NEWCASTLE GRAMMAR SCHOOL

SPECTEMUR AGENDO

YEAR 12
2004
EXTENSION 2 MATHEMATICS
TRIAL EXAMINATION

Time allowed – Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet **Marks**

a) Evaluate $\int_0^1 te^{-t}\,dt$ **3**

b) i) Find the real numbers a, b and c such that **2**

$$\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$$

ii) Hence find $\int \frac{dx}{x(1+x^2)}$ **2**

c) Evaluate $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$ **3**

d) i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ show that, for $n > 1$ **3**

$$I_n = \left(\tfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

ii) Hence find the area of the region bounded by the curve $y = x^4 \cos x$ and the x-axis for $0 \le x \le \frac{\pi}{2}$ **2**

QUESTION 2 Use a SEPARATE Writing Booklet **Marks**

a) The complex number z moves such that $\operatorname{Im}\left(\frac{1}{z-i}\right)=1$. **3**

Show that the locus of z is a circle and find its centre and radius.

b) i) Find the square roots of the complex number $5-12i$ **2**

ii) Given that $z=\dfrac{1+\sqrt{5-12i}}{2+2i}$ and is purely imaginary, **2**

find z^{400}

c) i) Shade the region on the Argand diagram containing all of the points representing the complex numbers z such that **3**

$$|z-1-i|\le 1 \quad \text{and} \quad \frac{-\pi}{4}\le \arg(z-i)\le \frac{\pi}{4}$$

ii) Let w be the complex number of <u>minimum modulus</u> satisfying the inequalities of part i) above. **1**

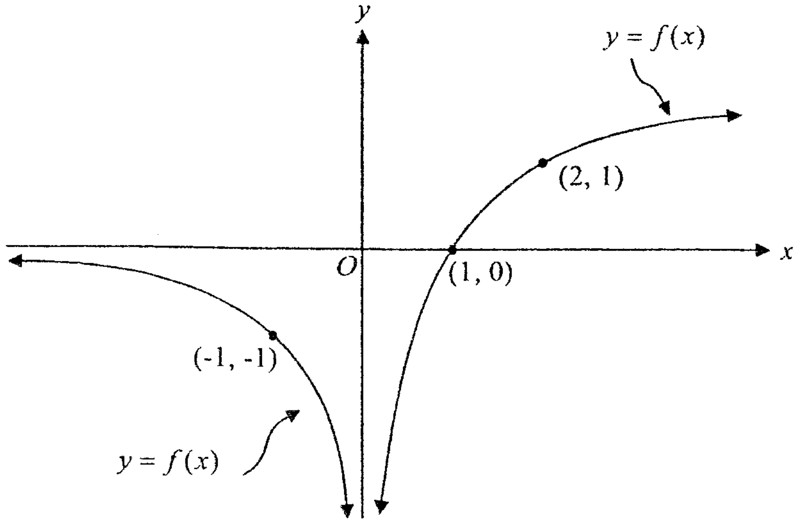
Express w in the form $x+iy$.

d) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus/argument form and hence **4**

evaluate $\cos\frac{7\pi}{12}$ in surd form.

QUESTION 3 Use a SEPARATE Writing Booklet **Marks**

a) The diagram below shows the graph of the discontinuous function $y = f(x)$



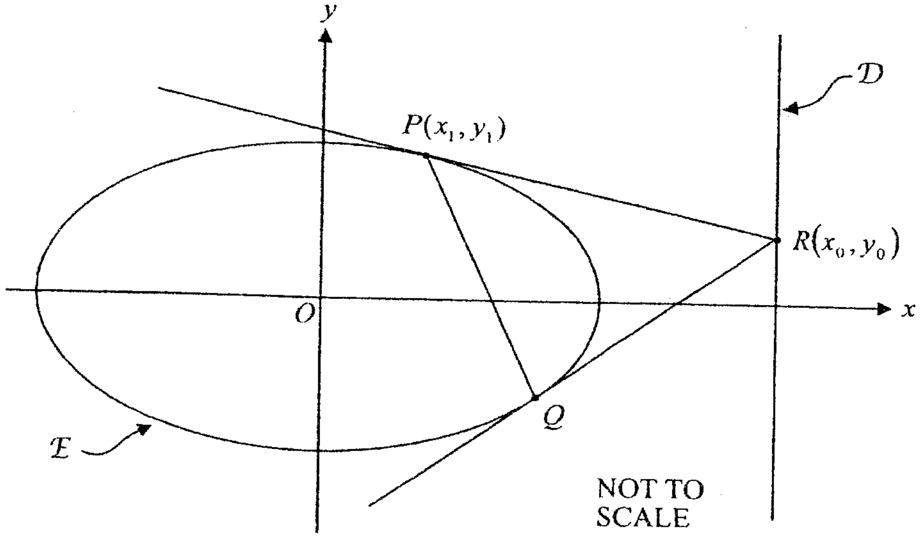
Draw large (half page), separate sketches of the following

i) $y = -\sqrt{f(x)}$ **3**

ii) $y = \left|f(|x|)\right|$ **3**

iii) $y = \dfrac{1}{f(x)}$ **3**

b)



The ellipse $\mathcal{E}$ with equation $\frac{x^2}{25}+\frac{y^2}{16}=1$ has a directrix $\mathcal{D}$ as shown in the diagram. Point $R(x_0, y_0)$ lies on $\mathcal{D}$. PQ is the chord of contact from R where P is the point (x_1, y_1).

i) Write down the equation of $\mathcal{D}$ **1**

ii) Show that the equation of the tangent at P is **3**

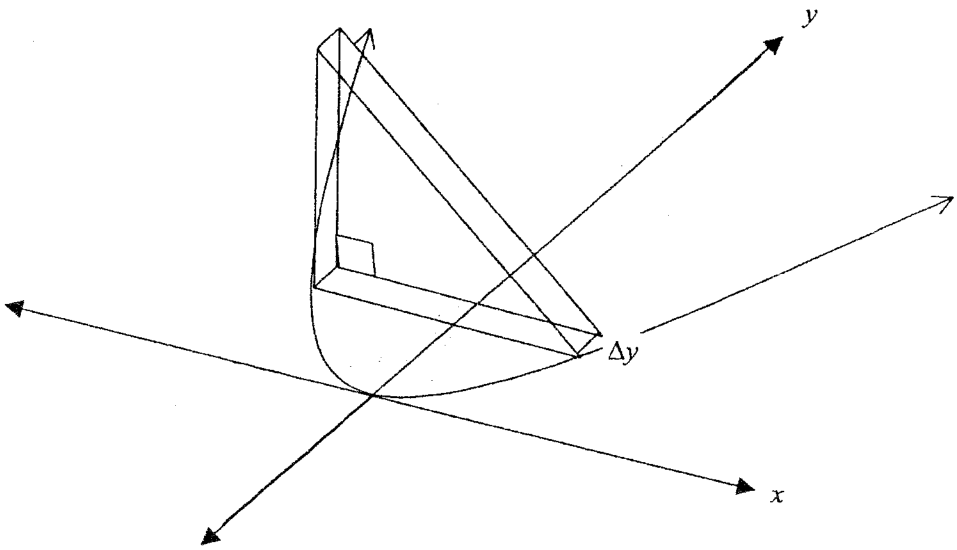
$$\frac{x_1 x}{25}+\frac{y_1 y}{16}=1$$

iii) The equation of PQ is $\frac{x_0 x}{25}+\frac{y_0 y}{16}=1$ **2**

Show that the focus of $\mathcal{E}$ lies on PQ

QUESTION 4 Use a SEPARATE Writing Booklet **Marks**

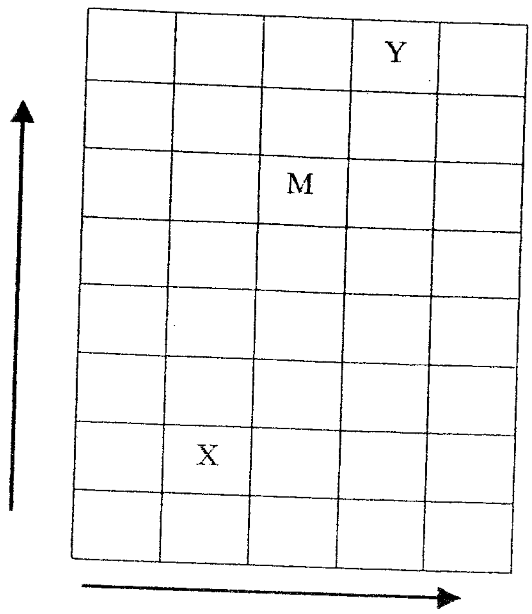
a) A solid is formed as shown below. Its base is in the xy-plane and is in the shape of the parabola $y = x^2$. The vertical cross-section is in the shape of a right angled isosceles triangle. **4**



By using the method of slicing, calculate the volume of the solid between the values $y = 0$ and $y = 4$.

b) Find, using the method of cylindrical shells, the volume of the solid generated by rotating the region bounded by the curve $y = (x-2)^2$ and the line $y = x$ about the x-axis. **6**

c) On a special chess board, the squares are arranged in 8 rows and 5 columns as shown



A player can only move forwards or across in the directions shown by the arrows, one square at a time.

i) If a player is situated at X, in how many ways can the player reach the square labelled Y ? 3

ii) In how many ways can a player move from X to Y if they <u>must</u> pass through M ? 2

QUESTION 5 Use a SEPARATE Writing Booklet **Marks**

a) The cubic equation $x^3 - x^2 + 4x - 2 = 0$ has roots α, β and γ

i) Find the equation with the roots α^2, β^2 and γ^2 **3**

ii) Find the value of $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ **3**

b) If $P(x) = 4x^3 + 4x^2 + x + k$ for some real number k, find the values of x for which $P'(x) = 0$. Hence find the values of k for which the equation $P(x) = 0$ has more than one real root. **4**

c) If $P(x) = 3x^4 - 11x^3 + 14x^2 - 11x + 3$ show that **5**

$$P(x) = x^2\left\{3\left(x + \frac{1}{x}\right)^2 - 11\left(x + \frac{1}{x}\right) + 8\right\}$$

and hence solve $P(x) = 0$ over C (complex numbers)
and factorise $P(x)$ over R (real numbers)

QUESTION 6 Use a SEPARATE Writing Booklet **Marks**

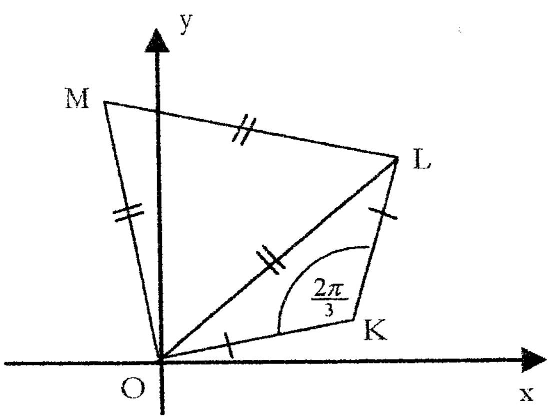
a) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is **4**

$$a\sin\theta\, x + by = (a^2+b^2)\tan\theta$$

ii) The normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the x-axis at G. PN is the perpendicular from P to the x-axis. Prove that $OG = e^2 \times ON$, where O is the origin. **5**

b) The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle\text{OKL} = \frac{2\pi}{3}$. The triangle OLM is equilateral. **6**

Show that $3\alpha^2 + \beta^2 = 0$



QUESTION 7 Use a SEPARATE Writing Booklet **Marks**

a) Prove by induction that, for $n \geq 1$ **5**

$$\cos\frac{90^\circ}{2^n} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\ldots+\sqrt{2}}}}}}_{n-terms}$$

b) i) Prove that: **3**

$$\tan^{-1}(n+1) - \tan^{-1}(n) = \cot^{-1}\left(1+n+n^2\right)$$

ii) Hence, sum the series **3**

$$\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \ldots + \cot^{-1}\left(1+n+n^2\right)$$

c) Using a graph, find the values of x for which $f(x) > (f(x))^3$ where $f(x) = \frac{1}{2} + \sin x$ and $0 \leq x \leq 2\pi$ **4**

QUESTION 8 Use a SEPARATE Writing Booklet **Marks**

a) The tangent at $P(cp, \frac{c}{p})$ to the hyperbola $xy = c^2$ meets the lines $y = \pm x$ at A and B respectively. The normal at P meets the axes at C and D. If M represents the area of ΔOAB and N represents the area of ΔOCD show that M^2N is a constant. **6**

b) i) Determine whether $f(x) = \dfrac{1-|x|}{|x|}$ is even, odd or neither. Justify your answer. **1**

ii) Sketch $y = f(x)$ **3**

iii) Hence, or otherwise, solve $f(x) \geq 1$ **3**

iv) Sketch $y = e^{f(x)}$ **2**

a) $\int_0^1 t e^{-t}\,dt$ let $u = t$ $v' = e^{-t}$ $\therefore u' = 1$ $\therefore v = -e^{-t}$

$= [-t\, e^{-t}]_0^1 + \int_0^1 e^{-t}\,dt$

$= (-e^{-1} - 0) + -[e^{-t}]_0^1$

$= -e^{-1} - (e^{-1} - 1)$

$= 1 - 2e^{-1}$ or $1 - 2/e$ (3)

$\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$ $\times$ B.S. $x(1+x^2)$

$1 = a(1+x^2) + (bx+c)x$

$= ax^2 + a + bx^2 + cx$

$0x^2 + 0x + 1 = (a+b)x^2 + cx + a$

Equating coefficients: (2)

$\left.\begin{array}{l} a+b=0 \\ c=0 \\ a=1 \end{array}\right\} \Rightarrow a=1,\ b=-1,\ c=0$

$\int \frac{dx}{x(1+x^2)} = \int \frac{1}{x} - \frac{x}{1+x^2}\,dx$ from (i)

$= \int \frac{1}{x} - \frac{1}{2}\cdot\frac{2x}{1+x^2}\,dx$

$= \ln|x| - \frac{1}{2}\ln|1+x^2| + c$ (2)

$\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$ let $u = x+4$ $\therefore du = dx$

$x = 0 \Rightarrow u = 4$

$x = 4 \Rightarrow u = 8$

$= \int \frac{u-4}{\sqrt{u}}\,du$

$= \int \frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\,du$

$= [\frac{2}{3}u^{3/2} - 8u^{\frac{1}{2}}]_4^8$

$= (\frac{2}{3}(2\sqrt{2})^3 - 8(2\sqrt{2})) - (\frac{2}{3}(2)^3 - 8(2))$

$= \frac{32\sqrt{2}}{3} - 16\sqrt{2} - \frac{16}{3} + 16$

$= \frac{32 - 16\sqrt{2}}{3}$ or $\frac{16}{3}(2 - \sqrt{2})$ (3)

(d)

(i) For: $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$

let $u = x^n$ $v' = \cos x$

$\therefore u' = nx^{n-1}$ $\therefore v = \sin x$

$\therefore I_n = [x^n \sin x]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$

let $u = x^{n-1}$ $v' = \sin x$

$\therefore u' = (n-1)x^{n-2}$ $\therefore v = -\cos x$

$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n\left\{[-x^{n-1}\cos x]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx\right\}$

$= \left(\frac{\pi}{2}\right)^n - n[0] - n(n-1)I_{n-2}$

ie. $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ (QED) (3)

(ii) Area $= \int_0^{\frac{\pi}{2}} x^4 \cos x\,dx = I_4$

(above x-axis for $0 \le x \le \frac{\pi}{2}$ as x^4 and $\cos x$ both ≥ 0 for $0 \le x \le \frac{\pi}{2}$)

$I_4 = \left(\frac{\pi}{2}\right)^4 - 4(3)I_2$

$I_2 = \left(\frac{\pi}{2}\right)^2 - 2(1)I_0$

and $I_0 = \int_0^{\frac{\pi}{2}} x^0 \cos x\,dx$

$= \int_0^{\frac{\pi}{2}} dx$

$= [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$\therefore I_4 = \left(\frac{\pi}{2}\right)^4 - 12\left\{\left(\frac{\pi}{2}\right)^2 - 2\left(\frac{\pi}{2}\right)\right\}$

$= \frac{\pi^4}{16} - 3\pi^2 + 12\pi$ units^2 (3)

) a) Let $z = x + iy$

$\therefore \bar{z} = x - iy$

$\therefore \bar{z} - i = x - i(y+1)$

$$\therefore \frac{1}{\bar{z}-i} = \frac{1}{x - i(y+1)} \times \frac{x + i(y+1)}{x + i(y+1)}$$

$$= \frac{x + i(y+1)}{x^2 + (y+1)^2}$$

and we are given:

$$\operatorname{Im}\left(\frac{1}{\bar{z}-i}\right) = \frac{y+1}{x^2+(y+1)^2} = 1$$

i.e. $y+1 = x^2 + (y+1)^2$

$\therefore x^2 + y^2 + y = 0$

$x^2 + y^2 + y + \frac{1}{4} = \frac{1}{4}$

$x^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$ (3)

$\therefore$ locus of z is circle, centre $(0, -\frac{1}{2})$ radius $= \frac{1}{2}$ unit

b)
(i) let $x + iy = \sqrt{5 - 12i}$

$\therefore (x+iy)^2 = 5 - 12i$

i.e. $x^2 - y^2 + 2xyi = 5 - 12i$

equating parts:

$x^2 - y^2 = 5 \quad \cdots (1)$ $\qquad 2xy = -12$

or $x = -6/y \quad \cdots (2)$

(2) into (1): $\frac{36}{y^2} - y^2 = 5$

$\therefore y^4 + 5y^2 - 36 = 0$

$(y^2+9)(y^2-4) = 0$

giving $y^2 = 4$ or $y = \pm 2$

and $x = \mp 3$

NOTE:
$y^2 = -9$: y must be real.

$\therefore \sqrt{5-12i} = 3 - 2i$ or $-3 + 2i$ (2)

(ii) $z = \dfrac{1 + \sqrt{5-12i}}{2+2i}$

$$\therefore z = \frac{1+3-2i}{2+2i} \text{ or } z = \frac{1-3+2i}{2+2i}$$

$$= \frac{4-2i}{2+2i} \qquad = \frac{-2+2i}{2+2i}$$

$$= \frac{2-i}{1+i} \times \frac{1-i}{1-i} \qquad = \frac{-1+i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-3i}{2} \qquad = i$$

choose $z = i$ (for z purely imaginary)

$\therefore z^{400} = i^{400}$

$= (i^4)^{100}$

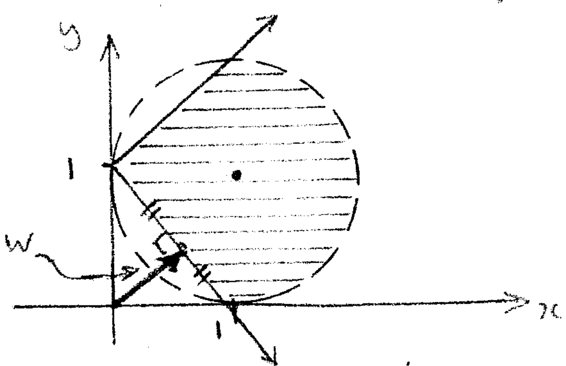
$= 1^{100}$

$= 1$ (2)

c)
(i) $|z - 1 - i| \le 1 \equiv |z - (1+i)| \le 1$

i.e. inside circle, centre $1+i$, radius $= 1$

$-\frac{\pi}{4} \le \arg(z - i) \le \frac{\pi}{4} \equiv$ angle from i between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$



(3)

(ii) w has minimum modulus

$\therefore$ shortest distance to line as indicated by w on diagram above.

i.e. to midpoint of $(0,1)$ and $(1,0)$ $\Big\} = (\frac{1}{2}, \frac{1}{2})$

$\therefore w = \frac{1}{2} + \frac{1}{2}i$ (1)

b) $z = \dfrac{-1+i}{\sqrt{3}+i}$

$= \dfrac{\sqrt{2}\,\text{cis}\,3\pi/4}{2\,\text{cis}\,\frac{\pi}{6}}$

$= \dfrac{\sqrt{2}}{2}\,\text{cis}\left(3\pi/4 - \frac{\pi}{6}\right)$

$\therefore \boxed{z = \frac{1}{\sqrt{2}}\,\text{cis}\,\frac{7\pi}{12}}$ . . . . . . (1)

AND $z = \dfrac{-1+i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \dfrac{1-\sqrt{3} + (1+\sqrt{3})i}{4}$ . . . (2)

Equating real parts in (1) and (2):

$\frac{1}{\sqrt{2}} \cos \frac{7\pi}{12} = \frac{1-\sqrt{3}}{4}$

i.e. $\boxed{\cos \frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}}$ (4)

(a) (i) • $y = \sqrt{f(x)}$ only defined

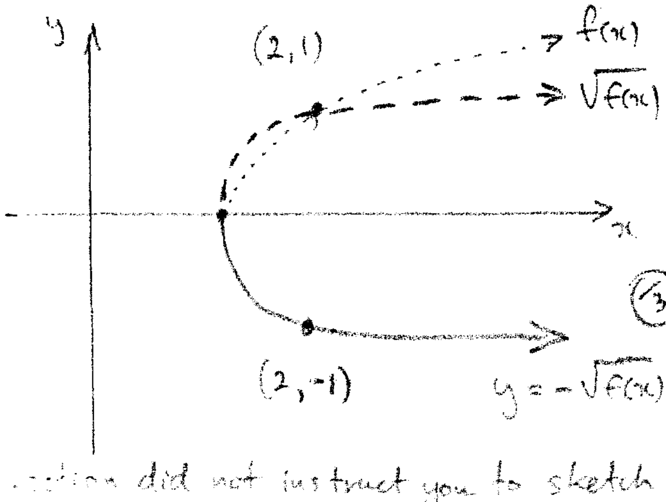
for $f(x) \geq 0$ (i.e. above/on x-axis)

• $y = \sqrt{f(x)}$ above $y = f(x)$

for $0 < f(x) < 1$, through $f(x) = 1$,

below $y = f(x)$ for $f(x) > 1$

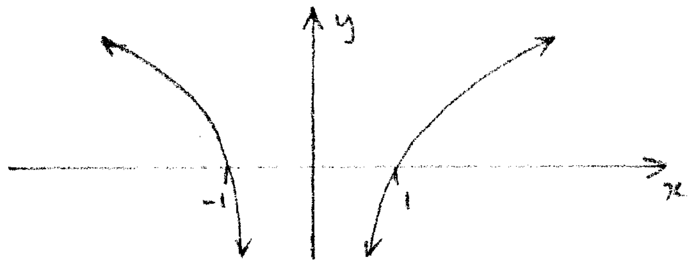
• $y = -\sqrt{f(x)}$ below x-axis.



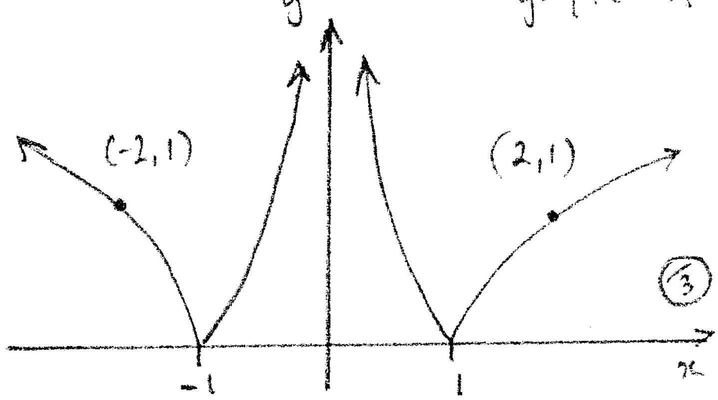
(3)

(Question did not instruct you to sketch $y = f(x)$ or $y = \sqrt{f(x)}$)

(ii) • $y = f(|x|)$ has right side reflected in y-axis.



• $y = |f(|x|)|$ has those parts of above graph below x-axis, reflected above x-axis:



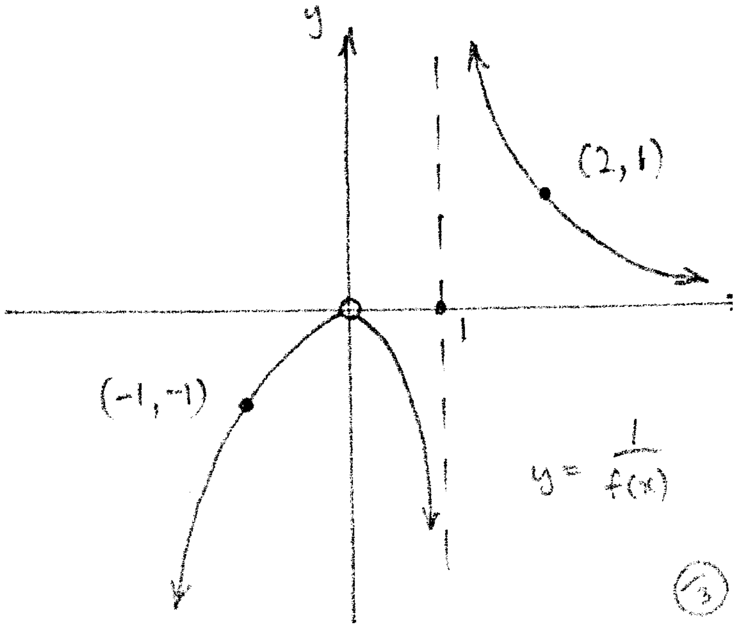
(3)

(iii) • $f(x) = 0$ at $x = 1 \Rightarrow$ asymptote

• $x \to 0$, $f(x) \to -\infty \Rightarrow \frac{1}{f(x)} \to 0$ from below (but undefined)

• $x \to -\infty$, $f(x) \to 0^- \Rightarrow \frac{1}{f(x)} \to -\infty$

• $x \to \infty$, $f(x) \to \infty \Rightarrow \frac{1}{f(x)} \to 0^+$



(3)

b) (i) D has equation: $x = a/e$

and for $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$\therefore e^2 = 1 - 16/25$

$\therefore e = 3/5 \quad (e>0)$

$\therefore x = \dfrac{5}{\frac{3}{5}} \quad (\text{for } a^2 = 25)$

$\therefore x = 25/3$ (1)

ii) Equation of tangent:

$y - y_1 = m(x - x_1)$

for m: $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ implicit diff'n

$\therefore \dfrac{2x}{25} + \dfrac{2y}{16} \cdot y' = 0$

$\therefore y' = \dfrac{-2x}{25} \times \dfrac{8}{y}$

$= \dfrac{-16x}{25y}$

$\therefore$ at $P(x_1, y_1)$, $m = \dfrac{-16x_1}{25y_1}$

$\therefore$ equation is:

$y - y_1 = \dfrac{-16x_1}{25y_1}(x - x_1)$

$\therefore 25y_1y - 25y_1^2 = -16x_1x + 16x_1^2$

$\therefore 16x_1x + 25y_1y = 16x_1^2 + 25y_1^2$

$(\div\ 16 \times 25)$

$\therefore \dfrac{x_1x}{25} + \dfrac{y_1y}{16} = \dfrac{x_1^2}{25} + \dfrac{y_1^2}{16} = 1$ (3)

i.e. $\dfrac{x_1x}{25} + \dfrac{y_1y}{16} = 1$ (QED)

(iii) Focus at $(ae, 0)$

i.e. at $(5 \times 3/5, 0) = (3, 0)$

For PQ: $\dfrac{x_0x}{25} + \dfrac{y_0y}{16} = 1$

$(x_0\ y_0)$ on D $\therefore x_0 = 25/3$ (from (i))

$\therefore$ PQ is: $\dfrac{x}{3} + \dfrac{y_0y}{16} = 1$

Substituting $(3, 0)$ gives:

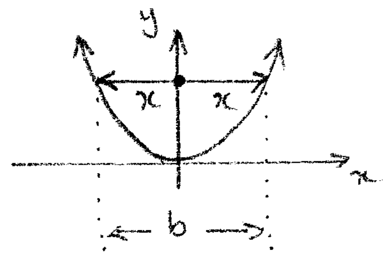
LHS $= \dfrac{3}{3} + 0$

$= 1$

$=$ RHS

$\therefore$ Focus lies on PQ (QED) (2)

(4) (a) Area of cross-section $= \frac{1}{2}bh$ where $b = h$ for isosceles



where $b = 2x$ and $x = \sqrt{y}$

$\therefore A = \frac{1}{2} \times 2\sqrt{y} \times 2\sqrt{y}$

$= 2y$

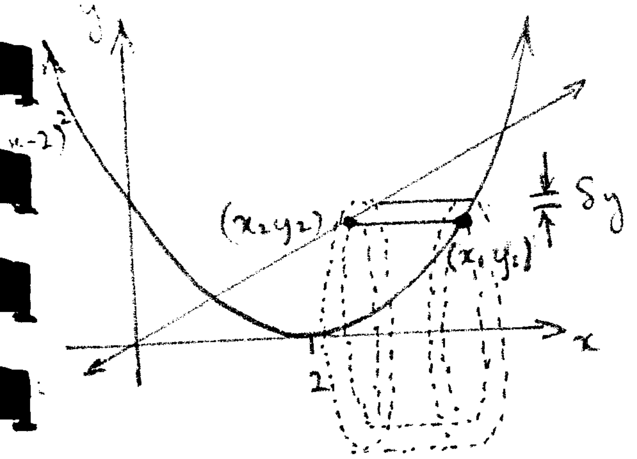
$\therefore V = \lim\limits_{\delta y \to 0} \sum_0^4 2y\,\delta y$

$= 2\int_0^4 y\,dy$

$= 2\left[\frac{1}{2}y^2\right]_0^4$

$= 4^2 - 0^2$

$= 16 \text{ units}^3$ (4)



oundary values:

$(x-2)^2 = x$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$\therefore x = 1$ or 4

$y = 1$ or 4

rea of annular base $= \pi[(y+\delta y)^2 - y^2]$

$= 2\pi y \delta y$ (δy^2 negligible)

and $v = Ah$ for shell

$\therefore V = \lim_{\delta y \to 0} \sum_1^4 2\pi y \, \delta y \times x$

where $x = x_1 - x_2$

$y_1 = (x_1 - 2)^2 \to x_1 = \sqrt{y_2} + 2$

$y_2 = x_2$

$\therefore x = \sqrt{y} + 2 - y$

$\therefore V = 2\pi \int_1^4 y(\sqrt{y} + 2 - y)\, dy$

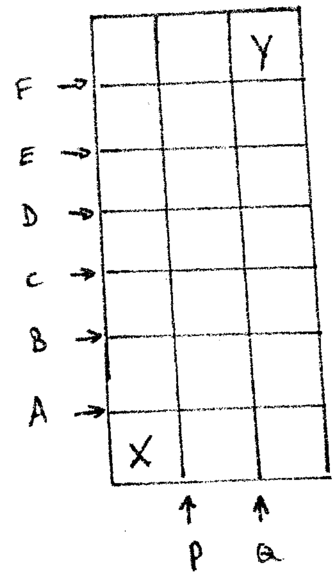
$= 2\pi \int_1^4 y^{3/2} + 2y - y^2 \, dy$

$= 2\pi \left[\frac{2}{5} y^{5/2} + y^2 - \frac{1}{3} y^3 \right]_1^4$

$\therefore$ $V = 64\pi/5$ units^3

(6)

(c)(i) Labelling the lines which have to be crossed from X to Y as shown:



and noting that they have to be crossed "in alphabetical order" both horizontally and vertically ...
THEN placing P first:

$\underline{P}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{Q in any of 7 possible positions}} = 7$ +

(the other 6 positions only possible in ONE way ie. alphabetical: A B C D E F)

or $\_\ \underline{P}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{Q in 6 positions}} = 6$

⋮

+

or $\_\ \_\ \_\ \_\ \_\ \_\ \underline{P}\ \underbrace{\_}_{\text{Q in 1 position}} = 1$

$\therefore$ Number of ways $= 7 + 6 + 5 + 4 + \cdots + 1$

$= 28$

(3)

6

(OR) All 8 letter "words" from

A B C D E F P Q = 8!

but within these arrangements each 6! arrangements (A to F) can only occur ONCE (alphabetical) and 2! (P, Q) arrangements can only occur ONCE

$\therefore$ Total ways $= \dfrac{8!}{6!\,2!}$

$= 28$

(ii) EITHER

X to M : $\underbrace{P\,\_\,\_\,\_\,\_}$

P in only 5 positions

M to Y : $\underbrace{Q\,\_\,\_}$

Q in only 3 positions

$\therefore$ Total $= 5 \times 3$

$= 15$ (2)

(OR) Total ways $= \dfrac{5!}{4!\,1!} \times \dfrac{3!}{2!\,1!}$

$(X \to M) \quad (M \to Y)$

$= 15$

(5) a) i)

$x^3 - x^2 + 4x - 2 = 0$

satisfied by $x = a^2$

i.e. $a = x^{\frac{1}{2}}$

$\therefore$ equation required given by:

$(x^{\frac{1}{2}})^3 - (x^{\frac{1}{2}})^2 + 4(x^{\frac{1}{2}}) - 2 = 0$

or $x\sqrt{x} - x + 4\sqrt{x} - 2 = 0$

$\therefore \sqrt{x}(x+4) = x+2$

$\therefore x(x^2+8x+16) = x^2+4x+4$

$\therefore x^3+8x^2+16x - x^2 - 4x - 4 = 0$

i.e. $\boxed{x^3+7x^2+12x-4=0}$ (3)

ii) Using:

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

then:

$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\{\alpha\beta\cdot\alpha\gamma + \alpha\beta\cdot\beta\gamma + \alpha\gamma\cdot\beta\gamma\}$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

where $\alpha+\beta+\gamma = -b/a = 1$

$\alpha\beta+\alpha\gamma+\beta\gamma = c/a = 4$

$\alpha\beta\gamma = -d/a = 2$

$\therefore$ Answer $= 4^2 - 2 \times 2 \times 1$

$= 12$ (3)

$P(x) = 4x^3 + 4x^2 + x + k$

$P'(x) = 12x^2 + 8x + 1$

For $P'(x) = 0$: $12x^2 + 8x + 1 = 0$

$$\Downarrow$$

$$(6x+1)(2x+1) = 0$$

$\therefore x = -\frac{1}{6}$ or $-\frac{1}{2}$ ..... (1)

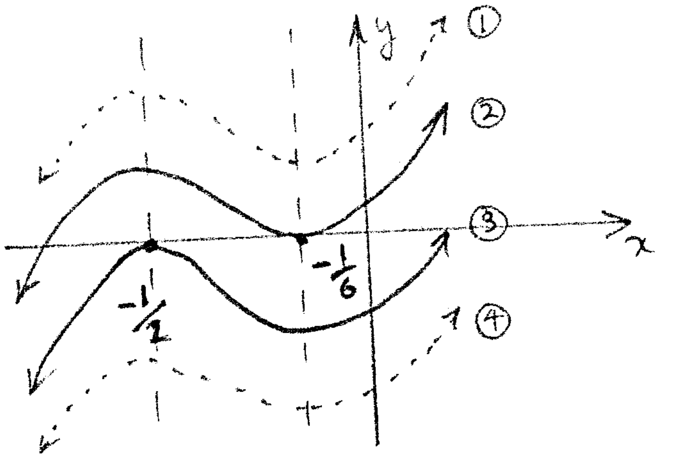
$\therefore P(-\frac{1}{6}) = 4(-\frac{1}{6})^3 + 4(-\frac{1}{6})^2 + (-\frac{1}{6}) + k$

$= -\frac{2}{27} + k$ ..... (2)

and $P(-\frac{1}{2}) = 4(-\frac{1}{2})^3 + 4(-\frac{1}{2})^2 + (-\frac{1}{2}) + k$

$= k$ .... (3)

Now, the graph of $y = P(x)$ has two turning points and could be:



For $P(-\frac{1}{6}) = 0$ (graph ②) $k = \frac{2}{27}$ (from (2))

$P(-\frac{1}{2}) = 0$ (graph ③) $k = 0$ (from (3))

$\therefore$ for one real root (graphs ① / ④) we need $k < 0$ or $k > \frac{2}{27}$ ④

c) $P(x) = 3x^4 - 11x^3 + 14x^2 - 11x + 3$

$= x^2 \left\{ 3x^2 - 11x + 14 - \frac{11}{x} + \frac{3}{x^2} \right\}$

$= x^2 \left\{ \left[3x^2 + 6 + \frac{3}{x^2} \right] + \left[-11x - \frac{11}{x} \right] + 8 \right\}$

$= x^2 \left\{ 3\left[x^2 + 2 + \frac{1}{x^2} \right] - 11\left[x + \frac{1}{x} \right] + 8 \right\}$

$= x^2 \left\{ 3\left(x + \frac{1}{x} \right)^2 - 11\left(x + \frac{1}{x} \right) + 8 \right\}$

(QED)

For $P(x) = 0$ $x^2 \neq 0$

i.e. $x = 0$ is not a solution

$\therefore 3(x + \frac{1}{x})^2 - 11(x + \frac{1}{x}) + 8 = 0$

Letting $A = x + \frac{1}{x}$

$\therefore 3A^2 - 11A + 8 = 0$

$(3A - 8)(A - 1) = 0$

$\therefore 3(x + \frac{1}{x}) - 8 = 0$ or $(x + \frac{1}{x}) - 1 = 0$

$\therefore 3x^2 - 8x + 3 = 0$ $x^2 - x + 1 = 0$

$\therefore x = \frac{8 \pm \sqrt{28}}{6}$ $x = \frac{1 \pm \sqrt{-3}}{2}$

$\therefore$ For $P(x) = 0$

$x = \frac{4 \pm \sqrt{7}}{3}, \frac{1 \pm i\sqrt{3}}{2}$ (over C)

and $P(x) = \left(x - \frac{4 - \sqrt{7}}{3}\right)\left(x - \frac{4 + \sqrt{7}}{3}\right)(x^2 - x + 1)$ (over R)

a) i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $P(a\sec\theta, b\tan\theta)$

Equation of normal:

$$y-y_1=m(x-x_1)$$

for m: $\frac{2x}{a^2}-\frac{2y}{b^2}\cdot y'=1$

$$\therefore y'=\frac{b^2x}{a^2y}$$

$$\therefore \text{at } P,\ m=\left.\begin{aligned}&\frac{b^2\cdot a\sec\theta}{a^2\cdot b\tan\theta}\\&=\frac{b}{a\sin\theta}\end{aligned}\right\}\text{TANGENT}$$

$$\therefore m=-\frac{a\sin\theta}{b}\ \Big\}\text{NORMAL}$$

$\therefore$ Equation of normal:

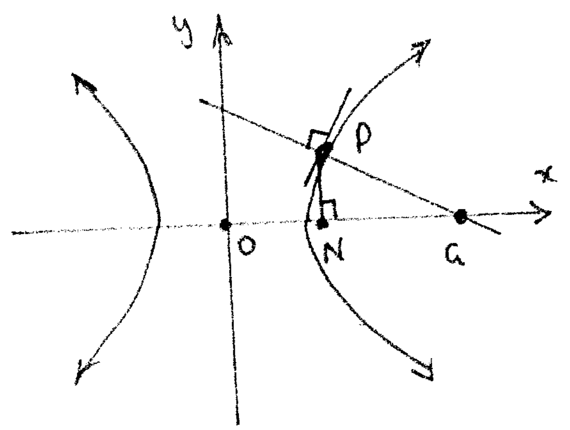
$$y-b\tan\theta=\frac{-a\sin\theta}{b}(x-a\sec\theta)$$

$$\therefore by-b^2\tan\theta=-a\sin\theta\, x+a^2\tan\theta$$

$$\therefore a\sin\theta\, x+by=(a^2+b^2)\tan\theta \quad \text{(QED)}$$

4

ii)



At N: $x=x_p=a\sec\theta$

$\therefore ON=a\sec\theta \quad \ldots\ (1)$

At G: let $y=0$

$$\therefore a\sin\theta\, x=(a^2+b^2)\tan\theta$$

$$\therefore x=\frac{a^2+b^2}{a}\sec\theta$$

$$\therefore OG=\frac{a^2+b^2}{a}\sec\theta$$

and $b^2=a^2(e^2-1)\rightarrow e^2=1+\frac{b^2}{a^2}$

$$=\frac{a^2+b^2}{a^2}$$

$$\therefore OG=a\times\frac{a^2+b^2}{a^2}\sec\theta$$

$$=a\times e^2\sec\theta$$

$$=e^2 a\sec\theta$$

ie $OG=e^2\times ON$ (QED) 5

b) In ΔOKL: $\angle LOK=\angle OLK$ (isos.)

$\therefore \angle LOK=\frac{\pi}{6}$ ($\angle$ sum of Δ)

In ΔOML: $\angle LOM=\frac{\pi}{3}$ (equil Δ)

$\therefore \angle KOM=\frac{\pi}{2}$

ie $\underset{(OM)}{\beta}\equiv\underset{(OK)}{\alpha}$ rotated $90^\circ\equiv$ (α by i) ...(1)

AND In ΔOKL:

$$\cos\angle LOK=\frac{\frac{1}{2}OL}{OK}=\frac{\frac{1}{2}OL}{\alpha}$$

$$\text{ie } \cos\frac{\pi}{6}=\frac{\frac{1}{2}OL}{\alpha}=\frac{\sqrt{3}}{2}$$

$$\therefore OL=\sqrt{3}\,\alpha$$

In ΔOML: $OM=OL$ (in magnitude)

$\therefore$ $\beta=\sqrt{3}\,\alpha$ ... (2)

from (1)/(2): $\sqrt{3}\,\alpha=i\beta$ (square b.s.)

$$3\alpha^2=-\beta^2$$

$$\therefore 3\alpha^2+\beta^2=0 \quad \text{(QED)}$$

a) Prove, by induction:

$$\cos\frac{90^\circ}{2^n} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}}}_{n \text{ terms}}$$

Prove true for $n=1$:

LHS $= \cos\frac{90^\circ}{2^1}$ $\qquad$ RHS $= \frac{1}{2}\sqrt{2}$ ← 1 term

$= \cos 45^\circ$

$= \frac{1}{\sqrt{2}}$

(exact ∠)

$$= \frac{\sqrt{2}\times\sqrt{2}}{2\times\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

∴ LHS = RHS

∴ true for $n=1$

Assume true for $n=k$:

$$\cos\frac{90^\circ}{2^k} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}}_{k \text{ terms}}$$

(here: we see 3 terms)

To prove true for $n=k+1$:

We want:

$$\cos\frac{90^\circ}{2^{k+1}} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}}}_{k+1 \text{ terms}}$$

(here: we want to see 4 terms)

LHS: Let $\cos\frac{90^\circ}{2^k\times 2} = \cos\theta$

$$\therefore \cos 2\theta = \cos\frac{90^\circ}{2^k} = 2\cos^2\theta - 1$$

$$\therefore \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$= \tfrac{1}{2}\left(1+\cos\frac{90^\circ}{2^k}\right)$$

$$= \tfrac{1}{2}\left(1+\tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right)$$ from assumption

$\therefore \ldots\ldots\ldots$ (1)

$\therefore$ LHS $= \cos\theta$

$= \sqrt{(1)}$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(1+\tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right)}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\tfrac{1}{2}\left(2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right)}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}}$$

$$= \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}}}_{k+1 \text{ (or 4) terms}}$$

= RHS

i.e. LHS = RHS

∴ true for $n=k+1$ when true for $n=k$ and true for $n=1$

∴ true for $n = 2, 3, 4$ i.e. $n \geq 1$

i.e. true, by maths induction (5)

(QED).

7) (i) Prove $\tan^{-1}(n+1) - \tan^{-1}(n) = \cot^{-1}(1+n+n^2)$

Let $\tan^{-1}(n+1) = \alpha$

$\therefore \tan\alpha = n+1$

and $\tan^{-1}(n) = \beta$

$\therefore \tan\beta = n$

Now: LHS $= \alpha - \beta$

and $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \dfrac{n+1-n}{1+(n+1)n}$

$= \dfrac{1}{1+n+n^2}$

$\therefore \cot(\alpha-\beta) = 1+n+n^2$

$\therefore \alpha-\beta = \cot^{-1}(1+n+n^2)$
$= \text{RHS}$

i.e. LHS = RHS (QED) 3

(ii) $\cot^{-1}3 = \cot^{-1}(1+1+1^2)$
$= \tan^{-1}2 - \tan^{-1}1$ (from (i))

$\cot^{-1}7 = \cot^{-1}(1+2+2^2)$

$= \tan^{-1}3 - \tan^{-1}2$

$\cot^{-1}13 = \cot^{-1}(1+3+3^2)$

$= \tan^{-1}4 - \tan^{-1}3$

$\therefore \cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cdots$
$+ \cot^{-1}(1+n+n^2)$

$= \tan^{-1}2 - \tan^{-1}1 + \cancel{\tan^{-1}3} - \cancel{\tan^{-1}2}$
$+ \cancel{\tan^{-1}4} - \cancel{\tan^{-1}3} + \cdots$
$+ \tan^{-1}(n+1) - \cancel{\tan^{-1}(n)}$

giving:

$\tan^{-1}(n+1) - \tan^{-1}1$

or $\boxed{\tan^{-1}(n+1) - \pi/4}$ 3

c) Sketch:

- $y_1 = \frac{1}{2} + \sin x$ i.e. $\sin x$ raised $\frac{1}{2}$.

- $y_2 = \left(\frac{1}{2}+\sin x\right)^3 = (y_1)^3$

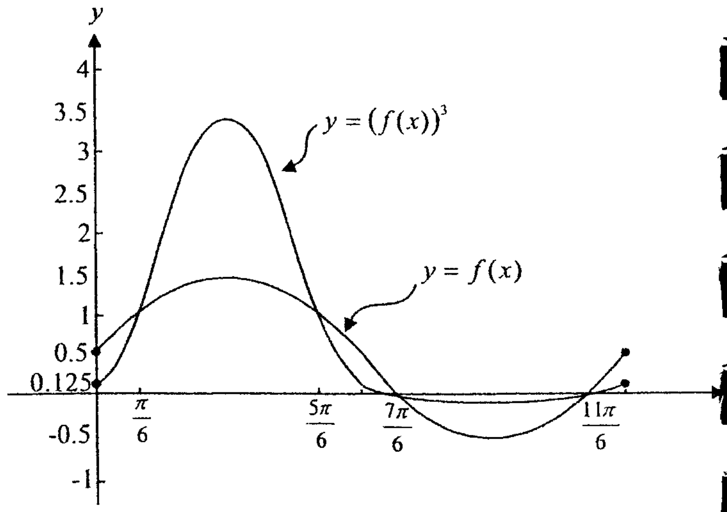
i.e. $(y_1)^3$:

critical points for $f(x) > (f(x))^3$

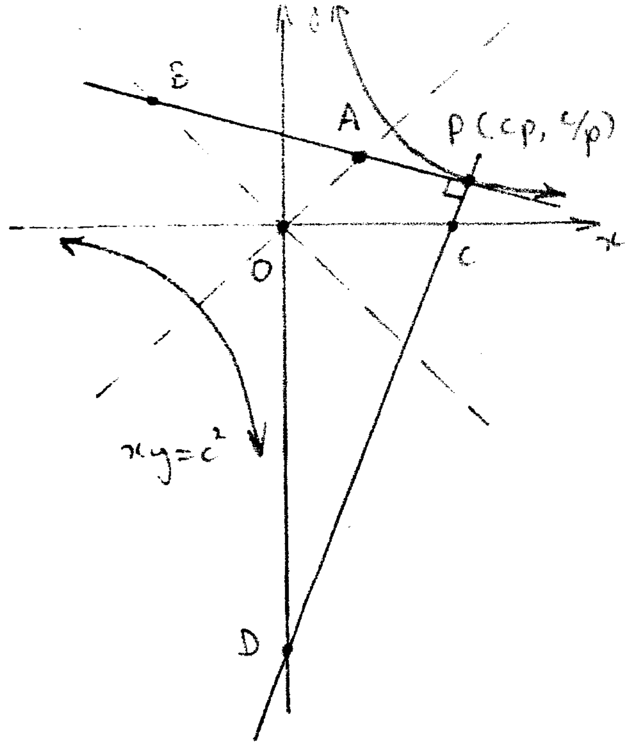
y_1	$(y_1)^3$	
0.5	0.125	
(*) ← 1	1	⇐ (cross over)
1.5	3.375	
0	0	(x-int's)
−0.5	−0.125	

see graph

(*) $\frac{1}{2} + \sin x = 1 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or ...



clearly: $f(x)$ "above" $(f(x))^3$ for
$0 < x < \frac{\pi}{6}$, $\frac{5\pi}{6} < x < \frac{7\pi}{6}$, $\frac{11\pi}{6} < x < 2\pi$



quations of Tangent / Normal

$$y - y_1 = m(x - x_1)$$

for m: $xy = c^2$
$\therefore \; xy' + y = 0$ } implicit diff'n

$$\therefore y' = -y/x$$

$$\therefore \text{ at } P: \; m_T = \frac{-c/p}{cp} = -1/p^2$$

$$\therefore m_N = p^2$$

Tangent: $y - c/p = -1/p^2 (x - cp)$

$$\therefore x + p^2 y - 2pc = 0 \quad \cdots (1)$$

Normal: $y - c/p = p^2 (x - cp)$

$$\therefore p^3 x - py - c(p^4 - 1) = 0 \quad \cdots (2)$$

$\therefore$ At A: $y = x$ in (1)

$$\therefore x + p^2 x - 2pc = 0$$

$$\therefore A \text{ is: } \left(\frac{2pc}{1+p^2}, \frac{2pc}{1+p^2} \right) \cdots (3)$$

Similarly, at B, $y = -x$

$$\therefore B \text{ is: } \left(\frac{2pc}{1-p^2}, \frac{2pc}{1-p^2} \right) \cdots (4)$$

At C: $y = 0$

$$\therefore p^3 x = c(p^4 - 1)$$

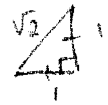
$$x = \frac{c(p^4-1)}{p^3} \Rightarrow OC$$

At D: $x = 0$

$$\therefore py = -c(p^4 - 1)$$

$$y = \frac{-c(p^4-1)}{p}$$

(ie $OD = \frac{c(p^4-1)}{p}$: for distance)

from (3)/(4) 45° right Δ's : 

give: $OA = \frac{2\sqrt{2}pc}{1+p^2}$, $OB = \frac{2\sqrt{2}pc}{1-p^2}$

$$\therefore M^2 N = \left(\frac{1}{2} \times \frac{2\sqrt{2}pc}{1+p^2} \times \frac{2\sqrt{2}pc}{1-p^2} \right)^2 \times \left(\frac{1}{2} \times \frac{c(p^4-1)}{p^3} \times \frac{c(p^4-1)}{p} \right)$$

$$= \left(\frac{4p^2c^2}{1-p^4} \right)^2 \times \frac{c^2(p^4-1)^2}{2p^4}$$

$$= \frac{16\cancel{p^4}c^4}{\cancel{(1-p^4)^2}} \times \frac{c^2\cancel{(p^4-1)^2}}{2\cancel{p^4}}$$

$$= 8c^2$$

ie. M^2N is constant (QED)
(for constant c).

6

$$f(x) = \frac{1-|x|}{|x|}$$

i) $$f(a) = \frac{1-|a|}{|a|}$$

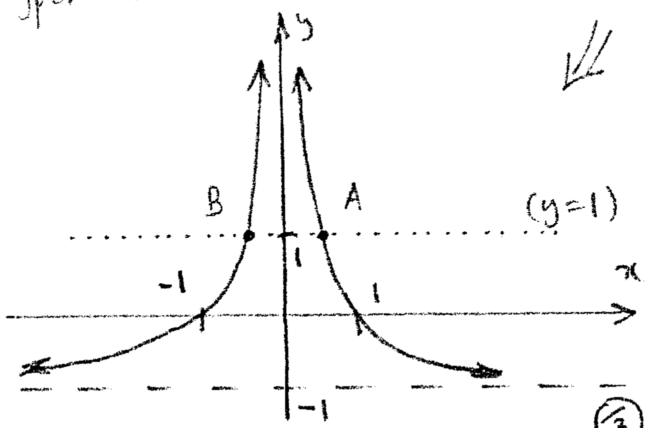
$$f(-a) = \frac{1-|-a|}{|-a|} = \frac{1-|a|}{|a|}$$

$\therefore f(a) = f(-a) \quad \therefore$ EVEN (1)

(ii) $$f(x) = \frac{1-|x|}{|x|} = \frac{1}{|x|} - \frac{|x|}{|x|} = \frac{1}{|x|} - 1 \quad \left(\frac{1}{|x|} = \left|\frac{1}{x}\right|\right)$$

down 1

hyperbola reflected above x-axis



(3)

(iii) For $f(x) \geq 1$, find A / B

A : intersection with $\frac{1}{x} - 1$ and 1

ie. $\frac{1}{x} = 2$

$x = \frac{1}{2}$

B : intersection with $-\frac{1}{x} - 1$ and 1

ie $\frac{-1}{x} = 2$

$\therefore x = -\frac{1}{2}$

and $y = f(x)$ ABOVE/on $y=1$ $(\geq)$

for $\boxed{-\frac{1}{2} \leq x \leq \frac{1}{2}}$ (3)

(iv) • For $\left.\begin{matrix} x<-1 \\ x>1 \end{matrix}\right\} : -1 < f(x) < 0$

$\therefore e^{-1} < e^{f(x)} < e^{0}$

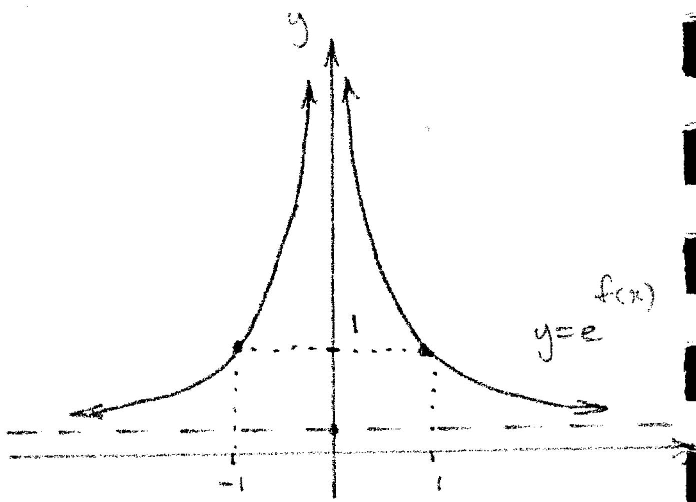
$\frac{1}{e} < e^{f(x)} < 1$

• For $x = \pm 1$: $f(x) = 0$

$\therefore e^{f(x)} = 1$

• For $-1 < x < 1$: $f(x) \to \infty$

$\therefore e^{f(x)} \to \infty$



(2)