MEREWETHER HIGH SCHOOL

2011
HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes.
- Working Time – 3 hours.
- Write using a black or blue pen.
- Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question. A correct answer without appropriate working may not receive full marks.
- Begin each question on a separate sheet of paper.

Total marks (120)

- Attempt Questions 1-8.
- All questions are of equal value.

TOTAL MARKS - 120 **ATTEMPT QUESTION 1-8** **ALL QUESTIONS ARE OF EQUAL VALUE**

Question 1 (15 marks) **Use a separate page** **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{3}} \tan x \; dx$ **3**

(b) Find $\int \frac{5x}{\sqrt{1+15x^2}} dx$ **3**

(c) Find $\int 2x\, e^{3x}\, dx$ **2**

(d) Using the substitution $t = \tan\frac{x}{2}$, or otherwise

Evaluate $\int_0^{\frac{\pi}{2}} \frac{4}{1+\sin x} dx$ **4**

(e) Find $\int \frac{1}{(4+x)\sqrt{x}} dx$ **3**

Question 2 (15 marks) **Use a separate page**

Let $z = \sqrt{3} - i$

(a) (i) Find z^2 in the form $x + iy$ **1**

(ii) Find $\bar{z}$ - 3z in the form $x + iy$ **1**

(iii) Find $\frac{3i}{2z}$ in the form $x + iy$ **2**

Question 2 (cont) **Use a separate page** **Marks**

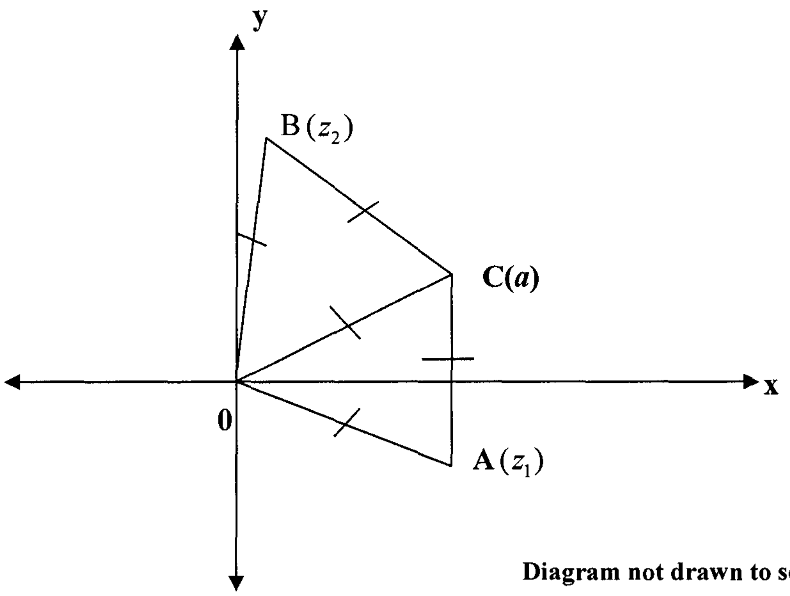
(b) Sketch the region in the complex plane where the inequalities

$1 \leq |z+2| \leq 2$ and $0 \leq \arg(z+2) \leq \dfrac{\pi}{4}$ hold simultaneously. **3**

(c) (i) Express $1+i$ and $1-i$ in modulus argument form **1**

(ii) Use De Moirve's Theorem to evaluate $(1+i)^{10}+(1-i)^{10}$ **2**

(d)



The points A,B and C on an Argand diagram represent the complex numbers z_1, z_2 and a respectively. The triangles OAC and OBC are equilateral with unit sides so $|z_1|=|z_2|=|a|=1$.

Let $\omega=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$.

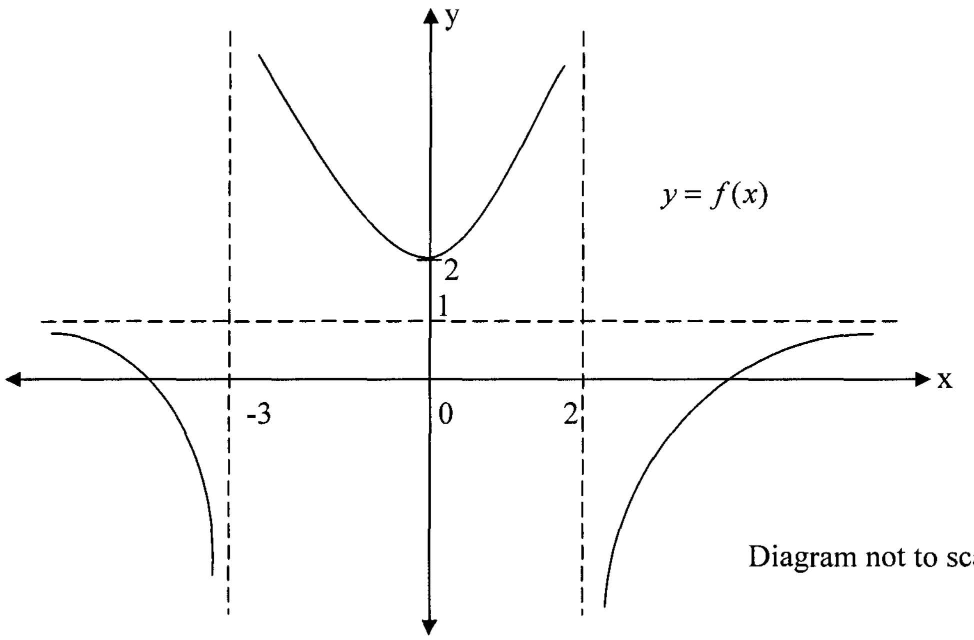
(i) Explain why $z_2=\omega a$ **1**

(ii) Show that $z_1 z_2=a^2$ **2**

(iii) Show that z_1 and z_2 are the roots of $z^2-az+a^2=0$ **2**

Question 3 (15 marks) **Use a separate page** **Marks**

(a) The sketch of $y = f(x)$ is shown below.



Draw separate one third page sketches of the graphs of the following:

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y^2 = \dfrac{1}{f(x)}$ **3**

(b) Given the curve $3x^2 + 6xy + 9y^2 = 54$, find the coordinates of the points where the tangent to the above curve is horizontal. **3**

(c) Sketch the graph of $y = \dfrac{3x^2}{x^2 - 4}$, showing clearly any turning points and asymptotes. **3**

Question 4 (15 marks) **Use a separate page** **Marks**

(a) Find the eccentricity, foci and equation of directrices of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. **3**

(b) The region bounded by the curve $y=(x-1)(3-x)$ and the x-axis is rotated about the line $x=3$ to form a solid. When the region is rotated, the horizontal line segment l at height y sweeps out an annulus.

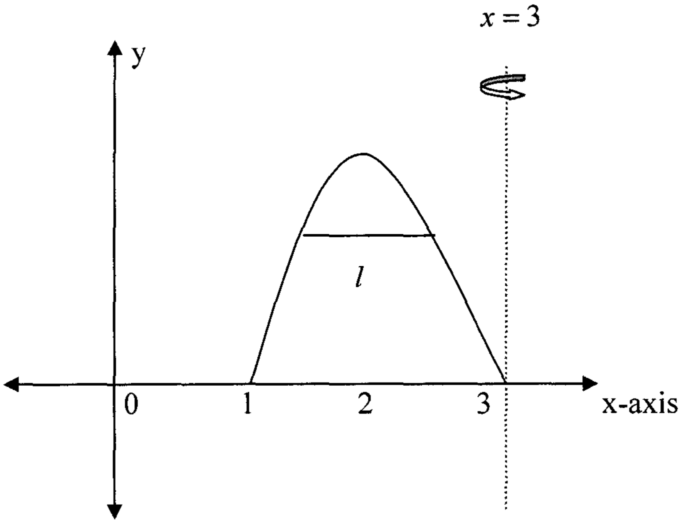


Diagram not drawn to scale

(i) Show that the area of annulus at height y is given by $4\pi\sqrt{1-y}$ **3**

(ii) Find the volume of the solid. **2**

(c) Suppose $\alpha, \beta, \gamma,$ and δ are the four roots of the polynomial equation

$$P(x)=x^4+px^3+qx^2+rx+s$$

(i) Find the values of $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma+\alpha\delta\beta+\alpha\gamma\delta+\beta\gamma\delta$ in terms of p, q, r and s. **2**

(ii) Show that $\alpha^2+\beta^2+\gamma^2+\delta^2=p^2-2q$ **2**

(iii) Apply the result in part (ii) to show that $x^4-3x^3+5x^2+7x-8=0$ cannot have four real roots. **1**

(iv) By evaluating the polynomial at x = 0 and x = 1 deduce that the polynomial

$x^4-3x^3+5x^2+7x-8=0$ has exactly two real roots. **2**

Question 5 (15 marks) **Use a separate page** **Marks**

(a)

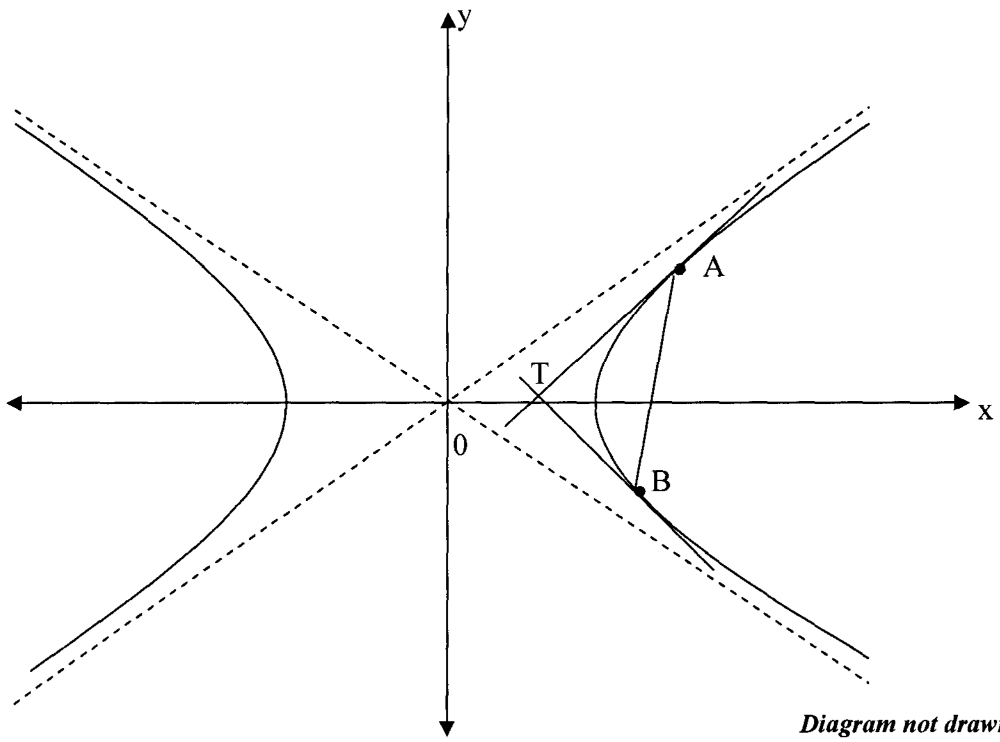


Diagram not drawn to scale

The points at $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the same branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The tangents at A and B meet at $T(x_0, y_0)$.

(i) Show that the equation of the tangent at A is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. **2**

(ii) Hence show that the chord of contact, AB, has equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. **2**

(iii) The chord AB passes through the focus $S(ae, 0)$ where e is the eccentricity of the hyperbola. Prove that T lies on the directrix of the hyperbola. **1**

(b)

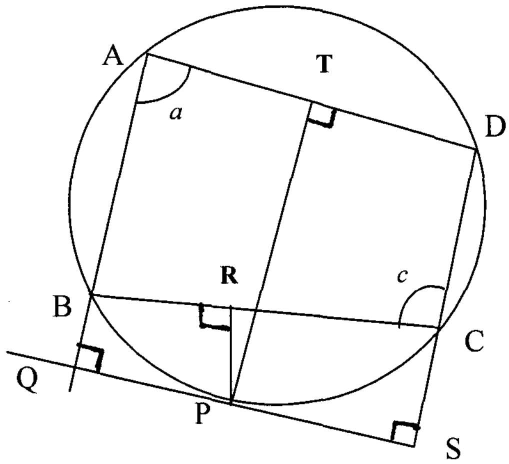


Diagram not drawn to scale

ABCD is a cyclic quadrilateral and from point P on the circle PQ,PR,PS and PT are perpendicular to the sides AB,BC,CD.DA respectively.

(i) Prove PSCR and AQPT are cyclic quadrilaterals. **1**

(ii) Prove that $\angle RPS = \angle TPQ$ and $\angle PSR = \angle PTQ$. **3**

(iii) Prove that $\Delta RPS ||| \Delta TPQ$. **1**

(c) (i) Find real numbers a ,b and c such that $\dfrac{x^2+2x}{(x^2+4)(x-2)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{x-2}$ 2

(ii) Hence find $\displaystyle\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$ 2

Question 6 (15 marks) **Use a separate page** **Marks**

(a) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$

3

(b) Show that $\sin^3\theta = 3\sin\theta\cos^2\theta - \sin 3\theta$

2

(c) Let ω be the complex number satisfying $\omega^3 = 1$ and $\text{Im}(\omega) > 0$.

The cubic polynomial, $q(z) = z^3 + az^2 + bz + c$, has zeroes $1, -\omega$ and $-\bar{\omega}$. Find $q(z)$.

3

(d) A sequence s_n is defined by $s_1 = 1$, $s_2 = 2$ and, for $n > 2$, $s_n = s_{n-1} + (n-1)s_{n-2}$.

(i) Find s_3 and s_4.

1

(ii) Prove that $\sqrt{x} + x \geq \sqrt{x(x+1)}$ for all real numbers $x \geq 0$.

2

(iii) Prove by induction that $s_n \geq \sqrt{n!}$ for all integers $n \geq 1$.

3

Question 7 (15 marks) **Use a separate page** **Marks**

a) For $n = 0, 1, 2, ...$ let $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta.$

(i) Show that $I_1 = \frac{1}{2}\ln 2.$ **1**

(ii) Show that, for $n \geq 2$,

$$I_n + I_{n-2} = \frac{1}{n-1}$$

3

(iii) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

3

(iv) By using the recurrence relation of part (ii), find I_5 and deduce

$$\frac{2}{3} < \ln 2 < \frac{3}{4}$$ **2**

(b) The depth of water y metres in a tidal creek is given by

$$4\frac{d^2y}{dt^2} = 5 - y, \text{ the time measured in hours.}$$

(i) Prove that the vertical motion of water is simple harmonic. **1**

(ii) Find the period and centre of motion. **2**

(c) A projectile is fired from a point O with a speed 20 m/s. T seconds later a second projectile is fired at the same point but with a different angle of inclination.
The two particles collide at a point whose horizontal and vertical distances from O are 8 *m* and 12 *m* respectively. Find the value of T to the nearest tenth of a second.
(Neglect air resistance and take $g = 10 m/s$) **3**

Question 8 (15 marks) **Use a separate page** **Marks**

(a) *(i)* Find the five fifth roots of 1. **2**

(ii) Let ω be a non real complex number such that $\omega^5 = 1$. By factorizing $Z^5 - \omega^5$,

show that $Z^4 + aZ^3 + a^2Z^2 + a^3Z + a^4 = (z - ax)(1 - ax^2)(1 - ax^3)(1 - ax^4)$. **2**

(iii) Deduce that $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 5$. **3**

(b) Prove that $(a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2$ **2**

(c) The normal at $P(cp, \frac{c}{p})$ meets the hyperbola at $xy = c^2$ again at $Q(cq, \frac{c}{q})$.

(i) Prove that $p^3q = -1$ **2**

(ii) Hence show that the locus of the midpoint R of PQ is given by $c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$. **4**

End of paper