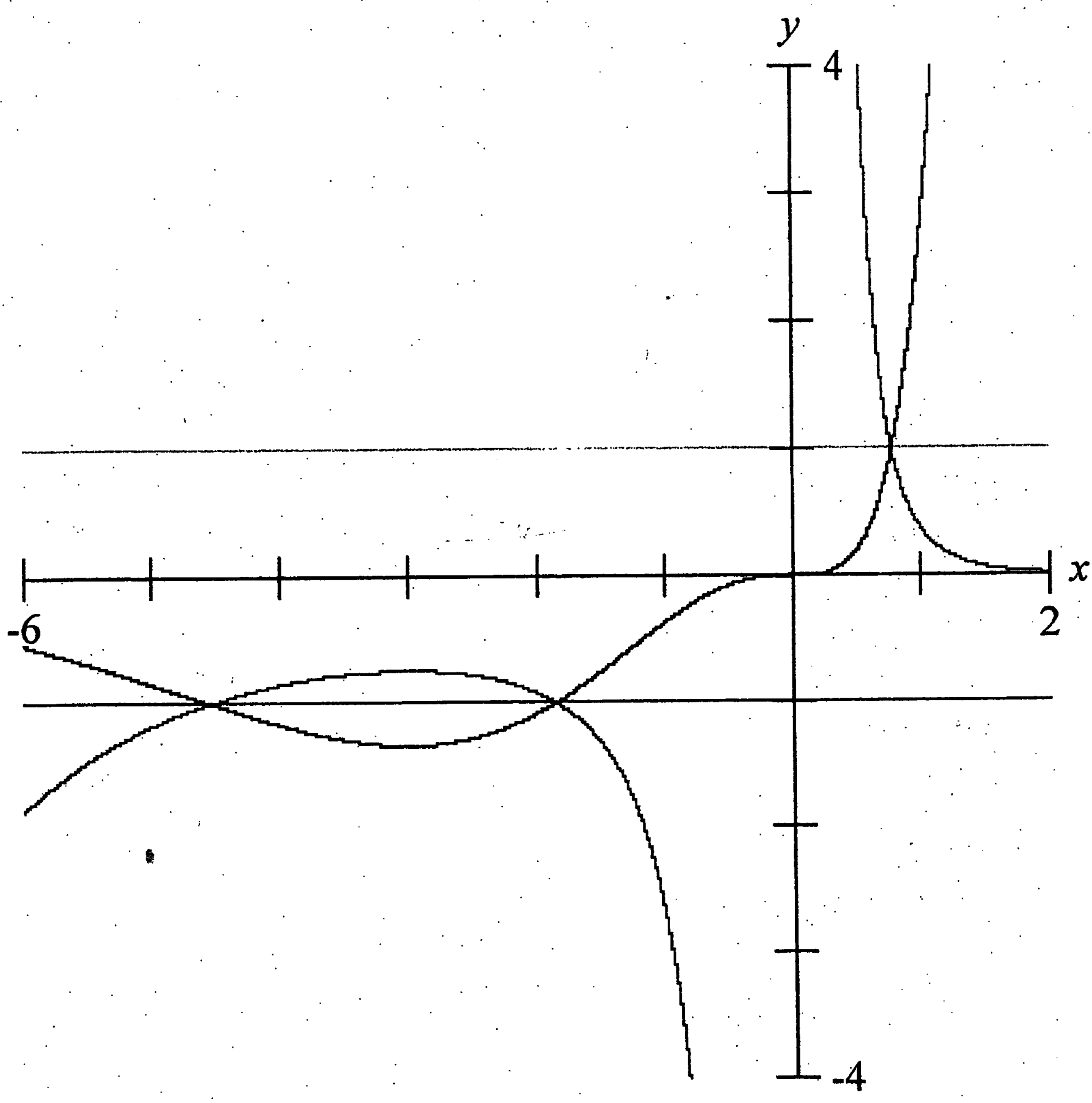
$= \frac{2i\sin n\theta}{2i\sin^n\theta} = \frac{\sin n\theta}{\sin^n\theta}.$ [4]

8a(i)



[1]

8 a(ii)



[1]

Question 8 continued.

b) $\dfrac{x-8}{20-x} = \dfrac{30-h}{h}$

$hx - 8h = 600 - 20h - 30x + hx$

$30x = 600 - 12h$

$x = 20 - \dfrac{2h}{5}$

$A = \pi x^2 = \pi\left(20 - \dfrac{2h}{5}\right)^2$

$\delta V = A\,\delta h = \pi\left(20 - \dfrac{2h}{5}\right)^2 \delta h$

$V = \pi\displaystyle\int_0^{30}\left(20 - \dfrac{2h}{5}\right)^2 dh$

$= \pi\left[\dfrac{\left(20 - \frac{2h}{5}\right)^3}{3\times\left(-\frac{2}{5}\right)}\right]_0^{30}$

$= \pi\left[\dfrac{8^3}{-\frac{6}{5}} - \dfrac{20^3}{-\frac{6}{5}}\right]$

$= \pi\left[\dfrac{512 - 8000}{-\frac{6}{5}}\right]$

$= 6240\pi$ cubic cm

$V = \pi\displaystyle\int_0^{30} 400 - \dfrac{80h}{5} + \dfrac{4h^2}{25}\, dh$

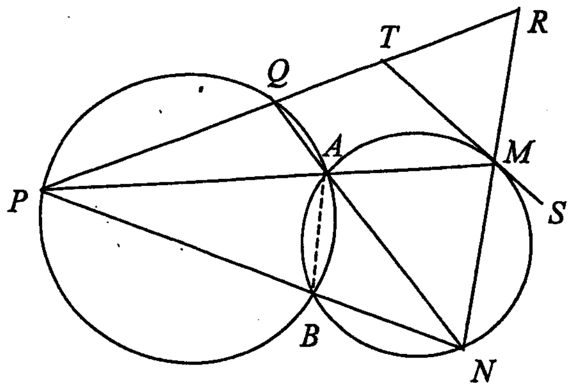
$= \pi\left[400h - \dfrac{40h^2}{5} + \dfrac{4h^3}{75}\right]_0^{30}$

$= 6240\pi$ cubic cm.

[4]

Question 8 continued

c.) i.



$\angle RMA = \angle ABN$ (*exterior angle of cyclic quad. ABNM is equal to interior opposite angle*)

Similarly

$\angle ABN = \angle AQP$ in cyclic quadrilateral $ABPQ$.

Hence quadrilateral $QAMR$ is cyclic.

(*exterior angle AQP is equal to interior opposite angle RMA*)

[2]

ii. Produce TM to S. Then

$\angle TMR = \angle SMN$ (*vertically opposite angles are equal*)

$\angle SMN = \angle MAN$ (*angle between tangent and chord drawn to point of contact is equal to angle subtended by that chord in the alternate segment*)

$\angle MAN = \angle PAQ$ (*vertically opposite angles are equal*)

$\angle PAQ = \angle TRM$ (*exterior angle of cyclic quad. QAMR is equal to interior opposite angle*)

Hence in ΔTMR, $\angle TMR = \angle TRM$ and hence $TM = TR$ (*sides opposite equal angles are equal*) [4]

d.)

$$a^2 + b^2 - 2ab = (a-b)^2$$
$$b^2 + c^2 - 2bc = (b-c)^2$$
$$c^2 + a^2 - 2ca = (c-a)^2$$

$$\therefore 2\{a^2 + b^2 + c^2 - (ab+bc+ca)\} = (a-b)^2 + (b-c)^2 + (c-a)^2$$

But $a,\ b,\ c$ are positive real numbers, as they are the lengths of triangle sides.

Hence $(a-b)$, $(b-c)$ and $(c-a)$ are also real numbers.

$\therefore (a-b)^2 \geq 0$ with equality if and only if $a = b$, and similarly for $(b-c)^2$, $(c-a)^2$.

Hence if $a^2 + b^2 + c^2 = ab + bc + ca$, then $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$$\therefore (a-b)^2 = (b-c)^2 = (c-a)^2 = 0$$
$$\therefore a = b,\quad b = c \text{ and } c = a$$

Hence $a = b = c$ and ΔABC is equilateral.

[3]