MORIAH COLLEGE

Year 12

MATHEMATICS

Extension 2

Date: **Wednesday 8th August, 2001**

Time Allowed: 3 hours, plus 5 minutes reading time.

Examiners: J. Taylor

Instructions:

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 6
- Board approved calculators may be used.
- Answer each question in a SEPARATE writing booklet.
- You may ask for extra Writing Booklets, if you need them.

Question 1 (15 marks)

a) Find

i) $$\int \frac{\cos^{-1}\frac{2x}{3}}{\sqrt{9-4x^2}}\,dx$$ **2**

ii) $$\int x^2 \tan^{-1}\frac{x}{2}\,dx$$ **3**

iii) $$\int \frac{dx}{\sqrt{2x^2+3x}}$$ **4**

b) If **6**

$$I_n = \int_0^1 x^n(1-x)^{\frac{1}{2}}\,dx \qquad (n>0)$$

prove that

$$I_n = \left(\frac{2n}{2n+3}\right)I_{n-1}$$

and evaluate I_2.

Question 2 (15 marks)

a) Sketch the region in the Argand Plane consisting of those points *z* for which

$|z+3-i|>4$ intersects with $-\frac{3\pi}{8}<\arg z\le-\frac{3\pi}{4}$ **3**

b) Given that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

i) Find, in terms of π, an approximations for $e^{i\pi}$ in the form $x + iy$ using the above four terms of the series. **1**

ii) Use your calculator where necessary to plot this approximation on an Argand diagram **2**

iii) If *z* is any complex number, prove that both **4**

$z^n + \bar{z}^n$

and

$e^z + e^{\bar{z}}$

are pure real numbers

c) i) Find the five fifth roots of unity. **2**

ii) If $\omega = \text{cis}\frac{2\pi}{5}$, show that $1+\omega+\omega^2+\omega^3+\omega^4=0$ **1**

iii) Show that $z_1=\omega+\omega^4$ and $z_2=\omega^2+\omega^3$ are the roots of the equation

$z^2+z-1=0$ **2**

Question 3 (15 marks)

a) Sketch $y = \sqrt{\cos 3x}$ in the domain: $-\pi \le x \le \pi$ **2**

b) The rational function *f*(*x*) is defined

$$f(x) = \frac{x - p}{(x - q)(x - r)}$$

i) Write *f*(*x*) as the sum of two partial fractions. **2**

ii) If *p* lies between *q* and *r*, use a neat sketch to explain why *f*(*x*) can assume all real values. **2**

iii) If *p* does not lie between *q* and *r*, use a neat sketch to explain why *f*(*x*) *cannot* assume all real values. **2**

iv) Show that in either case, the gradient of the tangent at the midpoint of the interval between *q* and *r* is independent of *p* **3**

c) If *x*, *y* are positive integers such that $x - y > 1$, then prove that

$$x! + y! > (x - 1)! + (y + 1)!$$ **3**

Question 4 (15 marks)

a) For the ellipse $\dfrac{y^2}{50}+\dfrac{x^2}{32}=1$, find

i) the eccentricity, **2**

ii) the coordinates of the foci *S* and *S′*. 1

b) Explain why $\dfrac{x^2}{\lambda-23}+\dfrac{y^2}{5-\lambda}=1$ cannot represent the equation of an ellipse. **1**

c) Normals to the ellipse $4x^2+9y^2=36$ at points $P(3\cos\alpha,2\sin\alpha)$ and $Q(3\cos\beta,2\sin\beta)$ are at right angles to each other. Show that

i) the gradient of the normal at *P* is $\dfrac{3\sin\alpha}{2\cos\alpha}$, **2**

ii) $4\cot\alpha\cot\beta=-9$. **2**

d) $P\left(5p,\dfrac{5}{p}\right), p>0$ and $Q\left(5q,\dfrac{5}{q}\right), q>0$ are two points on the hyperbola, *H*, $xy=25$.

i) Derive the equation of the chord *PQ*, **2**

ii)` State the equations of the tangents at *P* and *Q*, **1**

iii) If the tangents at *P* and *Q* intersect at *R*, find the co-ordinates of *R*. **2**

iv) If the secant *PQ* passes through the point $S(15,0)$, find the locus of *R*. **2**

Question 5 (15 marks)

a) When $x^3 - kx^2 - 10kx + 25$ is divided by $x - 2$ the remainder is 9. Find the value of *k*. **2**

b) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$. Factorise $P(x)$ over the field of

i) real numbers, **2**

ii) complex numbers. **1**

c) The equation $x^5 - 5x^4 - x^3 + 3x^2 + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equations with roots

i) $\frac{1}{\alpha} + 2, \frac{1}{\beta} + 2, \frac{1}{\gamma} + 2, \frac{1}{\delta} + 2$, **2**

ii) $\alpha^2 - 1, \beta^2 - 1, \gamma^2 - 1, \delta^2 - 1$ **3**

d) $\phi(x)$ is a polynomial of degree 5 such that $\phi(x) - 1$ is divisible by $(x-1)^3$ and $\phi(x)$ itself is divisible by x^3. Derive an expression for $\phi(x)$. **5**

Question 6 (15 marks)

a) i) Prove that $2\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx = 12\pi^2$ **3**

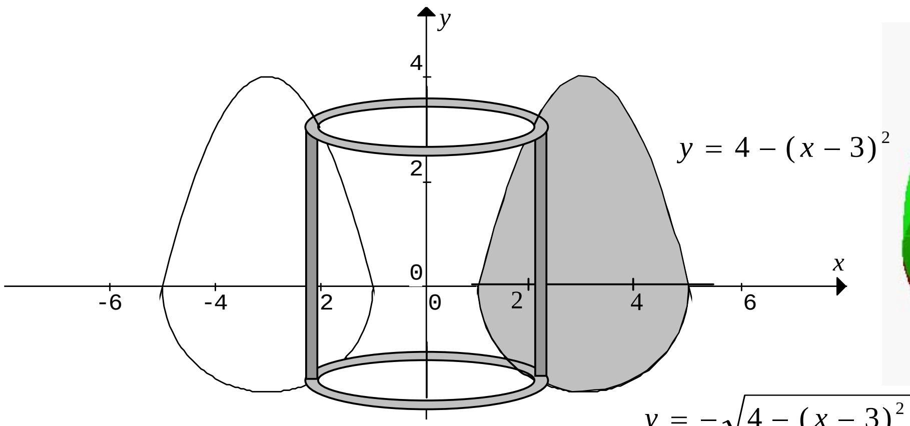


Figure 1

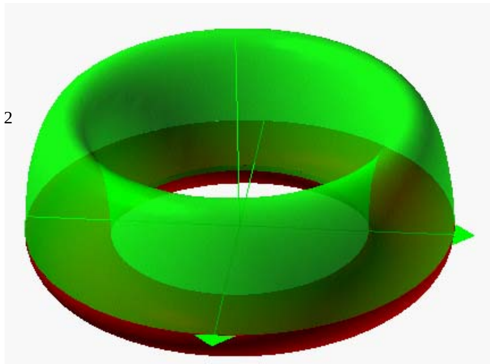


Figure 2

ii) The solid in fig.2 is formed by rotating about the y-axis the area bounded by the parabola $y = 4 - (x-3)^2$ and the semi-circle $y = -\sqrt{4-(x-3)^2}$.

Use the method of cylindrical shells to calculate the volume generated. **3**

Question 6 b) is on the next page.

Question 6

b) A circle in a horizontal *x*-*y* plane has centre *O*, radius *R* units . At each point *P*(*x*, *y*) on this circle, another circle is constructed perpendicular to the original circle in the plane containing the radius at that point. The radius *r* of such a circle (see the shaded circle, figure 1) is given by

$r = xy.$

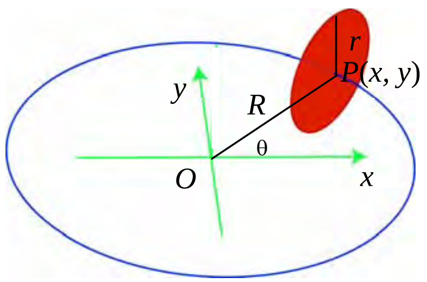


Figure 1

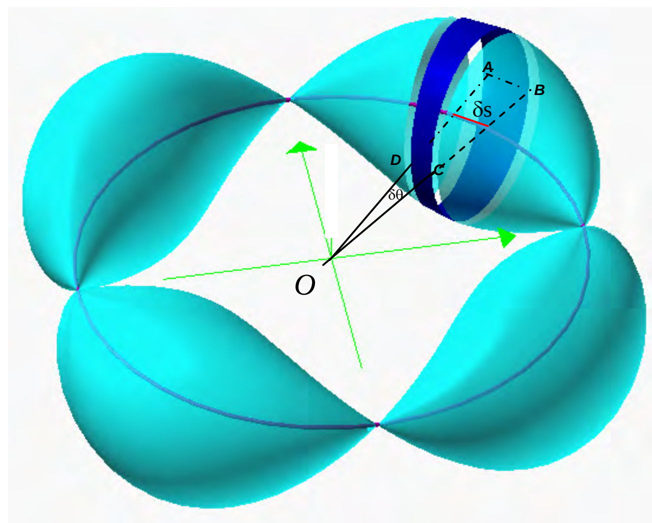


Figure 2

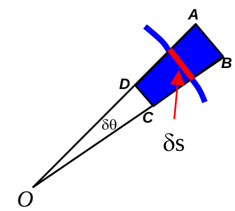
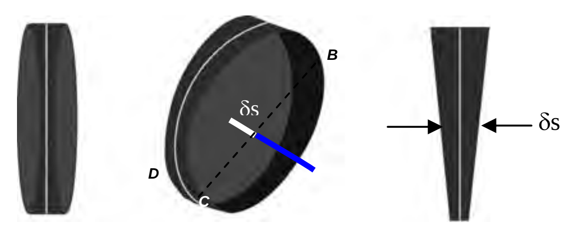


Figure 3

i) If $\angle POx = \theta$, prove that $r = \dfrac{R^2 \sin 2\theta}{2}$ **2**

As *P* moves around the horizontal circle, the vertical circles will form a surface drawn in figure 2.

A section is taken in the first quadrant by slicing the figure with two vertical planes from the centre of the horizontal circle. This section can be approximated by the wedge-shaped solid in figure 4:



Front *Perspective* *Top*

Figure 4

ii) If the thickness of the section at the centre is the straight line length δs, explain *briefly* why the volume δV of the section in figure 4 is $\delta V = \pi r^2 \delta s$. **1**

iii) If $\delta\theta$ is the angle between the radii used to slice the figure(see figure 3), prove that **3**

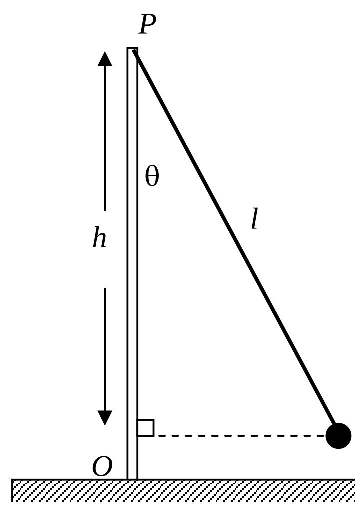
a) $\delta V \approx R\pi r^2 \delta\theta$

b) $$V = \frac{R^5}{4}\pi \lim_{\delta\theta \to 0} \sum_{\theta=0}^{2\pi} \sin^2 2\theta \delta\theta$$

iv) Find the volume of the solid. **3**

Question 7 (15 marks)

a) A vertical pole PO of height l units is standing on a flat plane. Suspended from the top P of the pole is a string, also of length l, at the end of which is attached a particle of mass m. The pole begins to rotate so that the mass describes a horizontal circle of radius r units with a uniform angular velocity so that the centre of the circle is h units below P.



The string makes an angle θ with the pole.

i) If the speed of the particle is v, prove that **2**

$$\tan\theta = \frac{v^2}{gr}$$

ii) When the particle is uniformly rotating at a height $h = \frac{l}{2}$ the string suddenly snaps, and the particle travels freely through the air to land at a point Q. **4**

Find the distance OQ in terms of l.

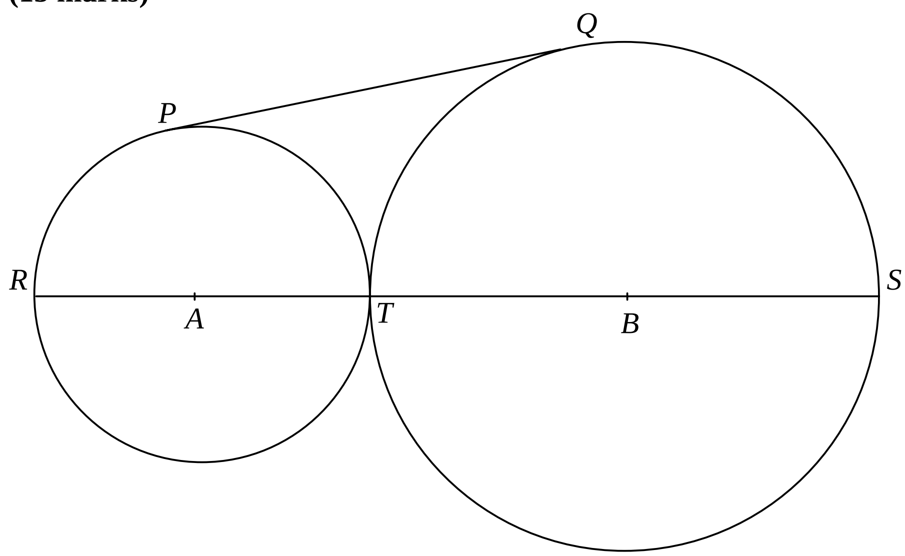
b) The only force acting on a particle moving in a straight line is a resistance $m\lambda(c+v)$ acting in the same line. The mass of the particle is m, its velocity is v, and λ and c are positive constants. The particle starts to move with velocity u (>0) and comes to rest in time T. At time $\tfrac{1}{2}T$ its velocity is $\tfrac{1}{4}u$. Show that

i) $c = \tfrac{1}{8}u$, **6**

ii) at time t, $8\dfrac{v}{u} = 9e^{-\lambda t} - 1$. **3**

Question 8 (15 marks)

a)



Copy the above diagram into your answer booklet.

In the diagram, circles with centres *A* and *B* touch each other at *T*. *PQ* is a direct common tangent. The line of centres cuts the circles at *R* and *S* as shown.

RP and *SQ* produced meet at *X*.

Prove that $\angle RXS$ is a right angle. **5**

b) A sequence of polynomials (called the *Bernoulli Polynomials*) is defined inductively by the three conditions:

1) $B_0(x) = 1$

2) $B_n'(x) = nB_{n-1}(x)$

3) $\int_0^1 B_n(x)dx = 0$ if $n \geq 1$

i) Prove that $B_1(x) = x - \frac{1}{2}$ **3**

ii) Prove that if $B_n(x+1) - B\ (x) = nx^{n-1}$ and

$$g(x) = B_{n+1}(x+1) - B_{n+1}(x)$$

then

$$g'(x) = (n+1)nx^{n-1}$$

and deduce an expression for $g(x)$. **4**

iii) Prove by induction that

$$B_n(x+1) - B_n(x) = nx^{n-1} \text{ if } n \geq 1.$$ **3**

End of Paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \quad x \neq 0, \text{ if } n < 0.$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0.$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \; a \neq 0, \; -a \leq x \leq a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\{x + \sqrt{(x^2 - a^2)}\right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left\{x + \sqrt{(x^2 + a^2)}\right\}$$

NOTE: $\ln x = \log_e x, \; x > 0.$

Question 1 (15 marks)

a) i)

Let $u = \cos^{-1} \frac{2x}{3}$

$$du = \frac{-2}{\sqrt{9-4x^2}}\,dx$$

$$I = \int -\frac{u}{2}\,du \qquad 2$$

$$= -\frac{u^2}{4} + c$$

$$= -\tfrac{1}{4}\left(\cos^{-1} \tfrac{2x}{3}\right)^2 + c$$

ii)

$$\int x^2 \tan^{-1} \tfrac{x}{2}\,dx \qquad v' = x^2 \qquad u = \tan^{-1} \tfrac{x}{2}$$

$$v = \frac{x^3}{3} \qquad u' = \frac{2}{4+x^2}$$

$$\int uv'\,dx = uv - \int vu'\,dx$$

$$= \frac{x^3}{3}\tan^{-1} \tfrac{x}{2} - \tfrac{2}{3}\int \frac{x^3}{4+x^2}dx \qquad 1$$

$$= \frac{x^3}{3}\tan^{-1} \tfrac{x}{2} - \tfrac{2}{3}\int \frac{4x+x^3-4x}{4+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1} \tfrac{x}{2} - \tfrac{2}{3}\int \left(x - \frac{4x}{4+x^2}\right)dx \qquad 1$$

$$= \frac{x^3}{3}\tan^{-1} \tfrac{x}{2} - \frac{x^2}{3} + \frac{4}{3}\ln\left(4+x^2\right) + c \qquad 2$$

iii)

$$I = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2 + \frac{3x}{2}}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2 - \frac{9}{16}}} \qquad 2$$

$$= \frac{1}{\sqrt{2}}\ln\left[x + \frac{3}{4} + \sqrt{x^2 + \frac{3x}{2}}\right] + C$$

1b)

$$I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}} dx$$

$$= \left[-\frac{2}{3} x^n (1-x)^{\frac{3}{2}} \right]_0^1 + \frac{2n}{3} \int_0^1 x^{n-1} (1-x)^{\frac{3}{2}} dx$$

$$= \frac{2n}{3} \int_0^1 x^{n-1} (1-x)(1-x)^{\frac{1}{2}} dx$$

$$= \frac{2n}{3} \int_0^1 x^{n-1} (1-x)^{\frac{1}{2}} - x^n (1-x)^{\frac{1}{2}} dx$$

$$= \frac{2n}{3} [I_{n-1} - I_n]$$

$$I_n + \frac{2n}{3} I_n = \frac{2n}{3} I_{n-1}$$

$$I_n \left(\frac{2n+3}{3} \right) = \left(\frac{2n}{3} \right) I_{n-1}$$ 4

$$I_n = \left(\frac{2n}{2n+3} \right) I_{n-1}$$

$$I_1 = 0 + \frac{2}{3} \int_0^1 (1-x)^{\frac{3}{2}} dx$$

$$= \frac{2}{3} \times -\frac{2}{5} \left[(1-x)^{\frac{5}{2}} \right]_0^1$$

$$= \frac{4}{15}$$

Hence

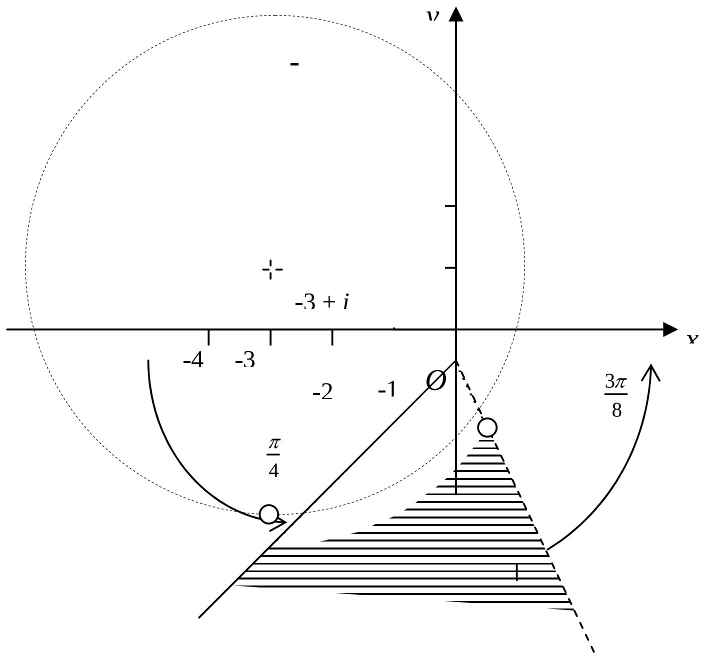
$$I_2 = \frac{4}{7} I_1$$

$$= \frac{4}{7} \times \frac{4}{15}$$ 3

$$I_2 = \frac{16}{105}$$

Question 2 (15 marks)

a)



3

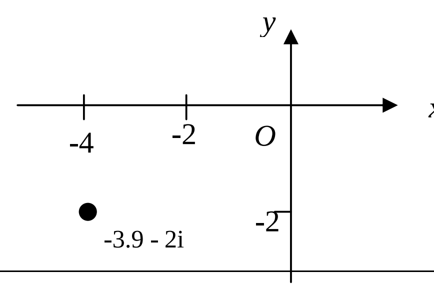
c) i)

$$e^{i\pi} \approx 1 + i\pi + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!}$$
$$= 1 + i\pi - \frac{\pi^2}{2} - \frac{i\pi^3}{6}$$
$$= 1 - \frac{\pi^2}{2} + i\left(\pi - \frac{\pi^3}{6}\right)$$

(1)

$$e^{i\pi} \approx 1 - \frac{\pi^2}{2} + i\left(\pi - \frac{\pi^3}{6}\right)$$ (1)

$$\approx -3.9438 + i.(-2.0261)$$



(1)

ii) a) Let $z = r(\cos\theta + i\sin\theta)$. Then

$$\bar{z} = r(\cos\theta - i\sin\theta)$$
$$= r(\cos(-\theta) + i\sin(-\theta))$$

Hence, by de Moivre's Theorem,

$$z^n + \bar{z}^n = r^n(\cos n\theta + i\sin n\theta) + r^n(\cos(-n\theta) + i\sin(-n\theta))$$
$$= r^n(\cos n\theta + i\sin n\theta + \cos(n\theta) - i\sin(n\theta))$$ (2)
$$= 2r^n\cos n\theta$$

which is totally real

b) Similarly

$$e^z + e^{\bar{z}} = 1 + z + \frac{z^2}{2!} + \frac{(z)^3}{3!} + \ldots + 1 + \bar{z} + \frac{\bar{z}^2}{2!} + \frac{\bar{z}^3}{3!} + \ldots$$ (2)
$$= 2 + (z + \bar{z}) + (z^2 + \bar{z}^2) + (z^3 + \bar{z}^3) + ..$$

which is totally real from above.

b) i) $z^5 = 1$

$(\cos\theta + i\sin\theta)^5 = 1$

$\cos 5\theta + i\sin 5\theta = 1$ 1

$\cos 5\theta = 1$ and $\sin 5\theta = 0$

$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta = 0.\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$z = cis\,\frac{2k\pi}{5} \qquad k = 0,1,2,3,4$ 1

ii) If $\omega = \text{cis}\,\dfrac{2\pi}{5}$

$\omega^2 = cis\,\frac{4\pi}{5}, \quad \omega^3 = cis\,\frac{6\pi}{5}, \quad \omega^4 = cis\,\frac{8\pi}{5}$

$1 + \omega + \omega^2 + \omega^3 + \omega^4 =$ sum of roots

$= 0$ 1

iii) $z_1 = \omega + \omega^4$ and $z_2 = \omega^2 + \omega^3$

$z_1 + z_2 = \omega + \omega^4 + \omega^2 + \omega^3 = -1$ 1

$z_1 z_2 = (\omega + \omega^4)(\omega^2 + \omega^3)$

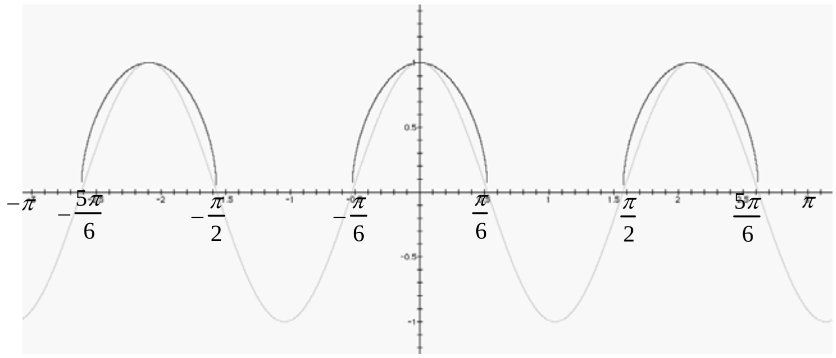
$= \omega^3 + \omega^4 + \omega^6 + \omega^7$ 1

$= \omega^3 + \omega^4 + \omega + \omega^2$

$= -1$

Question 3 (15 marks)

a) $y = \cos 3x$ (grey line) $\quad y = \sqrt{\cos 3x}$ (black line)



2

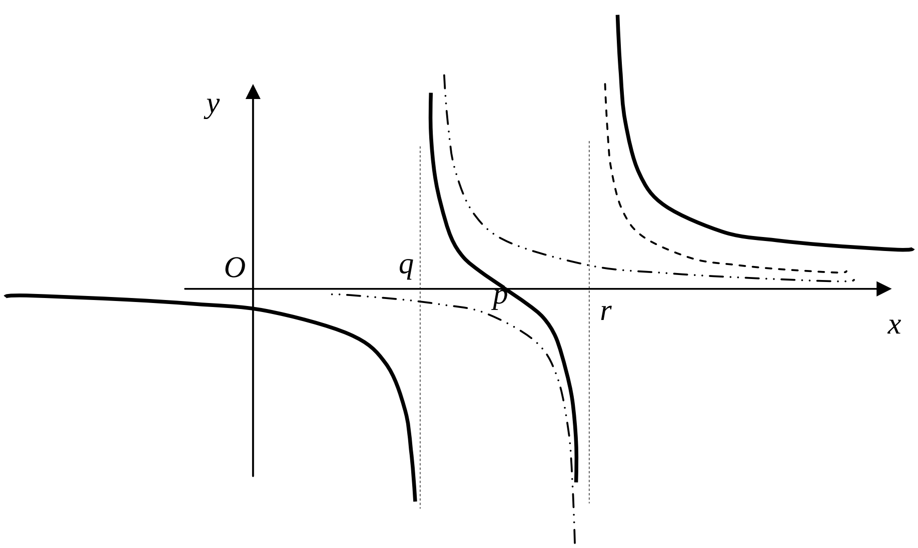
b) i)

$$\frac{x-p}{(x-q)(x-r)} = \frac{A}{(x-q)} + \frac{B}{(x-r)}$$

$$= \frac{\frac{q-p}{q-r}}{(x-q)} + \frac{\frac{r-p}{r-q}}{(x-r)} \qquad (\textit{Heaviside})$$ 2

$$= \frac{\frac{q-p}{q-r}}{(x-q)} - \frac{\frac{r-p}{q-r}}{(x-r)}$$

ii) The partial fractions show us the graph of the function can be viewed as the sum of two hyperbolas, both of which are defined for all values in the interval $q < x < r$ The sum of these graphs can only be zero in the given interval if one branch is above the x axis, and one is below the x axis, as in the diagram below. This shows that $f(x)$ approaches infinity in both directions and is defined for all x in that interval. Hence $f(x)$ assumes all real values in that interval.



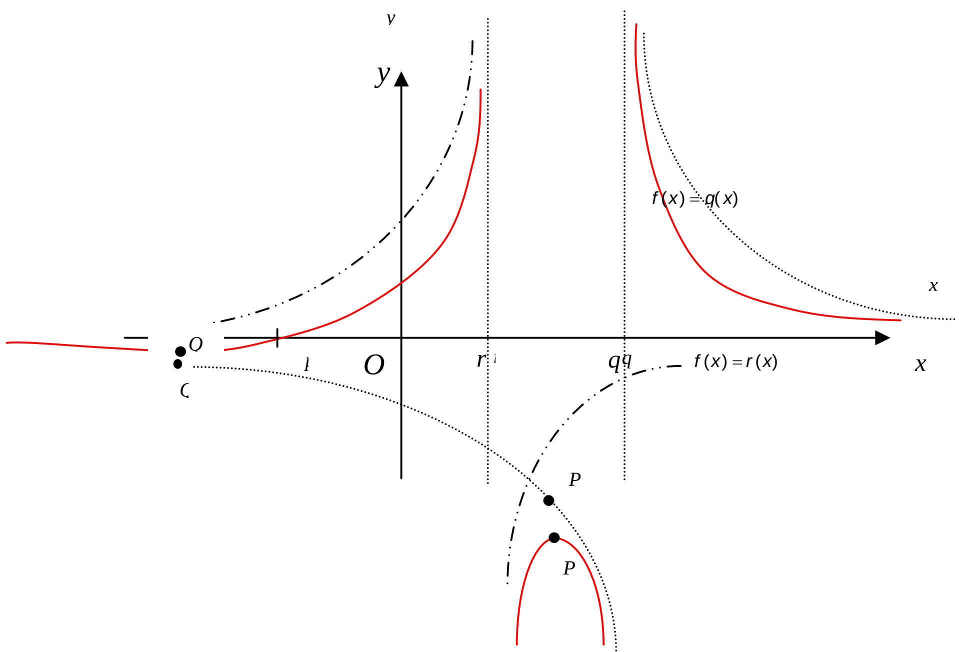
2

ii) From part i) we have

$$
\begin{aligned}
f(x) &= \frac{\frac{q-p}{q-r}}{(x-q)} + -\frac{\frac{r-p}{q-r}}{(x-r)} \\
&= \frac{L}{(x-q)} + -\frac{M}{(x-r)} \\
&= q(x) + r(x)
\end{aligned}
$$

where *L*, *M* are positive.if $p < r < q$

Let *P* be a point on $y = q(x), r < x < q$. Both $q(x)$ and $r(x)$ are negative, and so the point P' on $f(x)$ is lower than *P*. In a similar way, *Q* on $q(x)$ (where $x < r$) will be lower than Q' on $f(x)$ since $r(x) > 0$ and $q(x) < 0$ here. But *there will always be a vertical gap between* *P* and *Q*. Hence there will always be a gap between Q' and P' and the result follows.



The relative positions of *q* and *r* simply produce reflections of the above idea. 2

$$f'(x) = \frac{-\frac{q-p}{q-r}}{(x-q)^2} + \frac{-\frac{r-p}{r-q}}{(x-r)^2}$$

$$= \frac{p-q}{(q-r)(x-q)^2} + \frac{r-p}{(q-r)(x-r)^2}$$

$$= \frac{p-q}{(q-r)(x-q)^2} + \frac{r-p}{(q-r)(x-r)^2}$$

$$f'\left(\frac{q+r}{2}\right) = \frac{p-q}{(q-r)\left(\frac{r-q}{2}\right)^2} + \frac{r-p}{(q-r)\left(\frac{q-r}{2}\right)^2}$$ 3

$$= \frac{4}{(q-r)^3}\left[p-q+r-p\right]$$

$$= -\frac{4}{(q-r)^2}$$

which is independent of p

c) Consider $x! + y! - [(x-1)! + (y+1)!] = (x-1)![x-1] + y![1-(y+1)]$ 2

$= (x-1)![x-1] - y![y]$

$> y![y] - y![y]$

$= 0$

$\therefore\ x! + y! - [(x-1)! + (y+1)!] > 0$

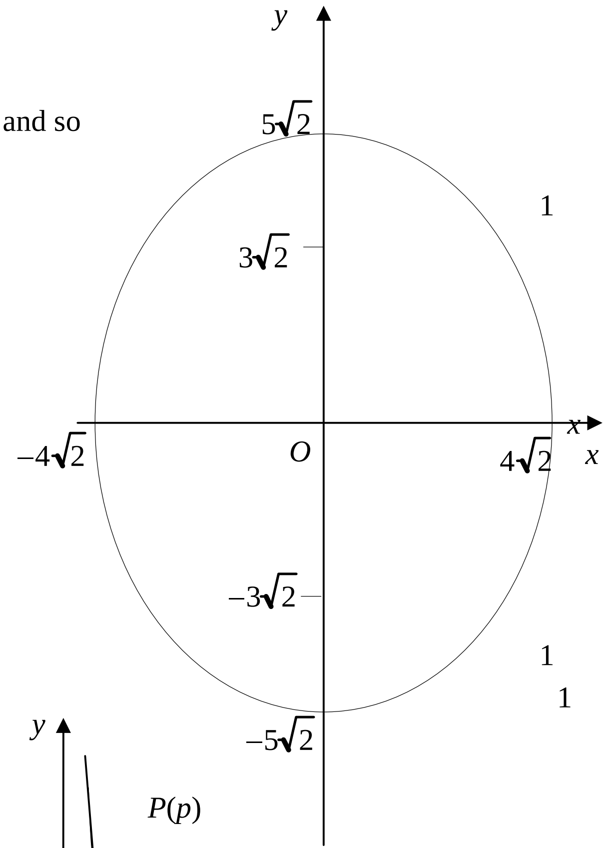
$\therefore\ x! + y! - [(x-1)! + (y+1)!] > [(x-1)! + (y+1)!]$

Question 4 (15 marks)

a) i) Since 50 > 32, the major axis is the y axis and so

$a^2 = 50 - 3\sqrt{2}$ $\qquad b^2 = 32$

$a = 5\sqrt{2}$ $\qquad b = 4\sqrt{2}$ 1



$$b^2 = a^2(1 - e^2)$$

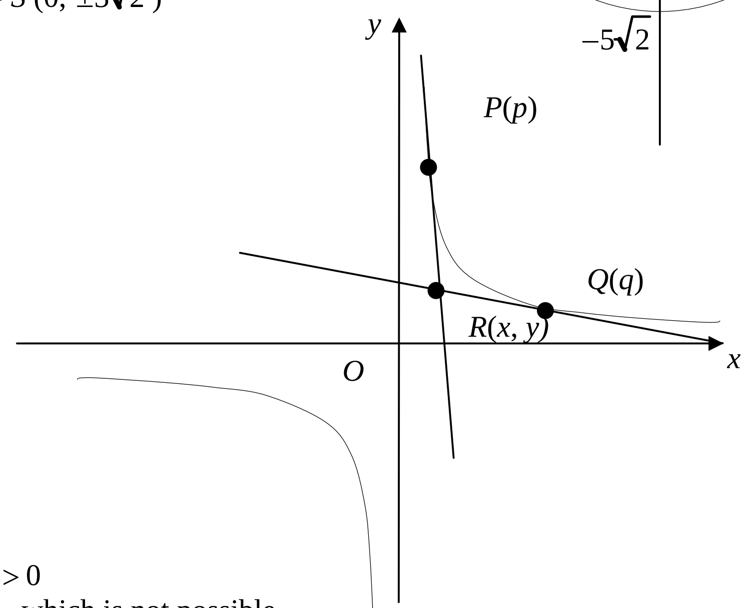
$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$= 1 - \frac{32}{50}$$

$$= \frac{18}{50}$$

$$e = \frac{3}{5}$$ 1

ii) $S(0,\ \pm ae) = S\,(0,\ \pm 3\sqrt{2}\,)$ 1



b) For an ellipse

$\lambda - 23 > 0$ and $5 - \lambda > 0$

i.e. $\lambda > 23$ and $\lambda < 5$, which is not possible. 1

c) $4x^2 + 9y^2 = 36$

i)

$$8x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x}{9y}$$ 1

At $P(3\cos\alpha, 2\sin\alpha)$, $\dfrac{dy}{dx} = -\dfrac{2\cos\alpha}{3\sin\alpha}$, $\quad m_{\text{normal}} = \dfrac{3\sin\alpha}{2\cos\alpha}$ $\qquad (m_1 m_2 = -1)$ 1

At $Q(3\cos\beta, 2\sin\beta)$, $\dfrac{dy}{dx} = -\dfrac{2\cos\beta}{3\sin\beta}$, $\quad m_{\text{normal}} = \dfrac{3\sin\beta}{2\cos\beta}$ $\qquad (m_1 m_2 = -1)$

ii)

$$\frac{3\sin\alpha}{2\cos\alpha}\cdot\frac{3\sin\beta}{2\cos\beta}=-1$$

$$\frac{4\cos\alpha\cos\beta}{9\sin\alpha\sin\beta}=-1$$ 2

$$4\cot\alpha\cot\beta=-9$$

d) .i)

$$m_{PQ}=\frac{\frac{5}{p}-\frac{5}{q}}{5p-5q}$$

$$=-\frac{1}{pq}$$ 1

PQ: $$y-\frac{5}{p}=-\frac{1}{pq}(x-5p)$$ 1

$$x+pqy=5(p+q)$$

ii)` Tangent at P is $x+p^2y=10p$

Tangent at Q is $x+q^2y=10q$ 1

iii) R: $(p^2-q^2)y=10(p-q)$

$$y=\frac{10}{p+q}$$ 1

$$x+\frac{10p^2}{p+q}=10p$$

$$x=10p-\frac{10p^2}{p+q}$$ 1

$$=\frac{10pq}{p+q}$$

iv) $S(15,0)$

$$15=5(p+q)$$

$$p+q=3$$ 1

$$x=\frac{10pq}{3} \qquad y=\frac{10}{3}$$

Locus of R is $y=\frac{10}{3}$ 1

Question 5 (15 marks)

a) $P(x) = x^3 - kx^2 - 10kx + 25 \qquad P(2) = 9$

$8 - 4k - 20k + 25 = 9$ 1

$-16k = -16$

$k = 1$ 1

b) $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$. Factorise $P(x)$ over the field of

i)

$P(x) = x^4(x+1) + 13x^2(x+1) - 48(x+1)$

$= (x+1)(x^4 + 13x^2 - 48)$ 1

$= (x+1)(x^2+16)(x^2-3)$

$= (x+1)(x^2+16)(x-\sqrt{3})(x+\sqrt{3})$ 1

ii) $P(x) = (x+1)(x+4i)(x-4i)(x-\sqrt{3})(x+\sqrt{3})$ 1

c) $x^5 - 5x^4 - x^3 + 3x^2 + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

i) The new equation has roots $x = \dfrac{1}{\alpha} + 2$ etc.

$\alpha = \dfrac{1}{x-2}$

Equation is $\left(\dfrac{1}{x-2}\right)^5 - 5\left(\dfrac{1}{x-2}\right)^4 - \left(\dfrac{1}{x-2}\right)^3 + 3\left(\dfrac{1}{x-2}\right)^2 + 1 = 0$ 1

$1 - 5(x-2) - (x-2)^2 + 3(x-2)^3 + (x-2)^5 = 0$ (full marks at this point) 1

$x^5 - 15x^4 + 53x^3 - 99x^2 + 115x - 49 = 0$

ii) The new equation has roots $x = \alpha^2 - 1$ etc.

$\alpha = \sqrt{x+1}$

Equation is $\left(\sqrt{x+1}\right)^5 - 5\left(\sqrt{x+1}\right)^4 - \left(\sqrt{x+1}\right)^3 + 3\left(\sqrt{x+1}\right)^2 + 1 = 0$ 1

$(x+1)^2\sqrt{x+1} - 5(x+1)^2 - (x+1)\sqrt{x+1} + 3\sqrt{x+1} + 1 - 0$

$\left[(x+1)^2 - (x+1)\right]\sqrt{x+1} = 5x^2 + 7x + 1$ 1

$(x^2+x)\sqrt{x+1} = 5x^2 + 7x + 1$

(full marks at this point) 1

$(x^2+x)^2(x+1) = (5x^2+7x+1)^2$

$x^5 - 22x^4 - 67x^3 - 58x^2 - 14x - 1 = 0$

d) $\phi(x) = x^3(ax^2 + bx + c)$

$\phi(x) - 1 = (x-1)^3(ax^2 + dx + e)$ 1

$x^3(ax^2 + bx + c) - 1 = (x-1)^3(ax^2 + dx + e)$

$$ax^5 + bx^4 + cx^3 - 1 = (x^3 - x^2 + x - 1)(ax^2 + dx + e)$$
$$= ax^5 + (d-3a)x^4 + (3a - 3d + e)x^3 + (3d - a - 3e)x^2 + (3e - d)x - e$$ 1

$a = a$

$b = d - 3a$

$c = 3a - 3d + e$

$3d - a - 3e = 0$ 1

$3e - d = 0$

$-e = -1$

Solving we get $e = 1$, $d = 3$, $a = 6$, $c = 10$. $b = -15$ 1

$\therefore \phi(x) = x^3(6x^2 - 15x + 10)$
$= 6x^5 - 15x^4 + 10x^3$ 1

Alternately,

$\phi(x) = ax^5 + bx^4 + cx^3$

$P(x) = ax^5 + bx^4 + cx^3 - 1$ 1

Since $P(x)$ is divisible by $(x-1)^3$, $x = 1$ is a triple zero of $P(x)$.

$P'(x) = 5ax^4 + 4bx^3 + 3cx^2$

$P''(x) = 20ax^3 + 12bx^2 + 6cx$ 1

$P''(1) = 5a + 4b + 3c = 0$

$P'(1) = 20a + 12b + 6c = 0$ 1

$P(1) = a + b + c - 1 = 0$

Solving simultaneously, $a = 6$, $b = -15$, $c = 10$ 1

$\phi(x) = 6x^5 - 15x^4 + 10x^3$ 1

Question 6 (15 marks)

Q6a)

i)

$$\int_1^5 x\sqrt{4-(3-x)^2}\,dx = \int_{-2}^{2}(3+u)\sqrt{4-u^2}\,du$$

$$= \int_{-2}^{2}3\sqrt{4-u^2}\,du + \int_{-2}^{2}u\sqrt{4-u^2}\,du$$

$$= 3\times \text{semicircle} + \text{integral of odd function}$$

$$= 3\times\frac{1}{2}\pi\times 2^2 + 0$$ 4

$$= 6\pi$$

$$2\pi\int_1^5 x\sqrt{4-(3-x)^2}\,dx = 12\pi^2$$

ii)

b)

$$\delta V = 2\pi r h\,\delta x$$

$$V = 2\pi\int_1^5 xy\,dx$$

$$= 2\pi\int_1^5 x\left[4-(x-3)^2+\sqrt{4-(3-x)^2}\right]dx$$

$$= 2\pi\int_1^5 x\left[4-(x-3)^2\right]dx + 2\pi\int_1^5 x\left[\sqrt{4-(3-x)^2}\right]dx$$ 4

$$= 2\pi\int_1^5 -x^3+6x^2-5x\,dx + 12\pi^2$$

$$= 64\pi + 12\pi^2$$

The volume is $64\pi + 12\pi^2$ units^3

b i)

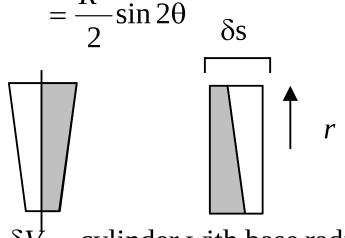
$$r = xy$$

$$= R\cos\theta R\sin\theta$$

$$= R^2\cos\theta\sin\theta$$ 2

$$= \frac{R^2}{2}\sin 2\theta$$

ii)



$$\delta V = \text{cylinder with base radius } r\text{, height } \delta s$$ 1

$$= \pi r^2\delta s$$

iii) a)

$$\text{Arc length} = R\delta\theta$$

$$\delta s \approx R\delta\theta$$

$$\delta V \approx \pi r^2 R\delta\theta$$ 1

$$= R\pi r^2\delta\theta$$

b)

$$V = \lim_{\delta\theta \to 0} \sum_{\delta\theta=0}^{2\pi} R\pi r^2 \delta\theta$$

$$= R\pi \lim_{\delta\theta \to 0} \sum_{\delta\theta=0}^{2\pi} r^2 \delta\theta$$

$$= R\pi \lim_{\delta\theta \to 0} \sum_{\delta\theta=0}^{2\pi} \frac{R^4}{4} \sin^2 2\theta\delta\theta \qquad 1$$

$$= \frac{R^5}{4} \pi \lim_{\delta\theta \to 0} \sum_{\delta\theta=0}^{2\pi} \sin^2 2\theta\delta\theta$$

$$V = \frac{\pi R^5}{4} \int_0^{2\pi} \sin^2 2\theta d\theta$$

$$= \frac{\pi R^5}{8} \int_0^{2\pi} 1 - \cos 4\theta d\theta$$

$$= \frac{\pi R^5}{8} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{2\pi} \qquad 2$$

$$= \frac{\pi^2 R^5}{4}$$

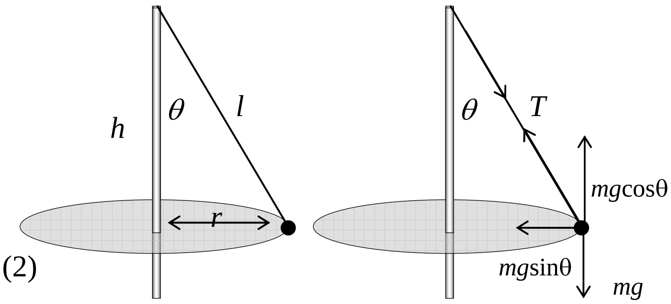
Question 7 (15 marks)

Solution

i) Let the tension in the string be T. Then

$T\sin\theta = m\dfrac{v^2}{r}$ (1)

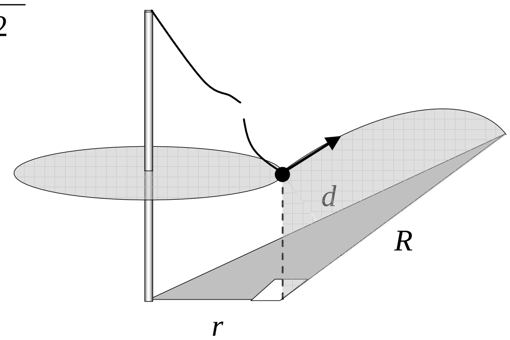
$T\cos\theta = mg$ (2)



Dividing (1) by (2) gives $\tan\theta = \dfrac{v^2}{gr}$ (2)

ii) When the string breaks, θ=60^0 and so

$h = \dfrac{l}{2},\ r = \dfrac{l}{2\sqrt{3}},\qquad \tan\theta = \sqrt{3},\qquad v = r\sqrt{\dfrac{g}{h}} = \sqrt{\dfrac{3l.g}{2}}$



When the string breaks, the particle will travel *horizontally* at a *tangent* to the circle of motion. Taking an origin at the point of breaking, the x and y coordinates will be

$x = vt,\ \ y = -\dfrac{1}{2}gt^2$

$y = -\dfrac{1}{2}g\dfrac{x^2}{v^2}$

This gives

$y = -\dfrac{1}{2}g\dfrac{x^2}{v^2}$

and so when $y = -l/2$, the distance R travelled before landing will be

$$\begin{aligned} R^2 &= \frac{l}{2g}.2v^2 \\ &= lv^2 \\ &= \frac{3l^2}{2} \end{aligned}$$

But the distance required is d, hence

$$\begin{aligned} d^2 &= r^2 + R^2 \\ &= \frac{3l^2}{4} + \frac{3l^2}{2} \\ &= \frac{9l^2}{4} \end{aligned}$$

Hence the particle will land a distance $\dfrac{3l}{2}$ m from

O(2)

$$m\frac{dv}{dt} = -m\lambda(c+v)$$

$$\frac{dv}{dt} = -\lambda(c+v)$$

$$\frac{dt}{dv} = -\frac{1}{\lambda(c+v)}$$

$$\int_0^t dt = -\frac{1}{\lambda}\int_u^v \frac{dv}{c+v}$$

$$t = -\frac{1}{\lambda}\log\left(\frac{c+v}{c+u}\right)$$

$$= \frac{1}{\lambda}\log\left(\frac{c+u}{c+v}\right)$$

At $T = T$, $v = 0$ and at $t = \frac{T}{2}$, $v = \frac{u}{4}$

$$T = \frac{1}{\lambda}\log\left(\frac{c+u}{c}\right), \qquad \frac{T}{2} = \frac{1}{\lambda}\log\left(\frac{c+u}{c+\frac{u}{4}}\right)$$ (4)

$$\frac{1}{\lambda}\ln\left(\frac{c+u}{c}\right) = \frac{2}{\lambda}\ln\left(\frac{c+u}{c+\frac{u}{4}}\right)$$

$$\left(\frac{c+u}{c}\right) = \left(\frac{c+u}{c+\frac{u}{4}}\right)^2$$ 1

$$\left(c+\tfrac{u}{4}\right)^2 = c(c+u)$$

$$c^2 + \frac{cu}{2} + \frac{u^2}{16} = c^2 + cu$$

$$\frac{cu}{2} = \frac{u^2}{16}$$ 1

$$c = \frac{u}{8}$$

ii)

$$t = \frac{1}{\lambda}\ln\left(\frac{c+u}{c+v}\right)$$ 1

$$\lambda t = \ln\left(\frac{\frac{u}{8}+u}{\frac{u}{8}+v}\right)$$

$$e^{\lambda t} = \frac{9u}{u+8v}$$

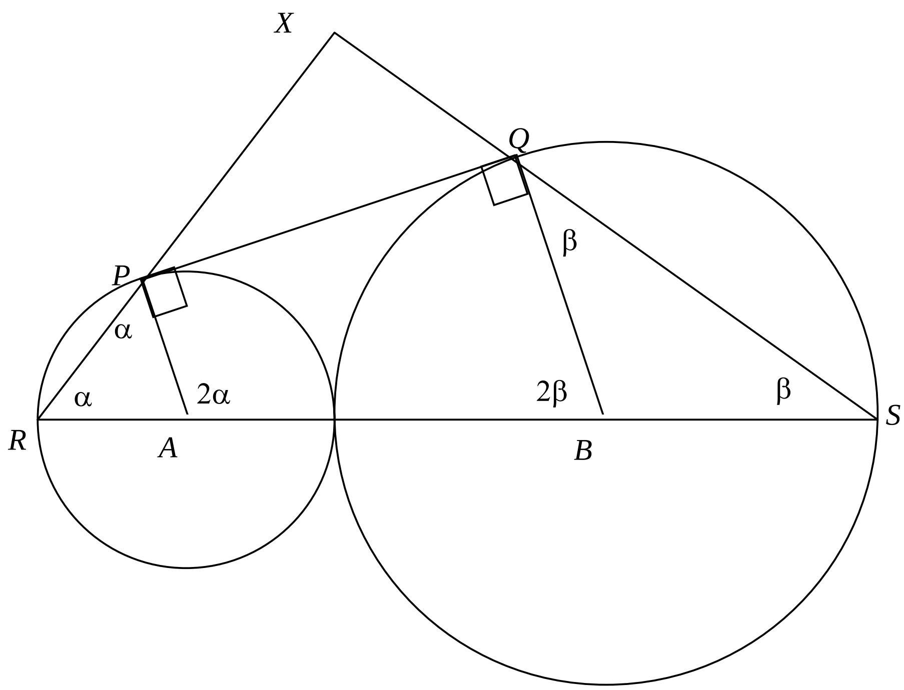
$$\frac{u+8v}{9u} = e^{-\lambda t}$$ 1

$$1 + \frac{8v}{u} = 9e^{-\lambda t}$$

$$\frac{8v}{u} = 9e^{-\lambda t} - 1$$ 1

Question 8 (15 marks)

a)



Construction:

Produce *RP* and *SQ* to *X*. Join *PA* and *QB*. Label angles as marked.

$\angle ARP = \angle APR = \alpha$ ($\angle$s of isos Δ, equal radii)

$\angle BSQ = \angle BQS = \beta$ ($\angle$s of isos Δ, equal radii) 1

$\angle PAB = 2\alpha$ (Exterior $\angle$s of Δ = sum of interior opp $\angle$s)

$\angle QBA = 2\beta$ (Exterior $\angle$s of Δ = sum of interior opp $\angle$s) 1

$2\alpha + 2\beta = 180^0$ (Cointerior $\angle$s of $\parallel$ lines supplementary, $PA \parallel QS$)

$\therefore \alpha + \beta = 90^0$

$\therefore \alpha + \beta + \theta = 180^0$ ($\angle$ sum of $\Delta = 180^0$) 1

$\therefore \theta = 90^0$

i.e. $\angle RXS = 90^0$ 1

b) i)

$B_1'(x) = 1.B_0(x)$

$= 1$

$B_1(x) = x + C$

To evaluate *C*:

$$\int_0^1 B_1(x)\,dx = \int_0^1 (x + C)\,dx$$

$$0 = \int_0^1 x\,dx + C \qquad \left[\text{since } \int_0^1 C\,dx = C\right]$$

$$C = -\int_0^1 x\,dx \qquad 3$$

$$= -\left[\frac{x^2}{2}\right]_0^1$$

$$= -\frac{1}{2}$$

$$\therefore B_1(x) = x - \frac{1}{2}$$

ii)

$$g(x) = B_{n+1}(x+1) - B_{n+1}(x)$$

$$g'(x) = B'_{n+1}(x+1) - B'_{n+1}(x)$$

$$= (n+1)B_n(x+1) - (n+1)B_n(x) \qquad 2$$

$$= (n+1)\left[B_n(x+1) - B_n(x)\right]$$

$$= (n+1)nx^{n-1} \quad \text{by assumption.}$$

$$\therefore g(x) = (n+1)x^n + C$$

iii)

$$B_1(x+1) - B_1(x) = \left(x + 1 - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)$$

$$= 1 \qquad 5$$

$$= 1x^0$$

and the proposition is true for $n = 1$.

Assuming the result is true for some positive integer $n = k$, we now consider the function

$$g(x) = B_{k+1}(x+1) - B_{k+1}(x)$$

$$g'(x) = B'_{k+1}(x+1) - B'_{k+1}(x)$$

$$= (k+1)B_k(x+1) - (n+1)B_k(x)$$

$$= (k+1)\left[B_k(x+1) - B_k(x)\right]$$

$$= (k+1)kx^{k-1} \quad \text{by assumption.}$$

$$\therefore g(x) = (n+1)x^n + C$$

Hence

$$B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^k + C$$

$$B_{k+1}(1) - B_{k+1}(0) = (k+1).0 + C$$

$$C = 0$$

and so $B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$, and the proposition is true for n + 1 if it is true for n.

But the proposition is true for n =1. Hence, by mathematical induction,

$$B_n(x+1) - B_n(x) = nx^{n-1} \text{ if } n \ge 1. \qquad (3)$$