NEWCASTLE GRAMMAR SCHOOL

Student Number: ______________________________

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Examination Date: Wednesday 17th August

Examiner: Mr. M. Brain

General Instructions

- Reading Time - 5 minutes
- Working Time - 3 hours
- Write using a blue or black pen
- Write your student number on every booklet
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Each question attempted is to be returned in a separate Writing Booklet clearly marked Questions 1 etc.
- If required, additional booklets may be requested

Total marks - 120

- Attempt Questions 1- 8
- All questions are of equal value

<u>Question 1</u> (Start a new booklet) **Marks**

a) Find **3**

(i) $\displaystyle\int \frac{dx}{x^2 - 16x + 60}$

(ii) $\displaystyle\int \frac{dx}{x^2 - 16x + 80}$

b) Evaluate **8**

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin\theta}\, d\theta$

(ii) $\displaystyle\int_0^{\pi} e^x \cos x \, dx$

c) Use the substitution $u = x - 2$ to find $\displaystyle\int \frac{2x}{\sqrt{4x - x^2}}\, dx$ **4**

Question 2 (Start a new booklet) **Marks**

a) (i) Express $\dfrac{-1+i}{\sqrt{3}+i}$ in mod-arg form **3**

(ii) Hence express $\cos\dfrac{7\pi}{12}$ in surd form

b) Evaluate $\arg\left((2+i)\overline{w}\right)$ given that $w=-1-3i$ **1**

c) (i) On an Argand diagram shade the region where both **4**
$|z-(1+i)|\le 1$ and $0\le \arg\left(z-(1+i)\right)\le\dfrac{\pi}{4}$

(ii) Find the sets of values of $|z|$ and $\arg z$ for the points in the shaded region

d) z_1 and z_2 are two complex numbers such that $\dfrac{z_1+z_2}{z_1-z_2}=2i$ **7**

(i) On an Argand diagram show vectors representing
z_1, z_2, z_1+z_2 and z_1-z_2

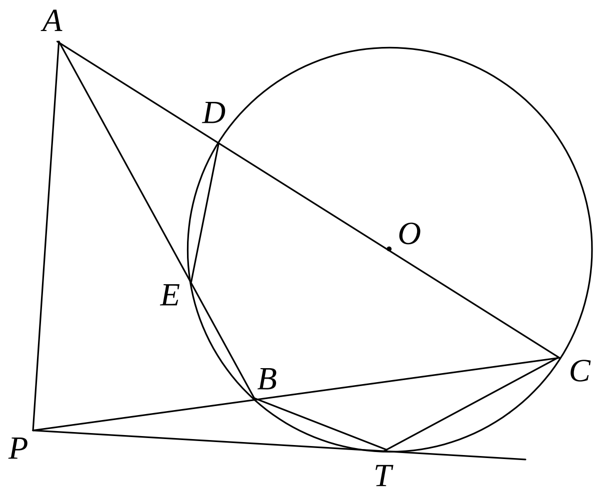
(ii) Show that $|z_1|=|z_2|$

(iii) If α is the angle between the vectors representing z_1 and z_2
show that $\tan\dfrac{\alpha}{2}=\dfrac{1}{2}$

(iv) Show that $z_2=\dfrac{1}{5}(3+4i)z_1$

Question 3 (Start a new booklet) **Marks**

a) A is a point outside a circle with centre O. P is a second point outside the circle such that $PT=PA$ where PT is a tangent to the circle at T. AO cuts the circle at D and C. PC cuts the circle at B. AB cuts the circle at E. **6**



Copy the diagram into your answer booklet

(i) Show that ΔPBT is similar to ΔPTC

(ii) Show that ΔAPB is similar to ΔCPA

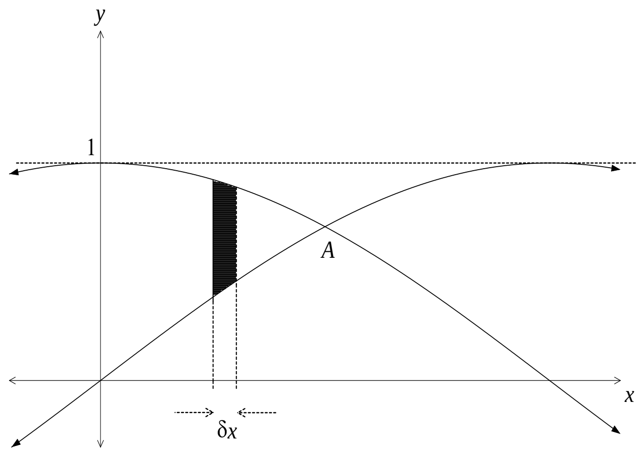
(iii) Hence show that DE is parallel to AP

b) (i) On the same number plane sketch the graphs of $y = |x| - 2$ and $y = 4 + 3x - x^2$ **3**

(ii) Hence, or otherwise, solve $\dfrac{|x| - 2}{4 + 3x - x^2} > 0$

Question 3 continued on next page

c) The area between $y = \sin x$ and $y = \cos x$, from the y-axis to the point of intersection, A, is rotated about the line $y = 1$ **4**



(i) Find the co-ordinates of point A

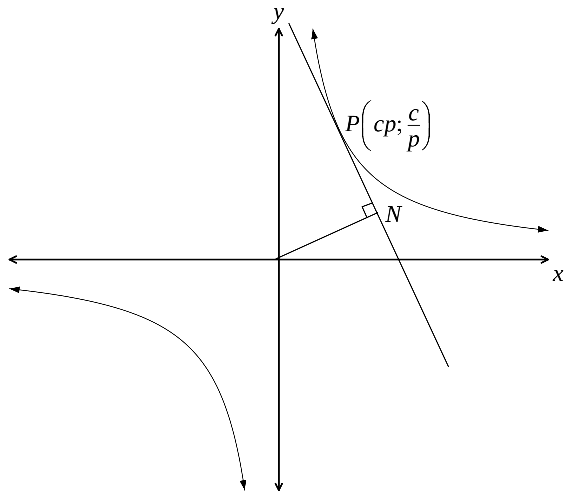
(ii) Calculate the generated volume of revolution

d) Solve the equation $8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$ given that it has a triple root **2**

Question 4 (Start a new booklet) **Marks**

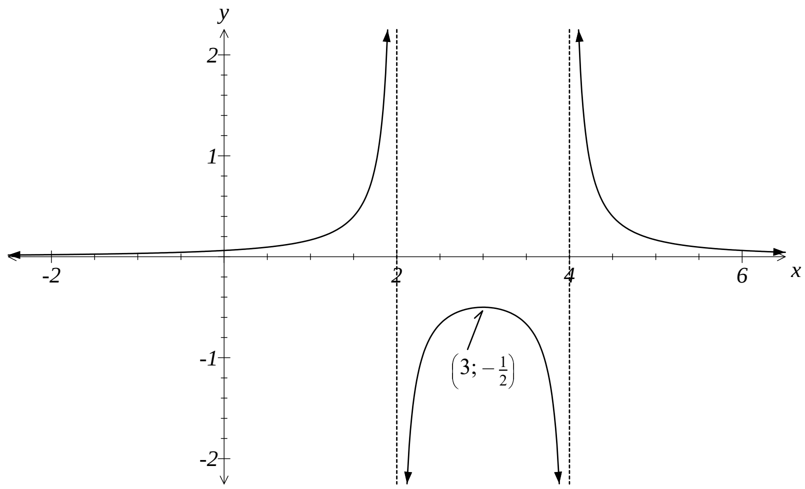
a) Find all the roots of $P(x) = x^4 - 8x^3 + 39x^2 - 122x + 170$ given that $3 - i$ is one of the roots **4**

b) The line through the origin which is perpendicular to the tangent at $P\left(cp, \frac{c}{p}\right)$ to the rectangular hyperbola $xy = c^2$ meets the tangent at N. **7**



Show that the locus of N has the equation $\left(x^2 + y^2\right)^2 = 4c^2xy$

c) Give a possible equation for the graph below: **4**



Question 5 (Start a new booklet) **Marks**

a) The equation of a curve is $x^2y^2 - x^2 + y^2 = 0$ **7**

(i) Show that the numerical value of y satisfies $|y| < 1$

(ii) Find the equations of the asymptotes

(iii) Show that $\dfrac{dy}{dx} = \dfrac{y^3}{x^3}$

(iv) Sketch the curve

b) Sketch the graph of each equation on a separate number plane: **8**

(i) (1) $y = \sqrt{x^2}$ (2) $y = \left(\sqrt{x}\right)^2$

(ii) (1) $y = \ln\left(e^x\right)$ (2) $y = e^{\ln x}$

(iii) (1) $y = \sin\left(\sin^{-1} x\right)$ (2) $y = \sin^{-1}\left(\sin x\right)$

<u>Question 6</u> (Start a new booklet) **Marks**

a) The inequality $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ is true for any real positive numbers *a*,*b* and *c*. Given that $a+b+c=1$ show: **5**

(i) $\frac{1}{abc} \geq 27$

(ii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$

(iii) $(1-a)(1-b)(1-c) \geq 8abc$

b) (i) Show that the area, *A*, of a regular pentagon of side length *P* is given by **4**

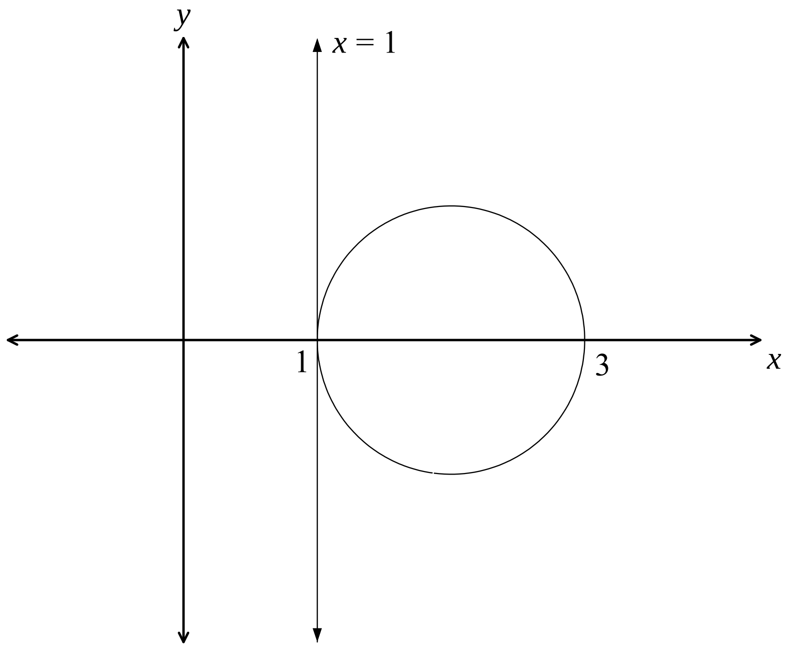
$$A = \frac{5}{2}P^2 \frac{\sin^2 54^\circ}{\sin 72^\circ}$$

(ii) The area enclosed by $y = x^2$ and $y = 3$ is the base of a solid. Cross-sections of the solid, parallel to the *x*-axis, are regular pentagons with one side of the pentagon on the base of the solid. Calculate the volume of the solid, correct to one decimal place

Question 6 continued on **next page**

Question 6 continued **Marks**

c) In the diagram below the circle with the equation $(x-2)^2 + y^2 = 1$ is drawn. The region bounded by the circle is rotated about the line $x = 1$ **6**



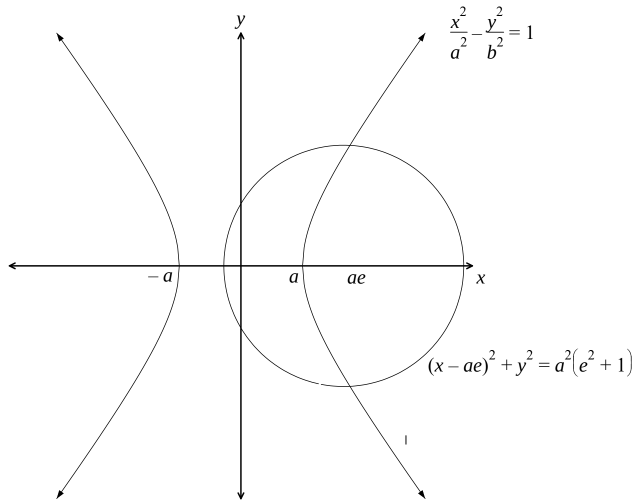
(i) Use the method of cylindrical shells to show that the volume, V, of the solid so formed is given by

$$V = 4\pi \int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$

(ii) By using the substitution $x - 2 = \sin\theta$ calculate the volume of the solid formed

<u>Question 7</u> (Start a new booklet) **Marks**

a) For the diagram below: **8**



(i) Show that the tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ has equation $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}-1=0$

(ii) Show that if the tangent at P is also a tangent to the circle with centre $(ae, 0)$ and radius $a\sqrt{e^2+1}$ then $\sec\theta=-e$

(iii) Given that $\sec\theta=-e$, deduce that the points of contact, P and Q on the hyperbola, of the common tangents to the circle and the hyperbola are the extremities of a latus rectum ($x=-ae$) of the hyperbola and state the coordinates of P and Q

(iv) Find the equations of the common tangents to the circle and the hyperbola and find the coordinates of their points of contact with the circle

b) (i) Show that $\dfrac{\sin(A+B)-\sin(A-B)}{2\sin B}=\cos A$ **7**

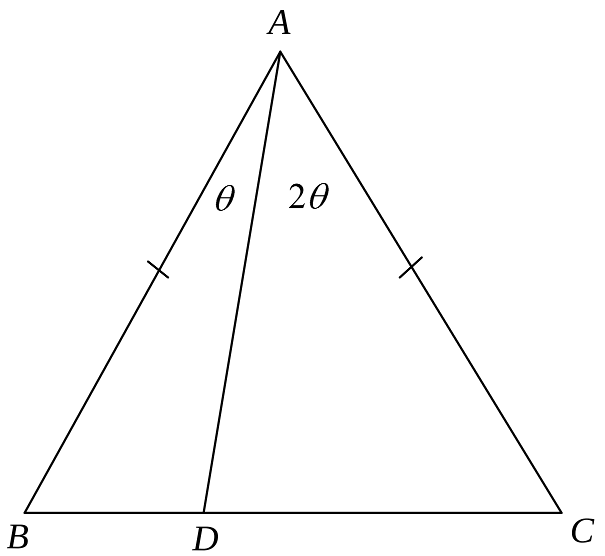
(ii) Hence show that

$$\cos x+\cos 3x+\cos 5x+...+\cos(2n-3)x+\cos(2n-1)x=\frac{\sin 2nx}{2\sin x}$$

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin 8x}{\sin x}\,dx$

<u>Question 8</u> (Start a new booklet) **Marks**

a) In the diagram below ABC is a triangle in which $AB=AC$ and $BC=1$. D is the point on BC such that $\angle BAD=\theta$, $\angle CAD=2\theta$ **6**



(i) Letting $BD=x$ show that $\cos\theta=\dfrac{1-x}{2x}$

(ii) Hence show that $\frac{1}{3}<x<\frac{1}{2}$

b) Let α,β and γ be the non-zero roots of $x^3+3px+q=0$ **5**

(i) Obtain the monic equation which has the roots $\dfrac{\alpha\beta}{\gamma},\dfrac{\alpha\gamma}{\beta}$ and $\dfrac{\beta\gamma}{\alpha}$

(ii) Show that if $\alpha\beta = \gamma$ then $(3p - q)^2 + q = 0$

c) (i) Prove that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ **4**

(ii) Given that $y = 5\sin(x + \alpha)$ prove, by mathematical induction,

that $\frac{d^n y}{dx^n} = 5\sin\left(x + \alpha + \frac{n\pi}{2}\right)$ for $n \geq 1$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}dx = \ln x, \quad x>0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

Year 12 TRIAL Ext 2 2011

① a)(i) $\int \frac{dx}{x^2-16x+60}$

$x^2-16x+60 = (x-6)(x-10)$

$\therefore \int = \int \frac{dx}{(x-6)(x-10)}$

Letting: $\frac{1}{(x-6)(x-10)} = \frac{A}{x-6} + \frac{B}{x-10}$

$\therefore 1 = A(x-10) + B(x-6)$

For $x=10$: $1 = 4B$ $\therefore B = \frac{1}{4}$

$x=6$: $1=-4A$ $\therefore A = -\frac{1}{4}$

$\therefore \int = \frac{1}{4}\int \frac{1}{x-10} - \frac{1}{x-6}\,dx$

$= \frac{1}{4}\{\ln|x-10| - \ln|x-6| + c\}$

or $= \frac{1}{4}\{\ln\left|\frac{x-10}{x-6}\right| + c\}$

or $= \ln\left|\frac{x-10}{x-6}\right|^{1/4} + c$ (2)

(ii) $\int \frac{dx}{x^2-16x+80}$

$= \int \frac{dx}{(x-8)^2+16}$

$= \frac{1}{4}\tan^{-1}\left(\frac{x-8}{4}\right) + c$ (1)

b) (i) Letting $\tan\frac{\theta}{2} = t$

$\therefore \frac{dt}{d\theta} = \frac{1}{2}(\sec^2\frac{\theta}{2})$

$= \frac{1}{2}(\tan^2\frac{\theta}{2}+1)$

$= \frac{1}{2}(t^2+1)$

$\therefore d\theta = \frac{2}{t^2+1}\,dt$

and $\sin\theta = \frac{2t}{t^2+1}$

and at $\theta = \frac{\pi}{2}$: $t = \tan\frac{\pi}{4} = 1$

$\theta = 0$: $t = \tan 0 = 0$

$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$

$= \int_0^1 \frac{1}{1+\frac{2t}{t^2+1}} \times \frac{2}{t^2+1}\,dt$

$= \int_0^1 \frac{2}{t^2+1+2t}\,dt$

$= \int_0^1 \frac{2}{(t+1)^2}\,dt$

$= 2\int_0^1 (t+1)^{-2}\,dt$

$= 2\left[\frac{-1}{t+1}\right]_0^1$

$= -2\left(\frac{1}{1+1} - \frac{1}{1+0}\right)$

$= -2\left(\frac{1}{2}-1\right)$

$= 1$ (4)

(ii) $\int_0^{\pi} e^x\cos x\,dx = I$ $u=e^x$ $v'=\cos x$, $u'=e^x$ $v=\sin x$

$= uv - \int vu'$

$= \left[e^x\sin x\right]_0^{\pi} - \int_0^{\pi} e^x\sin x\,dx$ $u=e^x$ $v'=\sin x$, $u'=e^x$ $v=-\cos x$

$= \left[e^x\sin x\right]_0^{\pi} - \left[\left[-e^x\cos x\right]_0^{\pi} + \int_0^{\pi} e^x\cos x\,dx\right]$

$\therefore I = \left[e^x\sin x + e^x\cos x\right]_0^{\pi} - I$

$\therefore 2I = \left[e^x(\sin x+\cos x)\right]_0^{\pi}$

$\therefore I = \frac{1}{2}\{(e^{\pi}(0-1)) - (e^0(0+1))\}$

$= \frac{1}{2}(-e^{\pi}-1)$ (4)

c) $\int \frac{2x}{\sqrt{4x-x^2}}\,dx = \int \frac{2(u+2)}{\sqrt{4(u+2)-(u+2)^2}}\,du$ → $u=x-2$ $\therefore du=dx$

$= 2\int \frac{u+2}{\sqrt{4-u^2}}\,du$

$= 2\int \frac{u}{\sqrt{4-u^2}}\,du + 4\int \frac{1}{\sqrt{4-u^2}}\,du$ → $u^2=w$, $dw=2u\,du$

$= 2\left\{\frac{1}{2}\int \frac{2u}{\sqrt{4-w}}\,du\right\} + 4\sin^{-1}\left(\frac{u}{2}\right)+c$

$= \int (4-w)^{-\frac{1}{2}}\,dw + 4\sin^{-1}\left(\frac{u}{2}\right)+c$

$$= \frac{(4-w)^{\frac{1}{2}}}{-\frac{1}{2}} + 4\sin^{-1}\left(\frac{u}{2}\right) + c$$

$$= -2\sqrt{4-u^2} + 4\sin^{-1}\left(\frac{u}{2}\right) + c$$

$$= -2\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + c$$

$$= -2\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + c \quad (4)$$

(2) a) (i) $\frac{-1+i}{\sqrt{3}+i} = \frac{\sqrt{2}\operatorname{cis}\frac{3\pi}{4}}{2\operatorname{cis}\pi/6}$

$= \frac{\sqrt{2}}{2}\operatorname{cis}(3\pi/4 - \pi/6)$

$= \frac{1}{\sqrt{2}}\operatorname{cis}(7\pi/12)$ (1)

(ii) $\frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{1-\sqrt{3}+(1+\sqrt{3})i}{4}$

$\therefore$ equating real parts of (i) and (ii):

$\frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{4}$

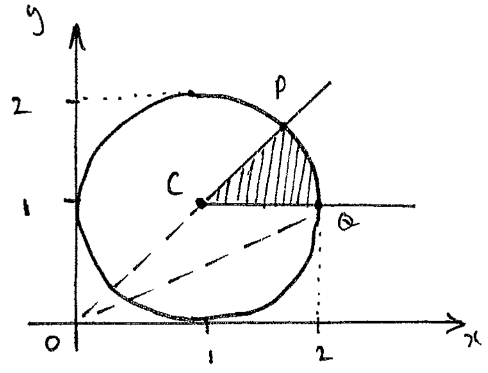
$\therefore \cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$ (2)

b) $\arg((2+i)-\bar{w}) = \arg((2+i)(-1+3i))$

$= \arg(-5+5i)$

$= 3\pi/4$ (1)

c) (i)



(2)

(ii) $OC < |z| < OP$

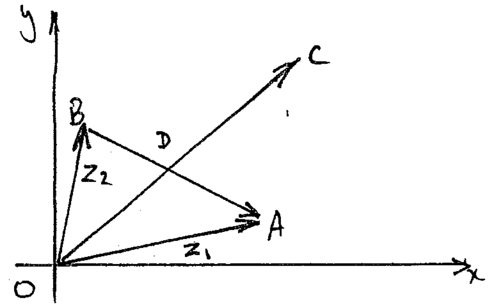
$\therefore \sqrt{2} < |z| < \sqrt{2}+1$

$\arg z_{min}$ at Q, $\arg z_{max}$ at P

$\therefore \tan^{-1}\frac{1}{2} < \arg z < \tan^{-1} 1$

or $\tan^{-1}\frac{1}{2} < \arg z < \pi/4$ (2)

d) (i)



(1)

$OA = z_1$, $OB = z_2$, $OC = z_1+z_2$, $BA = z_1 - z_2$

NOTE: AOBC is parallelogram (by addition of vectors)

(ii) $\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = \arg(2i)$

$\therefore \arg(z_1+z_2) - \arg(z_1-z_2) = \frac{\pi}{2}$

$\therefore$ angle between diagonals of parallelogram $= 90°$

$\therefore$ AOBC is rhombus

$\therefore |z_1| = |z_2|$ (QED) (2)

(iii) $\left|\frac{z_1+z_2}{z_1-z_2}\right| = |2i|$

$\therefore \frac{|z_1+z_2|}{|z_1-z_2|} = 2$

$\therefore |OC| = 2 \times |AB|$

$\therefore OD = 2 \times AD$ (2)

and $\tan \alpha/2 = \frac{AD}{OD} = \frac{AD}{2AD} = \frac{1}{2}$ (QED)

(iv) $\frac{1}{5}(3+4i) = \frac{1}{5} \times 5 \operatorname{cis}\left(\tan^{-1} \frac{4}{3}\right)$

$= \operatorname{cis}\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$\therefore \left|\frac{1}{5}(3+4i)\right| = 1$

$\arg\left(\frac{1}{5}(3+4i)\right) = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore z_1 \times \left(\frac{1}{5}(3+4i)\right)$ is same modulus as z_1, rotated through $\tan^{-1}\left(\frac{4}{3}\right)$

From diagram: $\tan\frac{\alpha}{2} = \frac{1}{2}$

$\therefore \tan\alpha = \dfrac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}$

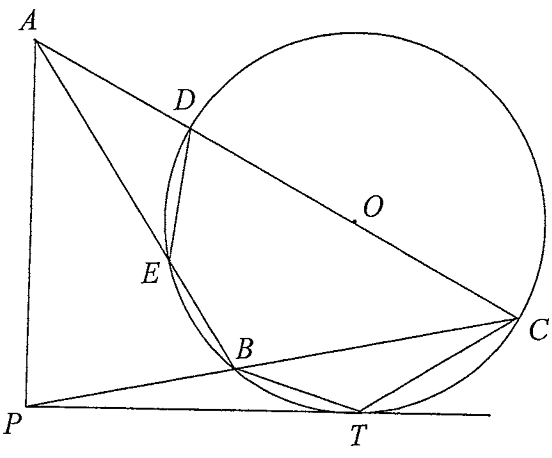
$= \frac{4}{3}$

$\therefore \alpha = \tan^{-1}\left(\frac{4}{3}\right)$ and $|z_1| = |z_2|$

$\therefore z_2 = \frac{1}{5}(3+4i)z_1$ (QED) (2)

(3)

a)



(i) $\angle BPT = \angle CPT$ (common) (A)

$\angle PTB = \angle PCT$ ($\angle$ bet. tang. + chord) (A)

$\therefore \angle PBT = \angle PTC$ ($\angle$ sum Δ) (A)

$\therefore \Delta PBT \parallel\!\mid \Delta PTC$ (AAA)

(QED) (1)

(ii) $\angle APB = \angle CPA$ (common) (A)

and $\dfrac{PB}{PT} = \dfrac{PT}{PC}$ (corres. sides $\parallel\!\mid \Delta$'s, from (i))

but $PT = PA$ (given)

$\therefore \dfrac{PB}{PA} = \dfrac{PA}{PC}$ (S, S)

$\therefore \Delta APB \parallel\!\mid \Delta CPA$ (S.A.S) (QED) (2)

(iii) $\angle PAE = \angle BCD$ (corres. $\angle$'s in $\parallel\!\mid \Delta$'s, from (ii))

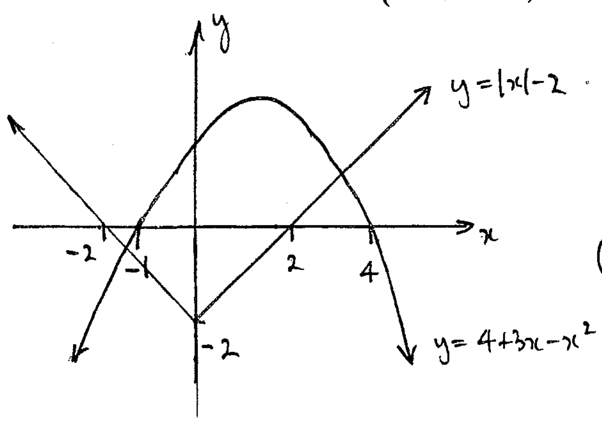
and $\angle BCD = \angle DEA$ (ext. $\angle$ of cyclic quad)

$\therefore \angle PAE = \angle DEA$

$\therefore DE \parallel AP$ (alt. $\angle$'s are =). (QED) (2)

b) (i) $y = |x| - 2$ $\quad y = 4 + 3x - x^2$

$= -(x+1)(x-4)$



(1)

(ii) For $\dfrac{|x|-2}{4+3x-x^2} > 0$

ie. $\dfrac{|x|-2}{4+3x-x^2}$ POSITIVE

ie when BOTH graphs on same side of x-axis

$\therefore$ Solution is:

$-2 < x < -1$ or $2 < x < 4$. (2)

c) (i) For A: $\sin x = \cos x$

$\therefore \tan x = 1$ (1)

$\therefore x = \pi/4, \ y = \frac{1}{\sqrt{2}}$

(ii) $A(x) =$ area of annulus

$= \pi\{(1-\sin x)^2 - (1-\cos x)^2\}$

$\therefore V = \lim_{\delta x \to 0} \sum_a^b A(x)\,\delta x$

$= \pi\int_0^{\pi/4} (1 - 2\sin x + \sin^2 x - 1 + 2\cos x - \cos^2 x)\,dx$

$= \pi\int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x)\,dx$

$= \pi\left[2\sin x + 2\cos x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \pi\left(\left(2\times\frac{1}{\sqrt{2}} + 2\times\frac{1}{\sqrt{2}} - \frac{1}{2}\right) - (0 + 2 - 0)\right)$

$\therefore V = \pi\left(\frac{4}{\sqrt{2}} - \frac{5}{2}\right)$ or $\pi(2\sqrt{2} - 2\cdot5)\ u^3$ (3)

d) For $P(x) = 8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$

$\therefore P'(x) = 32x^3 + 132x^2 + 108x + 25$

$\therefore P''(x) = 96x^2 + 264x + 108$

$= 12(4x+9)(2x+1)$

$\therefore (4x+9)$ or $(2x+1)$ is multiple factor

Now: $P(-\frac{1}{2}) = 8(-\frac{1}{2})^4 + 44(-\frac{1}{2})^3 + 54(-\frac{1}{2})^2 + 25(-\frac{1}{2}) + 4$

$= 0$

$\therefore P(x) = (2x+1)^3 \times Q(x)$ } by inspection

ie. $P(x) = (2x+1)^3(x+4)$

$\therefore x = -\frac{1}{2}$ or -4 (2)

(4) a) $P(x)$ has real coefficients

$\therefore$ roots in conjugate pairs

$\therefore$ $3-i$ is root $\Rightarrow$ $3+i$ is root

$\therefore (x-(3-i))(x-(3+i))$ is factor of $P(x)$

ie. $x^2 - 6x + 10$ is factor

$\therefore$ by division or by inspection:

$P(x) = (x^2 - 6x + 10)(x^2 - 2x + 17)$

For $x^2 - 2x + 17 = 0$: $x = \frac{2 \pm \sqrt{-64}}{2}$

$= 1 \pm 4i$

$\therefore$ Roots are: $3 \pm i, \ 1 \pm 4i$ (4)

b) $y = \frac{c^2}{x}$

$\therefore y' = \frac{-c^2}{x^2}$

$\therefore$ Equation of tangent at P:

$y - c/p = -\frac{1}{p^2}(x - cp)$

$\therefore x + p^2 y = 2cp$ ..... (1)

and: Equation of ON is

$y = p^2 x$ ..... (2)

$\therefore$ (2) into (1): $x + p^2(p^2 x) = 2cp$

$\therefore x = \frac{2cp}{1+p^4}$

$\therefore y = \frac{2cp^3}{1+p^4}$

Now: $x(1+p^4) = 2cp$ and $p^2 = y/x$

$\therefore x^2\left(1 + \frac{y^2}{x^2}\right)^2 = 4c^2(y/x)$ (× B.S. x^2)

$\therefore (x^2+y^2)^2 = 4c^2xy$ (QED) (4)

c) Graph is of the form:

$$y = \frac{1}{a(x-2)(x-4)}$$

ie. reciprocal graph of parabola

- with turning point at $x = 3$

But at $x=3$:

$$-\frac{1}{2} = \frac{1}{a(3-2)(3-4)}$$

$$-\frac{1}{2} = \frac{1}{-a}$$

$$\therefore a = 2$$

(4)

$\therefore$ Equation is: $\boxed{y = \frac{1}{2(x-2)(x-4)}}$

(5) a) (i) $x^2y^2 - x^2 + y^2 = 0$ ...... (1)

$$\therefore y^2(x^2+1) = x^2$$

$$\therefore y^2 = \frac{x^2}{x^2+1}$$

Now: RHS ≥ 0 for all x

and Denominator > Numerator

$\therefore$ RHS < 1

$$\therefore 0 \leq y^2 < 1$$

ie. $|y| < 1$ (QED) 2

(ii) as $x \to \infty$ $y^2 \to 1$

$$\therefore y \to \pm 1$$

$\therefore$ asymptotes: $\boxed{y = \pm 1}$ 1

(iii) By implicit differentiation:

$$x^2 2yy' + 2xy^2 - 2x + 2yy' = 0$$

$$\therefore y'(2x^2y + 2y) = 2x - 2xy^2$$

$$\therefore \frac{dy}{dx} = \frac{2x(1-y^2)}{2y(x^2+1)}$$

but from (1) $y^2(x^2+1) = x^2$

$$\therefore y(x^2+1) = x^2/y$$

and $x^2(1-y^2) = y^2$

$$\therefore x(1-y^2) = y^2/x$$

$$\therefore \frac{dy}{dx} = y^2/x \div x^2/y$$

ie $\frac{dy}{dx} = y^3/x^3$ 2

(iv) at $x = 0$ $y^2 = 0$ $\therefore y = 0$

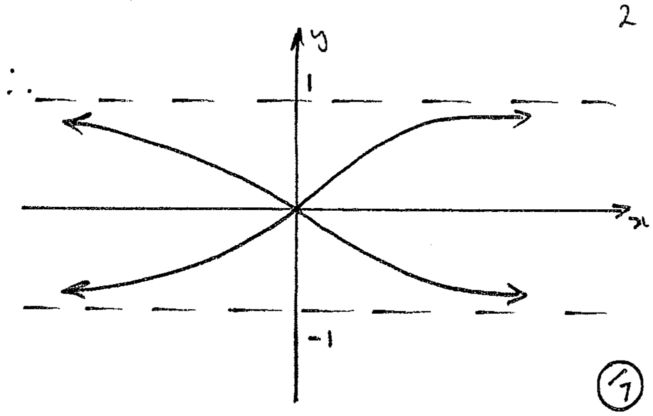
$\therefore$ as $x \to 0$, $y \to 0$

ie $y \to x$

$$\therefore \frac{dy}{dx} \to 1$$

as $x \to \pm\infty$ $\frac{dy}{dx} \to 0$

2



(7)

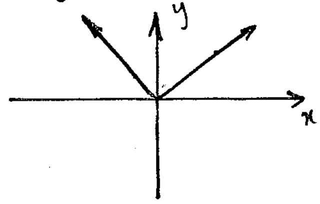
(NOTE: even in x and y

$\therefore$ symmetry about both axes)

b) (i) (1) $y = \sqrt{x^2}$ } Domain: $x \in \mathbb{R}$

$\therefore y = |x|$

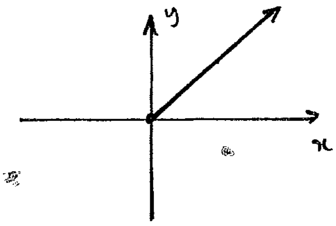
$\therefore$



1

(2) $y = (\sqrt{x})^2$ } Domain: $x \geq 0$

$\therefore y = x$

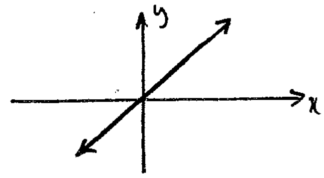


1

(ii) (1) $y = \ln(e^x)$ } Domain: $x \in \mathbb{R}$

$\therefore y = x$

$\therefore$

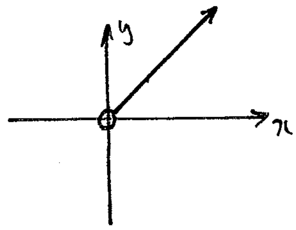


1

(2) $y = e^{\ln x}$ } Domain: $x > 0$

$\therefore y = x$

$\therefore$

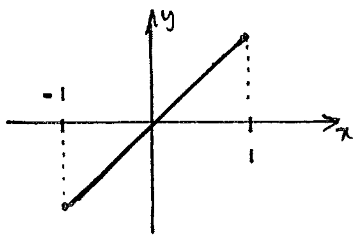


1

(iii) (1) $y = \sin(\sin^{-1} x)$ } Domain: $-1 \leq x \leq 1$

$\therefore y = x$

$\therefore$



2

(2) $y = \sin^{-1}(\sin x)$ } Domain: $x \in \mathbb{R}$ for $\sin x$

} BUT: Domain: $-1 \leq \sin^{-1} x \leq 1$

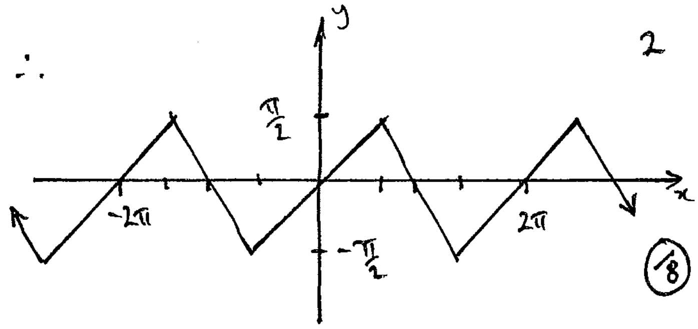
$\therefore y = x$ } Domain: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

increasing graph: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\therefore$ decreasing graph: $\pi/2 \leq x \leq \frac{3\pi}{2}$

AND periodic function.

$\therefore$



2

(8)

(6) a) (i) $\sqrt[3]{abc} \leq \frac{a+b+c}{3}$ } $a+b+c = 1$

$\therefore \quad \leq \frac{1}{3}$

$\therefore abc \leq \frac{1}{27}$

$\therefore \frac{1}{abc} \geq 27$ (QED) (1)

(ii) $\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \sqrt[3]{\frac{1}{a} \times \frac{1}{b} \times \frac{1}{c}}$

$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$

$\geq 3\sqrt[3]{27}$

ie $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ (QED) (2)

a) (iii)

$(1-a)(1-b)(1-c)$

$= 1-(a+b+c)+(bc+ca+ab)-abc$

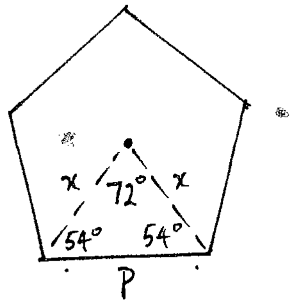
$= (bc+ca+ab)-abc$

$= abc\left\{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1\right\}$

$\geq abc\{9-1\}$

$\therefore (1-a)(1-b)(1-c) \geq 8abc$ (QED) (2)

b) (i)



Area of pentagon $= 5 \times \frac{1}{2} x^2 \sin 72°$

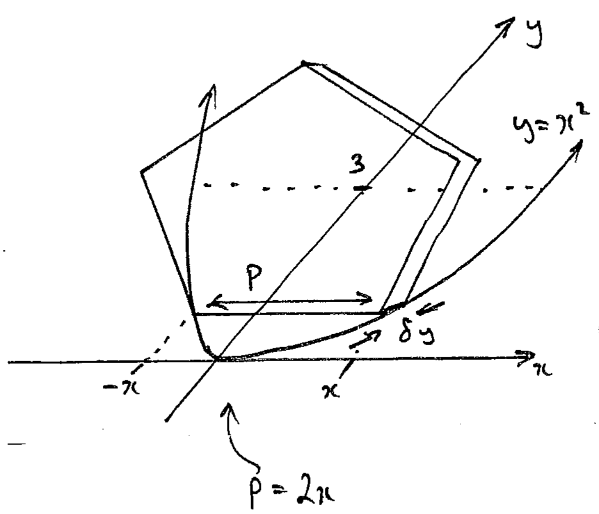
and $\frac{x}{\sin 54°} = \frac{p}{\sin 72°}$

$\therefore \text{Area} = \frac{5}{2} \times \left(\frac{p \sin 54°}{\sin 72°}\right)^2 \times \sin 72°$

$\therefore \text{Area} = \frac{5}{2} p^2 \frac{\sin^2 54°}{\sin 72°}$ (QED) (1)

(ii)

Sketch:



$A(x) = \frac{5}{2}(2x)^2 \frac{\sin^2 54°}{\sin 72°}$

$\therefore A(y) = 10y \frac{\sin^2 54°}{\sin 72°}$

$\therefore V = \lim_{\delta y \to 0} \sum_a^b 10y \frac{\sin^2 54°}{\sin 72°} \delta y$

$= 10 \frac{\sin^2 54°}{\sin 72°} \int_0^3 y \, dy$

$= 5 \frac{\sin^2 54°}{\sin 72°} [y^2]_0^3$

$= 45 \frac{\sin^2 54°}{\sin 72°}$ $(= 30.968\ldots)$

$\therefore V = 31.0 \text{ u}^3$ (1dp) (3)

c) (i) $\delta V = \pi\{(x+\delta x-1)^2-(x-1)^2\}2y$

$= 2\pi y\{2(x-1)+\delta x\}\delta x$

$= 4\pi(x-1)y\,\delta x$ (δx^2 negligible)

$= 4\pi(x-1)\sqrt{1-(x-2)^2}\,\delta x$

$\therefore V = \lim_{\delta x \to 0} 4\pi \sum_a^b (x-1)\sqrt{1-(x-2)^2}\,\delta x$ (3)

$= 4\pi \int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$ (QED)

(ii) let $x-2=\sin\theta$ $\therefore dx = \cos\theta\, d\theta$

at $x=1$ $\sin\theta=-1$ $\therefore \theta=-\frac{\pi}{2}$

$x=3$ $\sin\theta=1$ $\therefore \theta=\frac{\pi}{2}$

$\therefore V = 4\pi\int_{-\pi/2}^{\pi/2}(1+\sin\theta)\sqrt{1-\sin^2\theta}\cos\theta\, d\theta$

$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin\theta)\cos^2\theta\, d\theta$

$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta) + \sin\theta\cos^2\theta\, d\theta$

$= 4\pi\left[\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)-\frac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 4\pi\left(\frac{\pi}{4}+0-0+\frac{\pi}{4}+0-0\right)$

$= 2\pi^2 \text{ u}^3$ (3)

⑦ a) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{2x}{a^2} - \frac{2y}{b^2} y' = 0$

$\therefore y' = \frac{b^2 x}{a^2 y}$

$= \frac{b^2 \times a\sec\theta}{a^2 \times b\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ Eqn of tangent:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$\therefore a\tan\theta\, y - ab\tan^2\theta = b\sec\theta\, x - ab\sec^2\theta$

$\therefore b\sec\theta\, x - a\tan\theta\, y = ab(\underbrace{\sec^2\theta - \tan^2\theta}_{1}) = ab$

$\therefore \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$ (QED) ②

(ii) Perpendicular dist. from $(ae, 0)$ to circle tangent = radius

$\therefore \frac{|e\sec\theta + 0 - 1|}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}} = a\sqrt{e^2+1}$

$LHS = \frac{|e\sec\theta - 1|}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\sec^2\theta - 1}{b^2}}}$

$= \frac{|e\sec\theta - 1|}{\sqrt{\frac{\sec^2\theta(a^2+b^2)}{a^2b^2} - \frac{1}{b^2}}}$

but $b^2/a^2 + 1 = e^2 \therefore a^2 + b^2 = a^2e^2$

$= \frac{|e\sec\theta - 1|}{\sqrt{\frac{\sec^2\theta(a^2e^2)}{a^2b^2} - \frac{1}{b^2}}}$

$= \frac{|e\sec\theta - 1|}{\sqrt{\frac{e^2\sec^2\theta - 1}{b^2}}}$

$\therefore LHS^2 = RHS^2$

$\therefore a^2(e^2+1) = \frac{b^2(e\sec\theta - 1)^2}{e^2\sec^2\theta - 1}$

$\therefore \frac{a^2}{b^2}(e^2+1) = \frac{e\sec\theta - 1}{e\sec\theta + 1}$

but $\frac{a^2}{b^2} = \frac{1}{e^2-1}$

$\therefore \frac{e^2+1}{e^2-1} = \frac{e\sec\theta - 1}{e\sec\theta + 1}$

$\therefore e^3\sec\theta + e\sec\theta + e^2 + 1 = e^3\sec\theta - e\sec\theta - e^2 + 1$

$\therefore 2e\sec\theta = -2e^2$

$\therefore \sec\theta = -e$ (QED) ③

(iii) Coordinates of P and Q are $(a\sec\theta, b\tan\theta)$

$\therefore$ if $\sec\theta = -e$ then $\sec^2\theta = e^2$

$\therefore \tan^2\theta + 1 = e^2$

$\therefore \tan\theta = \pm\sqrt{e^2-1}$

$\therefore P, Q$ are $(-ae, \pm b\sqrt{e^2-1})$
i.e. on latus rectum ①

(iv) Tangents: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

$\therefore$ Common tangents:

$-\frac{xe}{a} - \frac{y(\pm\sqrt{e^2-1})}{b} = 1$

$\therefore -xe - y\left(\frac{\pm a\sqrt{e^2-1}}{b}\right) = a$

but $1 = \frac{a^2}{b^2}(e^2-1)$

$\therefore xe \pm y + a = 0$

or $y = \pm(xe + a)$ ①

a) (iv) continued:

solve simultaneously with circle equation:

ie with: $y = \pm\sqrt{a^2(e^2+1) - (x-ae)^2}$

$\therefore (xe+a)^2 = a^2(e^2+1) - (x-ae)^2$

$\therefore x^2e^2 + 2axe + a^2 = a^2e^2 + a^2 - x^2 - 2axe - a^2e^2$

$\therefore x^2(e^2-1) = 0$

$\therefore x = 0$

for $x=0$ $y = \pm a$

$\therefore$ contact pts: $(0, \pm a)$ ①

b) (i) Prove: $\dfrac{\sin(A+B) - \sin(A-B)}{2\sin B} = \cos A$

$LHS = \dfrac{\sin A\cos B + \cos A\sin B - \sin A\cos B + \cos A\sin B}{2\sin B}$

$= \dfrac{2\cos A\sin B}{2\sin B}$

$= \cos A$

$= RHS$ (QED) ②

(ii) Letting $A = (2n-1)x$, $B = x$, then:

$\cos(2n-1)x = \dfrac{\sin 2nx - \sin 2(n-1)x}{2\sin x}$

$= \dfrac{\sin 2nx}{2\sin x} - \dfrac{\sin 2(n-1)x}{2\sin x}$

$\therefore \cos x + \cos 3x + \cos 5x + \dots + \cos(2n-3)x + \cos(2n-1)x$

$= \left(\dfrac{\sin 2x}{2\sin x} - \dfrac{\sin 0}{2\sin x}\right) + \left(\dfrac{\sin 4x}{2\sin x} - \dfrac{\sin 2x}{2\sin x}\right)$
$+ \left(\dfrac{\sin 6x}{2\sin x} - \dfrac{\sin 4x}{2\sin x}\right) + \dots + \left(\dfrac{\sin 2(n-1)x}{2\sin x} - \dfrac{\sin 2(n-2)x}{2\sin x}\right)$
$+ \left(\dfrac{\sin 2nx}{2\sin x} - \dfrac{\sin 2(n-1)x}{2\sin x}\right)$

$= \dfrac{\sin 2nx}{2\sin x}$ (QED) ③

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx$

$= 2\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\times 4x}{2\sin x}\,dx$ (ie. $n=4$)

$= 2\displaystyle\int_0^{\frac{\pi}{2}} (\cos x + \cos 3x + \cos 5x + \cos 7x)\,dx$

(from (ii))

$= 2\left[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x\right]_0^{\frac{\pi}{2}}$

$= 2\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}\right)$

$= \dfrac{152}{105}$ ②

⑧ a) (i) In $\triangle ADB$: letting $BD = x$:

$\dfrac{x}{\sin\theta} = \dfrac{AB}{\sin ADB}$

In $\triangle ADC$: $DC = 1-x$:

$\dfrac{1-x}{\sin 2\theta} = \dfrac{AC}{\sin ADC}$

but $\sin ADC = \sin(180° - \angle ADB) = \sin ADB$

and $AB = AC$

$\therefore \dfrac{AC}{\sin ADC} = \dfrac{AB}{\sin ADB}$

$\therefore \dfrac{x}{\sin\theta} = \dfrac{1-x}{\sin 2\theta}$

$\therefore \dfrac{2\sin\theta\cos\theta}{\sin\theta} = \dfrac{1-x}{x}$ ④

$\therefore \cos\theta = \dfrac{1-x}{2x}$ (QED)

(ii) $0° < 3\theta < 180°$

$\therefore 0° < \theta < 60°$

$\therefore 1 > \cos\theta > \frac{1}{2}$

$\therefore 1 > \dfrac{1-x}{2x} > \dfrac{1}{2}$ $(x > 0)$ ②

$2x > 1-x > x$

$3x > 1 > 2x$ (QED)

$x > \frac{1}{3} ; x < \frac{1}{2} \Rightarrow \therefore \frac{1}{3} < x < \frac{1}{2}$

⑧ b)(i) $\alpha+\beta+\gamma=0$

$\alpha\beta+\alpha\gamma+\beta\gamma=3p$

$\alpha\beta\gamma=-q$

$$\frac{\alpha\beta}{\gamma}+\frac{\alpha\gamma}{\beta}+\frac{\beta\gamma}{\alpha}=\frac{\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{\alpha\beta\gamma}$$

$$=-\frac{9p^2}{q}$$

AND:

$$\frac{\alpha\beta}{\gamma}\cdot\frac{\alpha\gamma}{\beta}+\frac{\alpha\beta}{\gamma}\cdot\frac{\beta\gamma}{\alpha}+\frac{\alpha\gamma}{\beta}\cdot\frac{\beta\gamma}{\alpha}=\alpha^2+\beta^2+\gamma^2$$

$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$$

$$=-6p$$

AND

$$\frac{\alpha\beta}{\gamma}\times\frac{\alpha\gamma}{\beta}\times\frac{\beta\gamma}{\alpha}=\alpha\beta\gamma$$

$$=-q$$

∴ equation is:

$$x^3+\frac{9p^2}{q}x^2-6px+q=0$$

(3)

(ii) If $\alpha\beta=\gamma$ then the root: $\frac{\alpha\beta}{\gamma}$ is 1

$$\therefore 1+\frac{9p^2}{q}-6p+q=0$$

$$\therefore q+9p^2-6pq+q^2=0$$

$$\therefore (3p-q)^2+q=0 \quad \text{(QED)}$$

(2)

c)(i) $\cos x=\cos(-x)$

$=\sin\left(\frac{\pi}{2}-(-x)\right)$ 1

$=\sin\left(\frac{\pi}{2}+x\right)$ (QED)

{ OR $\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\frac{\pi}{2}+\cos x\sin\frac{\pi}{2}$

$=\cos x$ }

For $n=1$: Prove -

$$\frac{d}{dx}\left(5\sin(x+\alpha)\right)=5\sin\left(x+\alpha+\frac{\pi}{2}\right)$$

LHS $=5\cos(x+\alpha)$

RHS $=5\sin\left((x+\alpha)+\frac{\pi}{2}\right)$ } from (i)

$=5\cos(x+\alpha)$

∴ true for $n=1$ ½

Assume true for $n=k$

i.e. $\frac{d^ky}{dx^k}=5\sin\left\{(x+\alpha)+\frac{k\pi}{2}\right\}$

Now prove true for $n=k+1$

i.e. prove:

$$\frac{d^{k+1}y}{dx^{k+1}}=5\sin\left\{(x+\alpha)+(k+1)\frac{\pi}{2}\right\}$$

LHS $=\frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$

$=\frac{d}{dx}\left(5\sin\left\{(x+\alpha)+\frac{k\pi}{2}\right\}\right)$ from assumption

$=5\cos\left\{(x+\alpha)+\frac{k\pi}{2}\right\}$

$=5\sin\left\{(x+\alpha)+\frac{k\pi}{2}+\frac{\pi}{2}\right\}$ from (i)

$=5\sin\left\{(x+\alpha)+(k+1)\frac{\pi}{2}\right\}$ 2

$=$ RHS

∴ True for $n=1$ and true for $n=k+1$ when true for $n=k$ ½

∴ True for all $n\geq 1$ (QED) (4)