Student Number: ....

Moriah College

2003
TRIAL EXAMINATION

MATHEMATICS
Extension 2
Examiner: J. Taylor

Time Allowed - 3 hours
(plus 5 minutes reading time)

Directions to Candidates

- Start each question in a new booklet
- Attempt ALL questions
- Show all necessary working
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

Note there is a detachable sheet with the diagrams for Question 8. This should be submitted with your solutions to that question.

Marks

Question 1

a) Find $\int \frac{dx}{1+\sin x+\cos x}$ **3**

b) Find $\int \frac{x}{\sqrt{x^4-1}}dx$ **3**

c) Find $\int \frac{3u^2+2}{u(u^2+1)}du$ **4**

d) Evaluate **5**

$$\int_{-2}^{2} \frac{x^2}{\sqrt{4-x^2}}dx$$

Question 2

a) Find $\int x\sec x\tan x dx$ **2**

b) The equation $x^3-x^2+x+3=0$ has roots α, β and γ.

i) Find the equation with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$. **2**

ii) Find α, β, γ given that one root is $x=1+i\sqrt{2}$. **2**

c) You are given the curve $x^2+xy+y^2=12$

i) By solving for x, show that $|y|\le 4$ **2**

ii) Find the coordinates of the highest and lowest points. **2**

iii) Show the points in (ii) are stationary points. **3**

iv) By carefully examining its symmetry, sketch the curve. **2**

Question 3

a) A cylindrical container of radius r and height L is partially filled with a liquid whose volume is V. (Figure 1) If the container is rotated about its axis of symmetry with constant angular speed ω, then the surface of the liquid will be convex, as indicated in Figure 2. The closer the water is to the edge, the further it will rise up the container, causing the water at the centre to be lower.

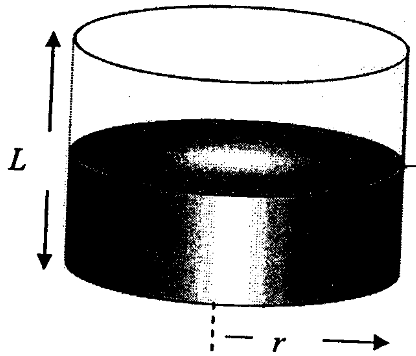


Figure 1

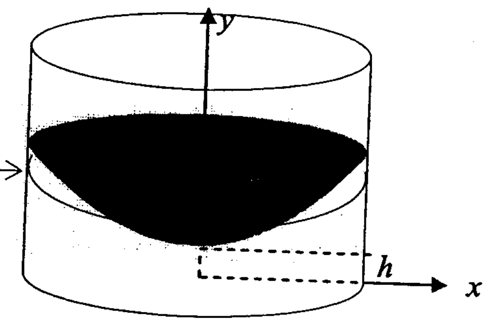


Figure 2

It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the y axis, where g is the acceleration due to gravity. Leave answers in terms of g.

(Note: you do not have to prove this, and the formulae for circular motion are not needed in this question).

i) By using the method of cylindrical shells, show that the volume of water under the parabola is **4**

$$V = \pi\left(r^2 h + \frac{\omega^2 r^4}{4g}\right)$$

ii) In terms of V and r, at what angular speed will the surface of the liquid touch the bottom? **2**

iii) The radius of a container is 2m, and the height of the container is 7m. The surface of the water, at the centre, is 5m below the top of the tank. At the edge of the container, the water is 4m below the top.

(1) Show that the volume of water is $10\pi \text{m}^3$. **2**

(2) Find the maximum angular speed of the container so that no water spills over the top. **3**

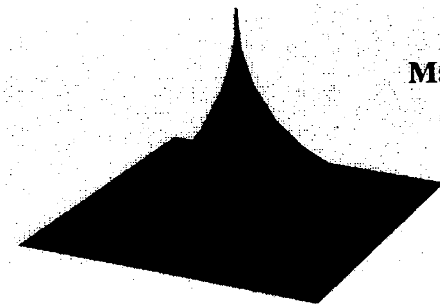
b) You are given the complex number $z = \dfrac{c + \sqrt{3}i}{c - \sqrt{3}i}$ where c is real. **4**

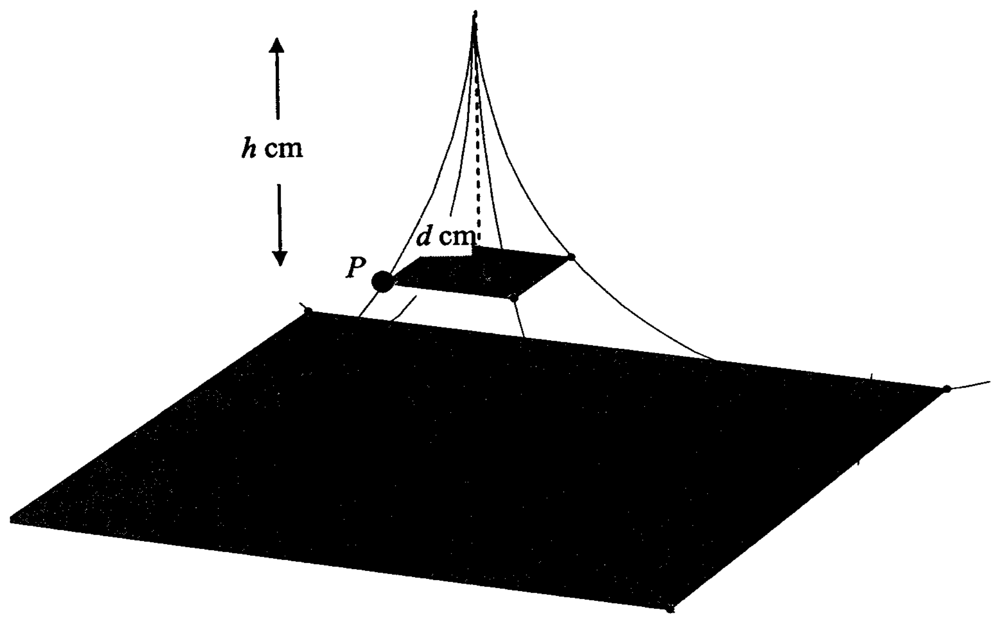
Find $|z|$ and hence describe the exact locus of z, if c varies from -1 to 1.

Marks

Question 4

a) A certain paperweight, which looks like a sharp Eiffel Tower, is drawn on the right and below. It is 10 cm high. The curved lines are all quarter circles, and P is a point on one of the curved lines. It has a square base.





A cross-section is drawn through P, parallel to the base. Let P be at a distance h cm below the peak.

i) Show that the distance d of a point P from the vertical axis is given by

$$d = 10 - \sqrt{100 - h^2}$$ **3**

ii) Hence find the volume of the paperweight. **8**

b) Sketch the graph of a function f such that **4**

$$f'(x) < 0 \text{ for all } x,$$

$$f''x) > 0 \text{ for } |x| > 1,$$

$$f''x) < 0 \text{ for } |x| < 1,$$

$$\text{and } \lim_{x \to \pm\infty} [f(x) + x] = 0$$

Question 5 **Marks**

a) If a, b are unequal positive numbers, prove that $a^a b^b > a^b b^a$ 2

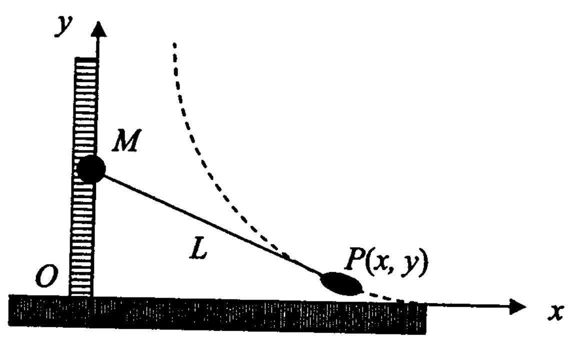
b) If x, y are positive and $x + y = 1$

i) prove that $xy \le \frac{1}{4}$ 2

ii) deduce that 3

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \ge \frac{25}{2}$$

c) A man M initially at the point O walks along a pier Oy pulling a row boat by a rope of length L. The boat is initially on the x axis L m from O. The man keeps the rope straight and taut. The path followed by the boat is such that the rope is always tangent to the curve (see the figure).



i) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then 2

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

ii) Prove that $\frac{d}{d\theta}\left(\log\left|\sec\theta + \tan\theta\right|\right) = \sec\theta$ 2

iii) Using the substitution $x = L\cos\theta$, or otherwise, determine the function $y = f(x)$. 4

Question 6 **Marks**

a) A particle of mass m is released so that it falls vertically, under the influence of gravity, in a medium whose resistance is mkv^2, where v is the velocity and k is a constant.

i) Show that if the velocity is v m/s after x metres, then $\frac{dx}{dv} = \frac{v}{g - kv^2}$ **2**

ii) By evaluating an integral, show that the particle has a limiting velocity given by $V = \sqrt{\frac{g}{k}}$. **4**

b) This same particle is projected from ground level vertically upwards in the same medium with a velocity U.

4

i) Show that the maximum height reached is

$$H = \frac{1}{2k} \log\left[\frac{kU^2 + g}{g}\right]$$

ii) Show that it returns to its starting point with a velocity **5**

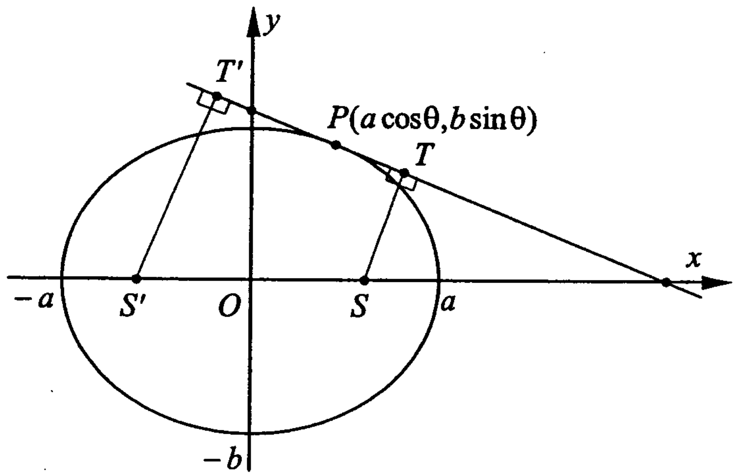
$$\frac{UV}{\sqrt{U^2 + V^2}}$$

where V is the terminal velocity.

Question 7

Marks

a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent.



i) Prove that the equation of the tangent at P is **3**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

ii) Prove that $ST=\dfrac{|e\cos\theta-1|}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$ where e is the eccentricity of the ellipse. **2**

iii) Show that $ST\times S'T'=b^2$. **4**

Question 7 continues over the page

b) A weight of mass m kg is placed at P, the middle of a string of length $2l$ metres. The string is fastened to two points A and B which are the same distance above a horizontal plane. A and B are so placed that the string makes an angle α with the horizontal (figure 1).

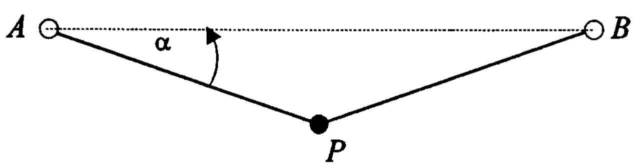


Figure 1

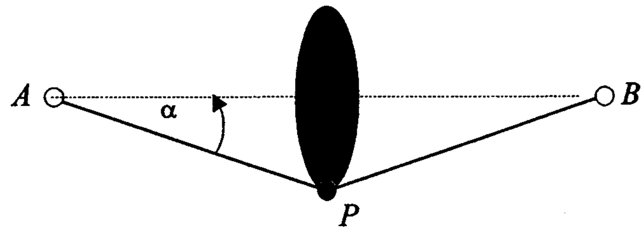
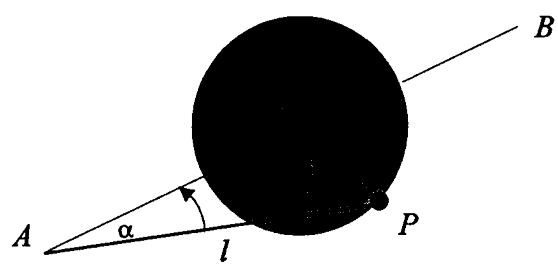


Figure 2

The mass is then twirled around the axis AB so that it traces out a circle with centre C, the midpoint of AB, and radius r. The mass moves with constant angular velocity $\omega = \frac{d\theta}{dt}$, where θ is the angle between the radius of the circle and the vertical (figure 2 and 3).



i) If T is the tension in the string, explain why $mr\omega^2 = 2T\sin\alpha - mg\cos\theta$ **2**

ii) By considering values of θ, deduce that the maximum tension in the string is **2**

$$\frac{m}{2}\left(l\omega^2 + \frac{g}{\sin\alpha}\right).$$

iii) Find the minimum angular velocity necessary so that the mass describes a complete circle. **2**

Note: The following diagrams are reproduced on the next page which can be removed and drawn upon. Insert this page into your Question 8 booklet.

Question 8 **Marks**

a) i) **4**

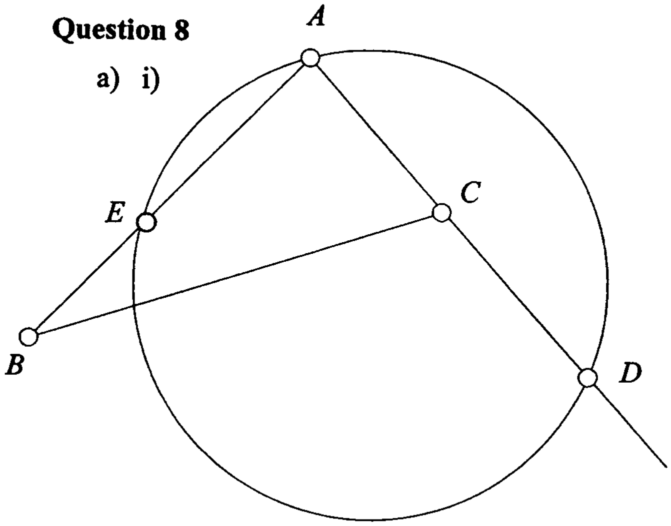


Figure 1

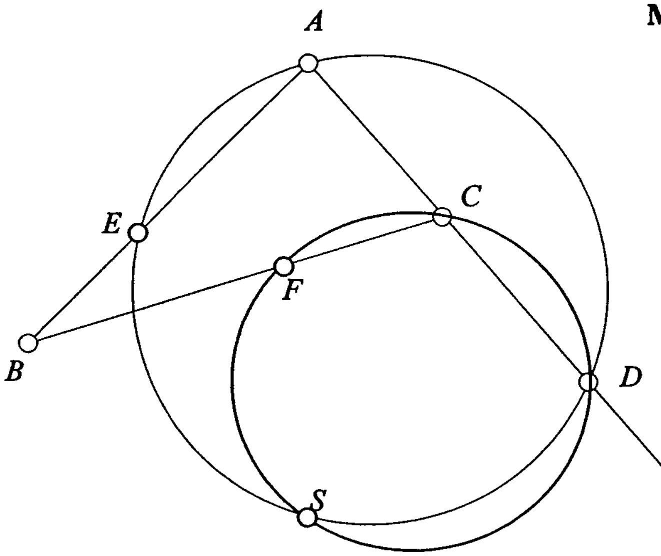
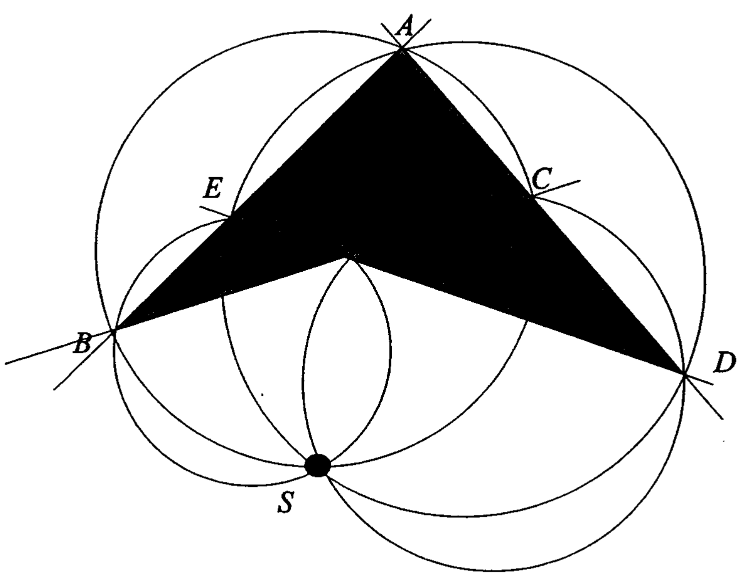


Figure 2

ABC is a triangle. *D* is a point on *AC* extended. A circle is drawn through *A* and *D* to cut *AB* at *E* (figure1). A second circle, with no particular radius, is now drawn to pass through *D* and *C* cutting *BC* at *F* and the first circle in *S* (figure 2).

Join *ES* and *FS* . Hence or otherwise prove that *BEFS* is a cyclic quadrilateral.

ii) Four general lines are drawn, intersecting to form four triangles. The circumcircles of the triangles are then drawn, as in the figure below. **2**

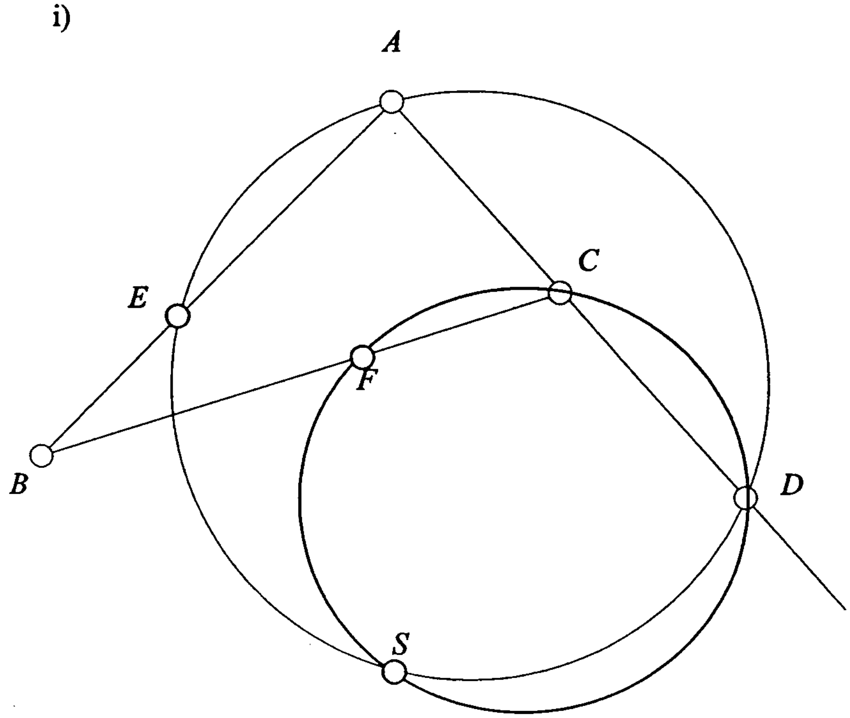


Prove these four circumcircles meet at a common point (*S* in the given diagram)

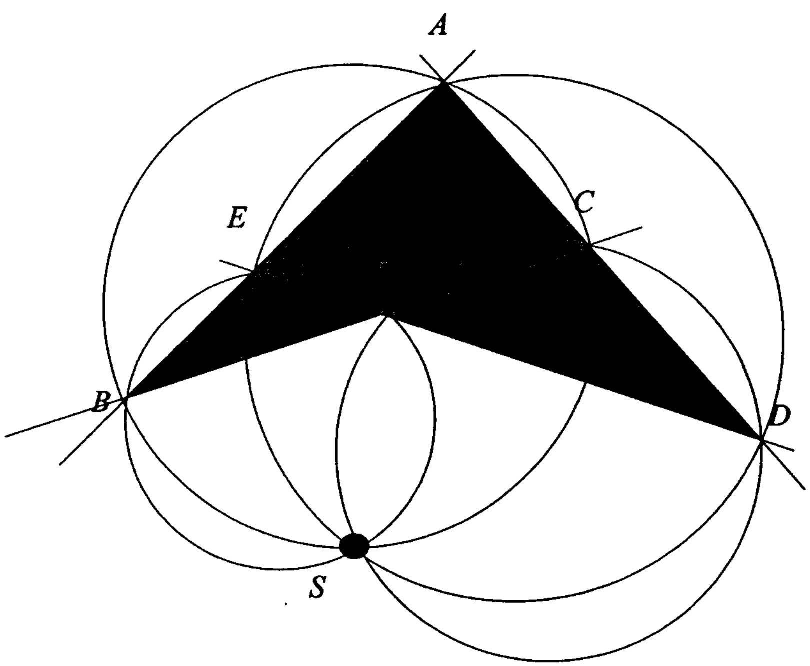
Question 8 continues over the page

Diagrams for Question 8 a) **Student Number ....**

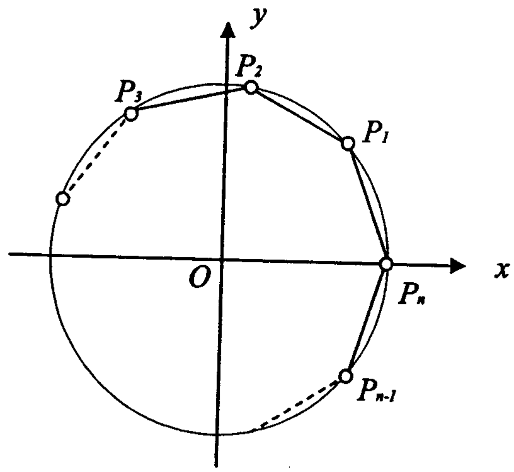
i)



ii)



b) i) Prove that the n nth roots of unity are integer powers of α, where $\arg(\alpha) = \frac{2\pi}{n}$. **Marks** 2



ii) $P_1, P_2, P_3, P_4, ..., P_n$ represent the complex numbers $z_1, z_2, z_3, z_4, ..., z_n$, and are the vertices of a regular polygon on a unit circle. Prove that 5

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + ... + (z_n - z_1)^2 = 0$$

iii) Deduce that 2

$$\sum_{r=1}^{n} z_r^{\ 2} = z_1 z_2 + z_2 z_3 + ... + z_{n-1} z_n + z_n z_1$$

Question 1 | **Marks**

a) 3

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}.\frac{2dt}{1+t^2} \qquad t=\tan\tfrac{x}{2}$$

$$= \int \frac{2}{1+t^2+2t+1-t^2}.dt$$

$$= \int \frac{1}{1+t}.dt$$

$$= \log|1+t|+C$$

$$= \log|1+\tan\tfrac{x}{2}|+C$$

b) 3

$$\int \frac{x}{\sqrt{x^4-1}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^4-1}}dx \qquad \text{Let } u = x^2 \therefore du = 2xdx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u^2-1}}du$$

$$= \frac{1}{2}\left(\log\left[u+\sqrt{u^2-1}\right]\right)+C$$

$$= \log\sqrt{x^2+\sqrt{x^4-1}}+C$$

4

c)

$$\text{Let } \frac{3u^2+2}{u(u^2+1)} = \frac{A}{u}+\frac{Bu+C}{u^2+1}$$

$$3u^2+2 = A(u^2+1)+Bu^2+Cu$$

Coeff. of $u \Rightarrow C=0$ 5

Constant: $\Rightarrow A=2$

Coeff. of $u^2 \Rightarrow B=1$

$$\int \frac{3u^2+2}{u(u^2+1)}du = \int \frac{2}{u}+\frac{u}{u^2+1}du$$

$$= 2\log|u|+\frac{1}{2}\log(u^2+1)+C$$

d)

$$\int_{-2}^{2} \frac{x^2}{\sqrt{4-x^2}}dx = -2\int_0^2 \frac{4-x^2}{\sqrt{4-x^2}}-\frac{4}{\sqrt{4-x^2}}dx$$

$$= -2\int_0^2 \sqrt{4-x^2}-\frac{4}{\sqrt{4-x^2}}dx$$

$$= -2\times\frac{\pi}{4}\times 2^2+8\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= -2\pi+8\left(\frac{\pi}{2}-0\right)$$

$$= 2\pi$$

Question 2

a) $\int x\sec x\tan x\,dx = [x\sec x] - \int 1\times\sec x\,dx$ 2

$= x\sec x - \log|\sec x + \tan x| + C$

b) i) If u is the solution of the required equation, then $u = \frac{1}{x}$. Replacing $x = \frac{1}{u}$ in the original equation gives 2

$$\frac{1}{u^3} - \frac{1}{u^2} + \frac{1}{u} + 3 = 0 \text{ or } 3u^3 + u^2 - u + 1 = 0$$ 2

ii) Another root is $x = 1 - i\sqrt{2}$. Since the sum of the roots is -1, the third root is -1. 2 2

d) I) 3

$$x^2 + xy + y^2 - 12 = 0$$

$$x = \frac{-y \pm \sqrt{y^2 - 4(y^2 - 12)}}{2}$$

$$= \frac{-y \pm \sqrt{48 - 3y^2}}{2}$$

For x to be real,

$$48 - 3y^2 \ge 0$$
$$y^2 - 16 \le 0$$
$$y^2 \le 16$$
$$|y| \le 4$$

ii) When y = 4, $x^2 + 4x + 4 = 0$ and so x = -2.

When y = -4, $x^2 - 4x + 4 = 0$ and so x = 2.

Hence the max point is (-2, 4) and the min point is (2, -4).

iii)

$$2x + (y + xy') + 2yy' = 0$$
$$y'(x + 2y) = -(2x + y)$$
$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

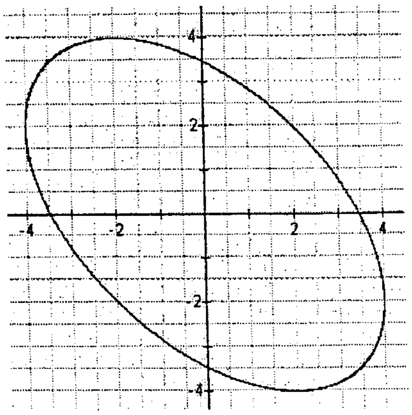
For (-2, 4), $2x + y = -4 + 4 = 0$ and for (2, -4), $2x + y = 4 - 4 = 0$.

Hence $\frac{dy}{dx} = 0$ for both points and so both points are stationary points. 2

iv) The curve remains unchanged if x and y are interchanged. Hence it is symmetric in $y = x$.

Hence $|x| \le 4$ *Also* (-4, 2) and (4, -2) are points on the curve

(Note also: The curve remains unchanged if x is replaced by $-y$, and y is replaced by $-x$. Hence it is symmetric in $y = -x$.)



Question 3a) i)

$\delta V = 2\pi r h \delta x$

$$V = 2\pi\int_0^r x\left(h+\frac{\omega^2x^2}{2g}\right)dx$$

$$= 2\pi\left[\frac{hx^2}{2}+\frac{\omega^2x^4}{8g}\right]_0^r$$

$$= 2\pi\left[\frac{hr^2}{2}+\frac{\omega^2r^4}{8g}\right]$$

$$= \pi\left(r^2h+\frac{\omega^2r^4}{4g}\right)$$

b) ii) when $h = 0$ 4

$$V = \pi\left(0+\frac{\omega^2r^4}{4g}\right)$$

$$\omega^2 = \frac{4Vg}{\pi r^4}$$

$$\omega = \frac{2}{r^2}\sqrt{\frac{Vg}{\pi}}$$

iii) (1)

We have $h = 2$, and the point (2,3) lies on the parabola. So:

$y = 2+\dfrac{\omega^2x^2}{2g}$ gives $3 = 2+\dfrac{\omega^2 2^2}{2g}$ or $\omega^2 = \dfrac{g}{2}$.

Substitute this in the formula for the volume:

$$V = \pi\left(r^2h+\frac{\omega^2r^4}{4g}\right)$$

$$= \pi\left(2^2\times 2+\frac{g}{2}\times\frac{2^4}{4g}\right)$$

$$= 10\pi$$

iii) (2)The maximum y value must be 7 at $x = 2$, and we now know the volume. We'll find h from the y equation: 3

$$7 = h+\frac{\omega^2 2^2}{2g}$$

$$h = 7-\frac{2\omega^2}{g}$$

Substitute this into the V equation:

$$10\pi = \pi\left(2^2\times\left(7-\frac{2\omega^2}{g}\right)+\frac{\omega^2 2^4}{4g}\right)$$

$$10 = 28-\frac{8\omega^2}{g}+\frac{4\omega^2}{g}$$

$$\frac{4\omega^2}{g} = 18$$

$$\omega^2 = \frac{18g}{4}$$

$$\omega = \frac{3\sqrt{2g}}{2}$$

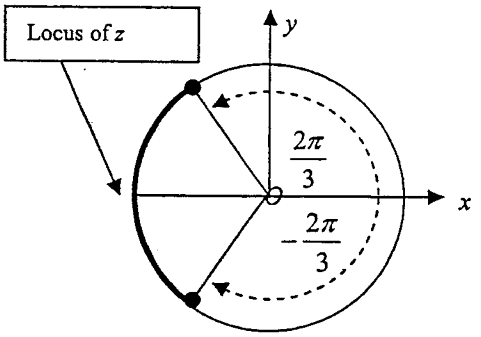
b)

$$|z| = \frac{|c+\sqrt{3}i|}{|c-\sqrt{3}i|}$$

$$= \frac{\sqrt{c^2+(\sqrt{3})^2}}{\sqrt{c^2+(-\sqrt{3})^2}}$$

$$= \frac{\sqrt{c^2+3}}{\sqrt{c^2+3}}$$

$$= 1$$



1

When $c = -1$, $z = \dfrac{-1+\sqrt{3}i}{-1-\sqrt{3}i}$ and

$$\arg z = \arg\left(\frac{-1+\sqrt{3}i}{-1-\sqrt{3}i}\right) = \arg\left(-1+\sqrt{3}i\right)-\arg\left(-1-\sqrt{3}i\right)$$

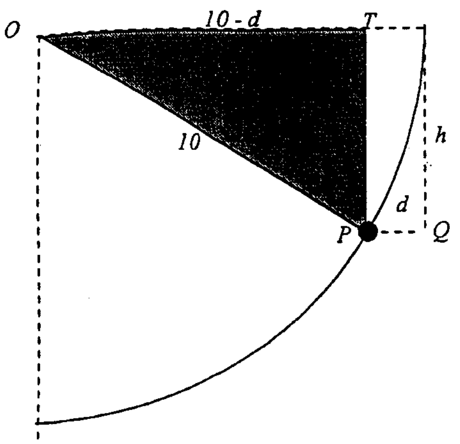
$$= \frac{2\pi}{3}--\frac{2\pi}{3} = \frac{4\pi}{3}$$

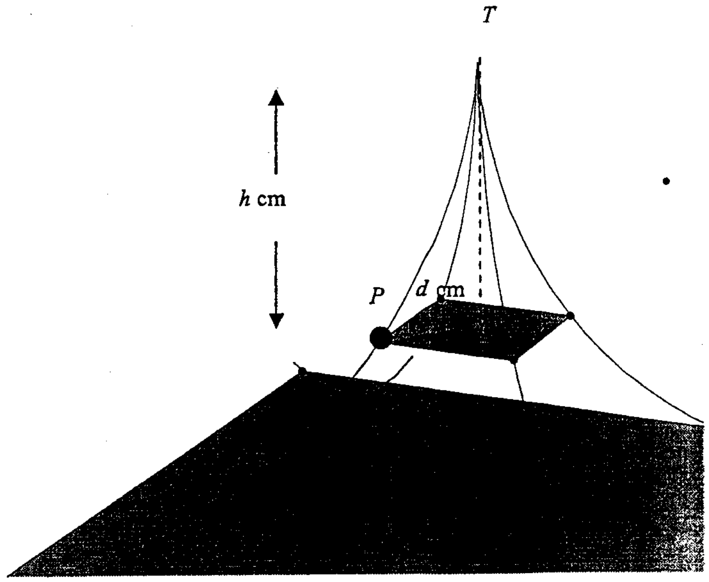
When $c = 1$, $z = \dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}$ and

$$\arg z = \arg\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right) = \arg\left(1+\sqrt{3}i\right)-\arg\left(1-\sqrt{3}i\right)$$

$$= \frac{\pi}{3}--\frac{\pi}{3} = \frac{2\pi}{3}$$

Question 4





Since the arcs are quarter circles, OP in the above diagram is 10 cm and $OT = 10 - d$. From Pythagoras' Theorem
$OP^2 = OT^2 + PT^2$

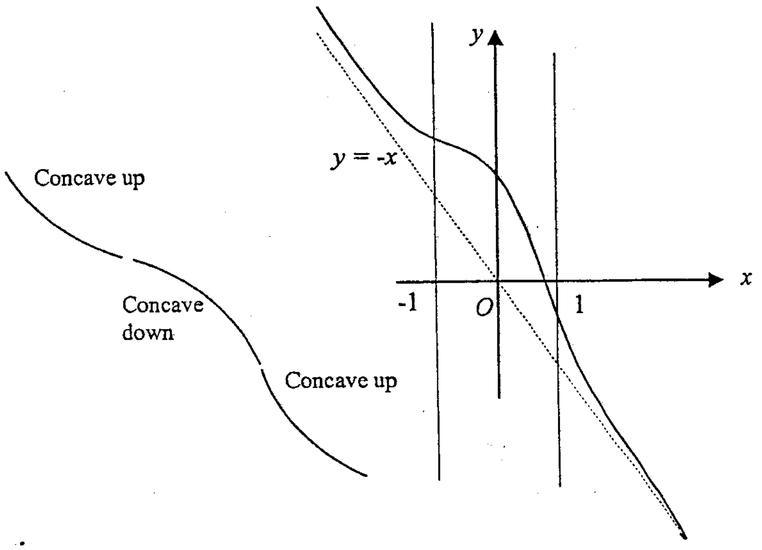
$$10^2 = h^2 + (10-d)^2$$
$$100 - h^2 = (10-d)^2$$
$$\sqrt{100-h^2} = 10 - d \qquad (10 - d > 0)$$
$$d = 10 - \sqrt{100-h^2}$$

3

$$A = \frac{1}{2}(2d)^2$$
$$= 2\left(10 - \sqrt{100-h^2}\right)^2$$
$$= 2\left(100 - 20\sqrt{100-h^2} + 100 - h^2\right)$$
$$= 2\left(200 - h^2 - 20\sqrt{100-h^2}\right)$$

8 4

$$\delta V = A\delta h$$
$$V = \int_0^{10} A\,dh$$
$$= 2\int_0^{10} 200 - h^2 - 20\sqrt{100-h^2}\,dh$$
$$= 2\left[200h - \frac{h^3}{3}\right]_0^{10} - 40\int_0^{10}\sqrt{100-h^2}\,dh$$
$$= 2\left[2000 - \frac{1000}{3}\right] - 40 \times \frac{1}{4}\pi \times 10^2$$
$$= \frac{10000}{3} - 1000\pi$$
$$= 1000\left[\frac{10-3\pi}{3}\right]$$



Hence the volume of the paperweight is $1000\left[\frac{10-3\pi}{3}\right]cm^3$

Question 5

a) This is equivalent to proving $\left(\frac{a}{b}\right)^a > \left(\frac{a^b}{b}\right)$ 2

i) If $a>b$, $\frac{a}{b}>1$ and a higher power will give a higher number. Hence the result is true.

ii) If $a<b$, $0<\frac{a}{b}<1$ and a higher power will give a lower number. Hence the result is true.

Hence the result is always true, if $a \neq b$.

b) i) A standard result is $a^2+b^2 \geq 2ab$. Replace $a^2 \to x$, $b^2 \to y$ and we get $x+y \geq 2\sqrt{xy}$

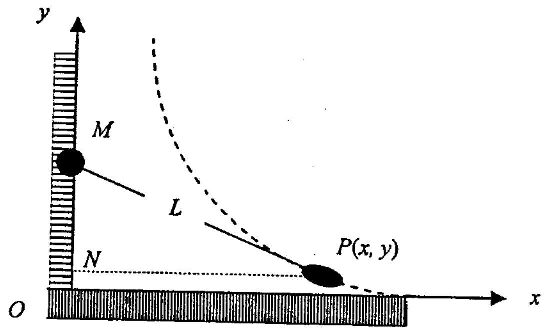
For our problem $x+y=1$ and so $1 \geq 2\sqrt{xy}$ or $xy \leq \frac{1}{4}$

ii): Replace $\frac{1}{x} \geq 4y$ and $\frac{1}{y} \geq 4x$: 2 3

$$
\begin{aligned}
LHS &= \left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \\
&\geq (x+4y)^2+(y+4x)^2 \\
&= x^2+8xy+16y^2+y^2+8xy+16x \\
&= 17(x+y)^2-18xy \\
&\geq 17-18\times\frac{1}{4} \\
&= \frac{25^2}{2}
\end{aligned}
$$

From the diagram, $PN = x$ and so $MN=\sqrt{L^2-x^2}$ by Pythagoras' Theorem. The gradient of the tangent is therefore

$-\frac{MN}{PN}=\frac{-\sqrt{L^2-x^2}}{x}$ and so $f'(x)=\frac{dy}{dx}=\frac{-\sqrt{L^2-x^2}}{x}$ 2



ii)

$$
\begin{aligned}
\frac{d}{d\theta}\left(\log|\sec\theta+\tan\theta|\right) &= \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta} \\
&= \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} \\
&= \sec\theta
\end{aligned}
$$

2

iii)Let $x = L\cos\theta$ which gives $dx = -L\sin\theta d\theta$

$$\int \frac{-\sqrt{L^2-x^2}}{x}dx = -\int \frac{\sqrt{L^2-L^2\cos^2\theta}}{L\cos\theta}(-L\sin\theta)d\theta$$

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$$= L\int \frac{\sin^2\theta}{\cos\theta}d\theta$$

$$= L\int \frac{1-\cos^2\theta}{\cos\theta}d\theta$$

$$= L\int \frac{1}{\cos\theta} - \cos\theta d\theta$$

$$= L\int \sec\theta - \cos\theta d\theta$$

$$= L[\log|\sec\theta + \tan\theta| - \sin\theta] + C$$

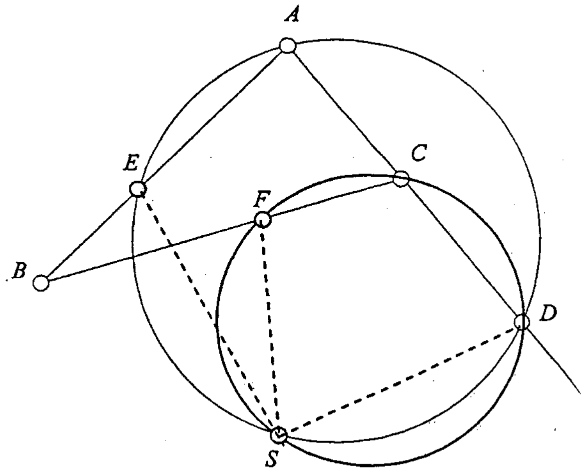
$$= L\left[\log\left|\frac{L}{x} + \frac{\sqrt{L^2-x^2}}{x}\right| - \frac{\sqrt{L^2-x^2}}{L}\right] + C$$

When $x = L$, f(x) = 0 so $0 = L\left[\log\left|\frac{L}{L} + \frac{\sqrt{L^2-L^2}}{L}\right| - \frac{\sqrt{L^2-L^2}}{L}\right] + C$ which gives $C = 0$. Hence

$$f(x) = L\left[\log\left|\frac{L}{x} + \frac{\sqrt{L^2-x^2}}{x}\right| - \frac{\sqrt{L^2-x^2}}{L}\right]$$

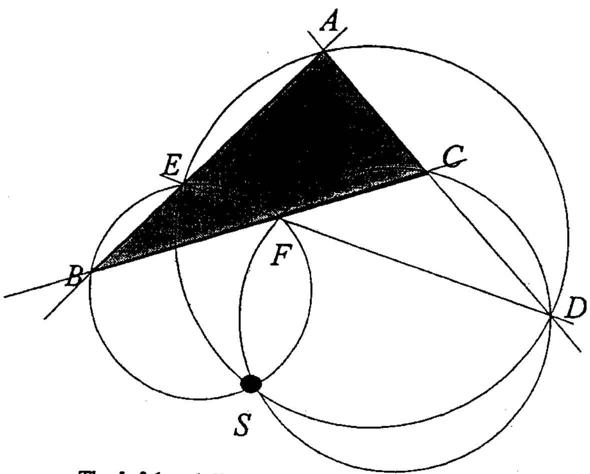
Question 7

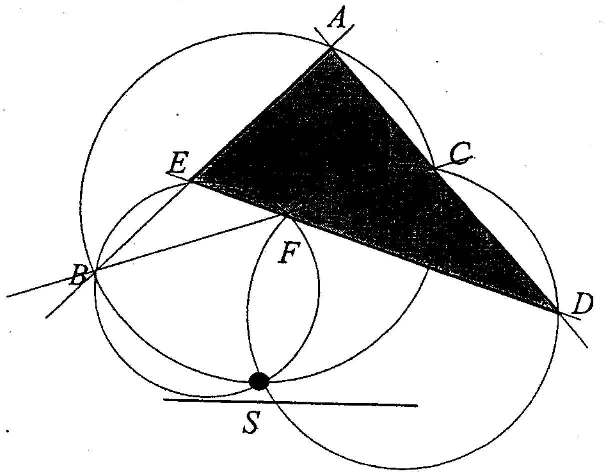
a) i) Join SD also. Then $CFSD$ and $AESD$ are cyclic quadrilaterals.



$\angle BES = \angle ADS$ (ext. $\angle$ of cyclic quad. $ADSE$ equals opposite interior $\angle$)
$\angle BFS = \angle CDS$ (ext. $\angle$ of cyclic quad. $CDSF$ equals opposite interior $\angle$)
But $\angle ADS$ and $\angle CDS$ are the same angle.
$\therefore \angle BES = \angle BFS$
$\therefore BEFS$ is a cyclic quadtrilateral (converse of angles in the same arc of a circle are equal)

ii)

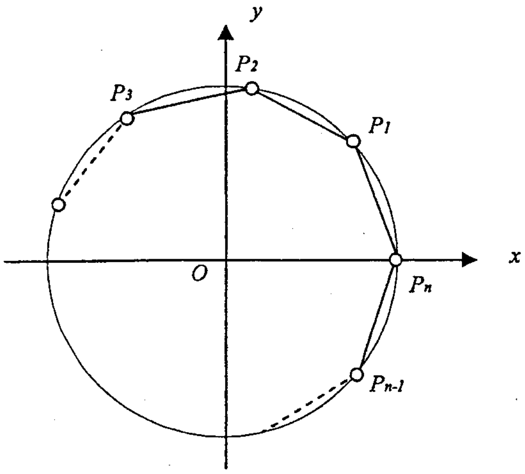




The left hand diagram is identical to the result proved in (i), which showed the three circles had a common point S. The right hand diagram is the same result with the triangle reflected. It is a different triangle, but our result from (i) shows that the circles $BEFS$ and $BACS$ form cyclic points $SDCF$. But this is the same circle as in the first diagram, so all four circles have the common point S.

Marks

b) I)



For $z^n = 1$, $z^n = cis(2k\pi)$ where k=0, 1, 2, ... 2

Hence

$$z = cis\left(\frac{2k\pi}{n}\right) \qquad where\ k=0,1,2...$$

$$= \left[cis\left(\frac{2\pi}{n}\right)\right]^k \qquad by\ De Moivre's\ Theorem$$

$$= \alpha^k$$

where $\arg(\alpha) = \frac{2\pi}{n}$, and so the result is proved.

4

ii)

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + ... + (z_n - z_1)^2 = (\alpha - \alpha^2)^2 + (\alpha^2 - \alpha^3)^2 + ... + (\alpha^{n-1} - \alpha^n)^2 + (\alpha^n - \alpha)^2$$

$$= \alpha^2(1-\alpha)^2 + \alpha^4(1-\alpha)^2 + ... + \alpha^{2n-2}(1-\alpha)^2 + (1-\alpha)^2$$

$$= \underbrace{\left[\alpha^2 + \alpha^4 + ... + \alpha^{2n-2} + 1\right]}_{GP\ a=1\ r=\alpha^2\ N=n}(1-\alpha)^2$$

$$= \frac{1-\alpha^{2n}}{1-\alpha^2}.(1-\alpha)^2$$

$$= 0 \qquad since\ \alpha^n = 1$$

iii) If we square out the LHS of the above result, we get a doubling of squares from adjacent terms as well as the normal doubling of products. Note how the last term gives the doubling for the z_1^2:

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + ... + (z_n - z_1)^2 = \left[z_1^2 - 2z_1z_2 + z_2^2\right] + \left[z_2^2 - 2z_2z_3 + z_3^2\right] + ... + \left[z_n^2 - 2z_nz_1 + z_1^2\right]$$

$$= 2\left[z_1^2 + z_2^2 + ... + z_n^2\right] - 2\left[z_1z_2 + z_2z_3 + ...z_nz_1\right]$$

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But we have proved the LHS = 0. So

$$2\left[z_1^2 + z_2^2 + ... + z_n^2\right] - 2\left[z_1z_2 + z_2z_3 + ...z_nz_1\right] = 0$$

$$z_1^2 + z_2^2 + ... + z_n^2 = z_1z_2 + z_2z_3 + ...z_nz$$

Or

$$\sum_{r=1}^{n} z_r^2 = z_1z_2 + z_2z_3 + ... + z_{n-1}z_n + z_nz_1$$