

Moriah College

HSC Trial Examination

Mathematics Extension 2

2002

1. (a) Find $\int \frac{1}{\sqrt{4x-x^2}} dx$
- (b) Find the exact value of k if $\int_1^{k^2} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 6e$
- (c) Let $I_n = \int_1^e x(\ln x)^n dx$ where n is a positive integer.
- (i) Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$ for $n \geq 1$.
- (ii) Hence find the value of $\int_1^e x(\ln x)^3 dx$
- (d) Find $\int_0^{\frac{\pi}{3}} \frac{1}{2+\cos 2x} dx$ using the substitution $u = \tan x$.
2. (a) Let $z = -\sqrt{3} + i$
- (i) Write z in modulus-argument form.
- (ii) Hence find z^8 in the form $x + iy$ where x and y are real.
- (iii) Find the least positive integer value of n such that z^n is real.
- (b) Sketch each of the following regions on separate diagrams.
- (i) $-2 < \Re(z) < 1$ and $-1 < \Im(z) \leq 3$
- (ii) $|z - 2 - i| \leq 2$
- (iii) $0 < \arg[(1+i)z] \leq \frac{\pi}{2}$
- (c) $OABC$ is a square. O represents 0, A represents $3 + i$, B represents z , C represents w and D is the point where the diagonals of the square meet.
- (i) Find the complex numbers represented by C and D in the form $x + iy$.
- (ii) Find the $\arg\left(\frac{w}{z}\right)$
3. (a) (i) Find real numbers B and C such that $\frac{2x^2+5x}{(x-2)(x+3)} = 2 + \frac{B}{(x-2)} + \frac{C}{(x+3)}$
- (ii) Hence find $\int \frac{2x^2+5x}{(x-2)(x+3)} dx$
- (b) Consider the function given by $f(x) = \frac{2x^2+5x}{(x-2)(x+3)}$
- (i) Noting that $f(-4) = 2$ sketch the graph of $y = f(x)$ showing all asymptotes and intercepts. There is no need for finding stationary points and inflexion points.
- (ii) Hence sketch the graph of $g(x) = \frac{(x-2)(x+3)}{2x^2+5x}$ showing all asymptotes and intercepts.
- (iii) By making x the subject of the expression $y = \frac{2x^2+5x}{(x-2)(x+3)}$ show that the range of the function is given by $\left\{y : y \leq \frac{29-6\sqrt{6}}{25} \vee y \geq \frac{29+6\sqrt{6}}{25}\right\}$.
- (iv) Hence, or otherwise find the values of k such that the equation $\frac{2x^2+5x}{(x-2)(x+3)} = k$ has no real solutions for x .

4. (a) Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers. If $x = \frac{1}{r}$ is a root of the polynomial equation $P(x) = 0$ then show that r is a factor of a .

(b) Consider the polynomial $P(x) = 8x^4 - 8x^2 - x + 1$. You are given that $P(x) = (x - 1)Q(x)$ where $Q(x)$ is a cubic polynomial.

(i) Find $Q(x)$

(ii) If the equation $Q(x) = 0$ has ane rational root then find the it four solutions to the equations $P(x) = 0$

(c) (i) Solve the equation $\cos 4\theta = \cos \theta$ giving the general solution.

(ii) Show that $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$

(iii) Show that the equation $\cos 4\theta = \cos \theta$ can be expressed in the form

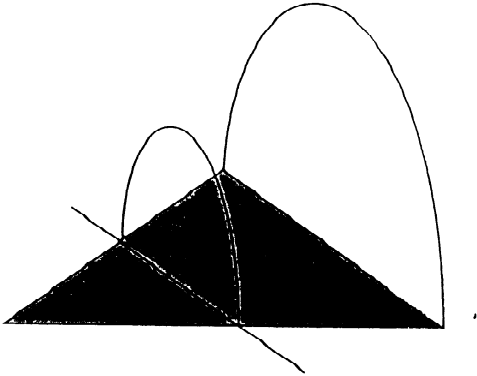
$$8x^4 - 8x^2 - x + 1 = 0 \text{ where } x = \cos \theta.$$

(iv) Hence using part (b) (ii) find the exact value of

(α) $\cos \frac{2\pi}{5}$ (β) $\cos \frac{\pi}{5}$

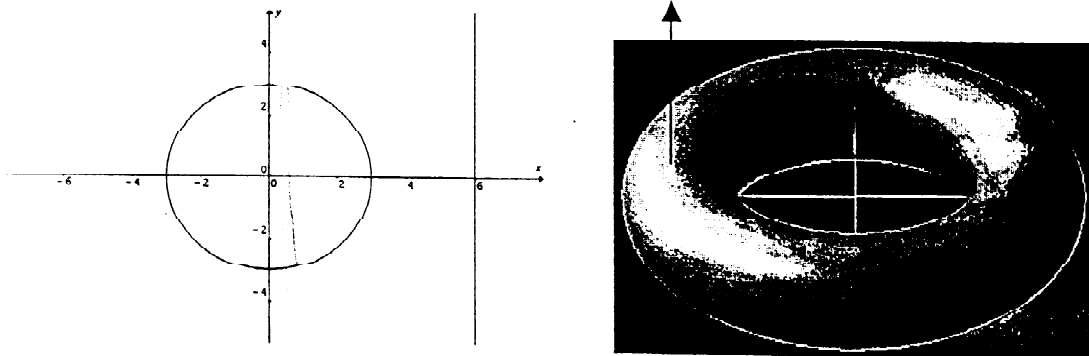
5. (a) Show that the area bounded by the parabola $x^2 = 4ay$ and the latus rectum $y = a$ is equal to $\frac{8a^2}{3}$.

(b) A particular solid has a triangular base with side lengths all 6 metres. Cross sections taken parallel to one side of the base are parabolas. Each parabolic cross-section is such that it has its latus rectum lying in the base of the solid. Using part (a) find the volume of the solid.



(c) (i) Show that the function $f(x) = x\sqrt{a^2 - x^2}$ is odd.

(ii) The circle with equation $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a torus.



Show, using the method of cylindrical shells that the volume V of the torus is given by $V = 4\pi \int_{-3}^3 (6-x)\sqrt{9-x^2} dx$

(iii) Hence find the volume of the torus.

6. (a) (i) Factorise the cubic polynomial $z^3 + 8$

(α) field of real numbers

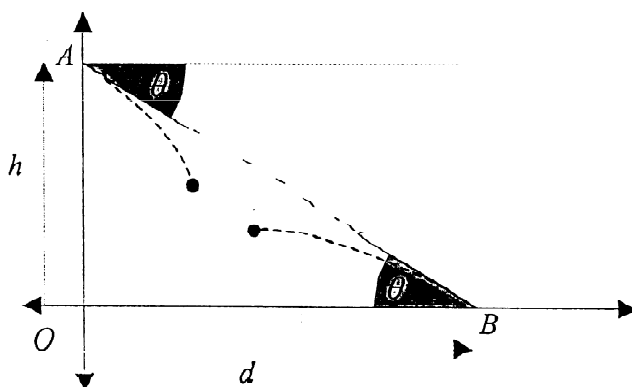
(β) over the field of complex numbers

Let W be one of the non-real complex roots of the equation $z^3 + 8 = 0$.

(ii) Show that $w^2 = 2w - 4$

(iii) Hence simplify $(2w - 4)^6$

(b) The diagram shows the point A at a height h vertically above the point O . It also shows the point B which is positioned at a horizontal distance d from O . A projectile is fired from A directly at point B with a velocity V . At the same instant a projectile is fired from point B directly at point A with the same velocity V .



Let θ be the angle between the horizontal at A and the angle of projection.

(i) Show carefully that the equations of motion of the two projectiles are given by

$$\begin{aligned}x_A &= Vt \cos \theta & x_B &= d - Vt \cos \theta \\ y_A &= h - Vt \sin \theta - \frac{gt^2}{2} & y_B &= Vt \sin \theta - \frac{gt^2}{2}\end{aligned}$$

(ii) Show that the two particles will always meet.

(iii) Show that the height H at which they meet is given by $H = \frac{h}{2} - \frac{g(h^2+d^2)}{gV^2}$

(iv) Find the range of values of V such that the two projectiles meet above the x -axis.

7. (a) The equation $x^3 + 2x^2 + bx - 16 = 0$ has roots α, β and γ such that $\alpha\beta = 4$.

(i) Show that $b = -20$.

(ii) Find the cubic equation with roots given by α^2, β^2 and γ^2 .

(iii) Hence find the value of $\alpha^3 + \beta^3 + \gamma^3$

(b) Consider the roots to the equation $z^n - 1 = 0$. These roots are plotted on an Argand diagram. The points represented by these roots are joined to form a regular n -sided polygon.

(i) Show that the area of this polygon is given by $A_n = \frac{n}{2} \sin \frac{2\pi}{n}$.

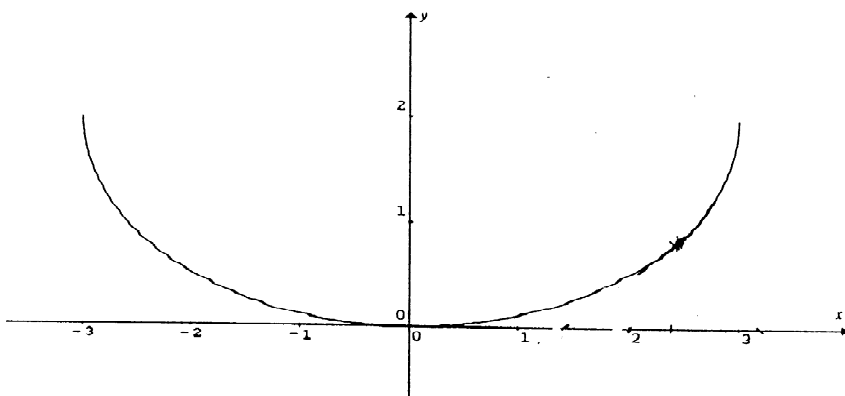
(ii) Show that the perimeter of the polygon is given by $P_n = 2n \sin \frac{\pi}{n}$.

(iii) Show that $P_n > 2A_n$ for all positive integers n .

(iv) Show that $\lim_{n \rightarrow \infty} A_n = \pi$ using the result that $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ and substituting an appropriate expression for n .

(v) Find the $\lim_{n \rightarrow \infty} P_n$.

8. (a) The semi-ellipse given by $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$ where $0 \leq y \leq 2$ is drawn below:



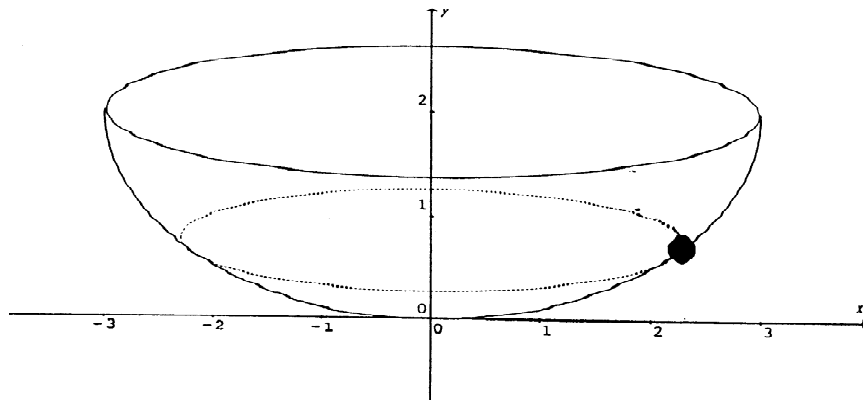
The point $P(r, h)$ lies on the ellipse where $r > 0$ and $0 < h < 2$. The tangent at P makes an angle α with the positive direction of the x -axis.

(i) Show that $\tan \alpha = \frac{4r}{9(2-h)}$

(ii) Hence, show that $\tan \alpha = \frac{2\sqrt{4-(2-h)^2}}{3(2-h)}$

(iii) Show that the acute angle between the normal at the point P and the vertical line $x = r$ is equal to the angle between the tangent at P and the positive direction of the x -axis.

(b) The semi-ellipse in part (a) is rotated about the y -axis to form a solid of revolution. A particle of unit mass slides smoothly in a horizontal circle on the inner surface of the solid so that it passes through the point P . The particle moves with a constant linear speed v .



(i) Draw a force diagram showing the forces acting on the particle at P .

(ii) By resolving these forces horizontally and vertically show that $v^2 = \frac{g(4-(2-h)^2)}{2-h}$.

(iii) If $h = 1$ then find the normal reaction of the surface of the solid on the particle.

(iv) If the normal reaction is equal to $\sqrt{2}$ times the weight of the particle then find the height h of the particle.
