Newington College

IN FIDE SCIENTIAM

HSC Trial Examination, 1999

12 MATHEMATICS
4 UNIT

HSC Trial Examination, 1999

Time allowed - Three hours

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- *All* questions may be attempted.
- *All* questions are of equal value.
- In every question, show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- A table of standard integrals is provided for your convenience. Approved silent calculators may be used.
- The answers to the eight questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc.
- Each bundle must show your Candidate's number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

QUESTION 1 (Start a new page.) (15 Marks)

(a) Evaluate (i) $\displaystyle\int_0^4 \frac{1}{\sqrt{x^2+9}}\,dx$ (5 marks)

(ii) $\displaystyle\int_2^4 \frac{dx}{x^2-2x+4}$.

(b) Find (i) $\displaystyle\int \frac{\left(\sqrt{x}-1\right)^6}{\sqrt{x}}\,dx$ (5 marks)

(ii) $\displaystyle\int e^{-x}\cos\frac{x}{2}\,dx$.

(c) (i) If $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n}x\,dx$ prove that (5 marks)

$$I_n + I_{n-1} = \frac{1}{2n-1},\ \text{for } n \geq 1.$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 x\,dx$.

QUESTION 2 (Start a new page.) (15 Marks)

(a) (i) If $Z = -1+\sqrt{3}i$, find $|Z|$ and arg Z. (4 marks)

(ii) Hence evaluate $\left(-1+\sqrt{3}i\right)^9$.

(b) (i) Express the value of $\left(-1+\sqrt{3}i\right)(1+i)$ in the form $a+ib$. (4 marks)

(ii) Hence, or otherwise, find the exact value of $\cos\frac{11\pi}{12}$.

(c) Graph the region in the complex plane for which (3marks)

$$2 < |z-1+2i| < 3.$$

(d) If $|z| < \frac{1}{2}$, show that $\left|(1+i)z^3+iz\right| < \frac{3}{4}$. (4 marks)

/ Q3...page 2

QUESTION 3 (Start a new page.) (15 Marks)

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

(i) If $P(x)$ has the zeros $a+bi$, $a-2bi$, where a and b are real, find the values of a and b. (7 marks)

(ii) Hence, find all the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two factors.

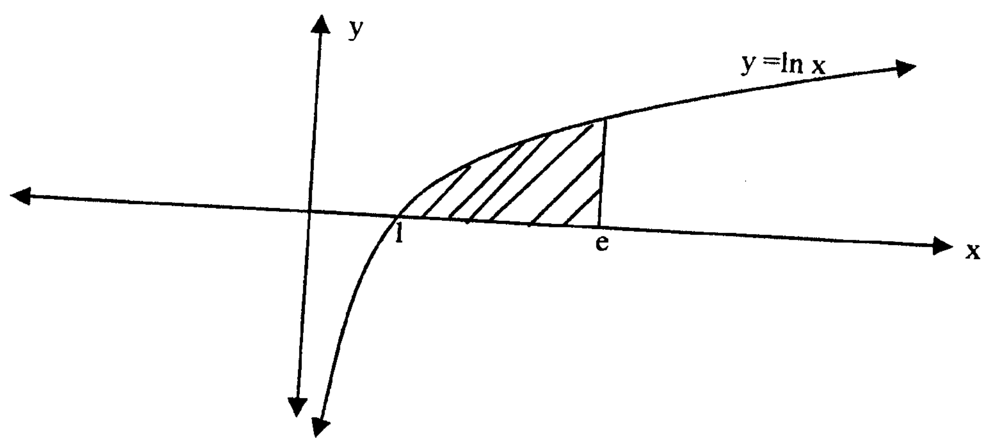
(b) (i) If α is a double root of $f(x)=0$, show that α is a root of $f'(x)=0$. (4 marks)

(ii) Show that if the equation $x^n + px + q = 0$ has a double root α (where α, p, q are real non-zero constants, and n is an integer with n $\geq$2), then:

$$\alpha = \frac{qn}{p(1-n)}.$$

(c) Using the method of cylindrical shells, find the volume generated by revolving the region bounded by $y = \ln x$, the x-axis and $1 \leq x \leq e$, about the y-axis.

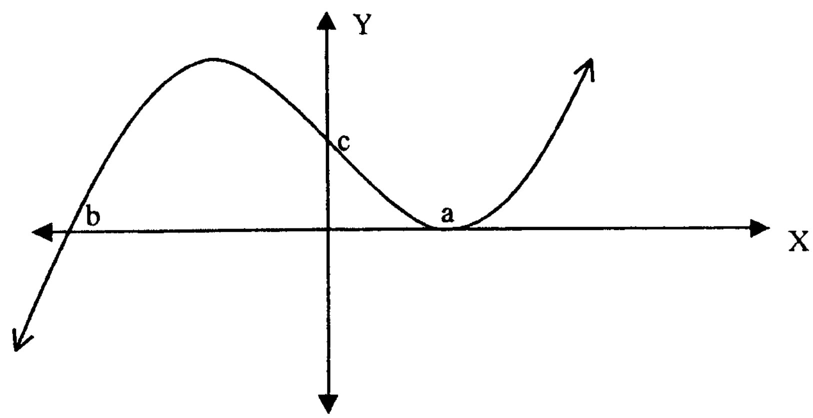
(4 marks)



/ Q4...page 3

QUESTION 4 (Start a new page.) (15 Marks)

(a)



The graph of the function $y = f(x)$ is sketched above. On separate number planes sketch the graphs of: (10 marks)

(i) $y = f(-x)$

(ii) $y^2 = f(x)$

(iii) $y = f(|x|)$

(iv) $y = \frac{1}{1 - f(x)}$.

(b) (i) Resolve $\frac{1}{(x+1)(x^2+4)}$ into partial fractions. (5 marks)

(ii) Use this result to show that

$$\int_0^2 \frac{1}{(x+1)(x^2+4)}\,dx = \frac{1}{10}\left(\frac{\pi}{4} + \log\frac{9}{2}\right).$$

/.Q5...page 4

QUESTION 5 (Start a new page.) (15 Marks)

(a) (i) Consider the rectangular hyperbola $xy = c^2$, where $c > 0$. Prove that the equation of the chord joining the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $0 < p < q$, is given by: $x + pqy = c(p+q)$. (8 marks)

(ii) The chord PQ intersects the x and y axes in A and B respectively.

Prove that AP=BQ.

(iii) Show that the area enclosed by the hyperbola $xy = c^2$ and the chord PQ is:

$$\frac{c^{2(}(q^2 - p^2)}{2pq} + c^2 \ln\frac{p}{q} \text{ square units.}$$

(b) A solid has as its base the area bounded by the hyperbola $16x^2 - 9y^2 = 144$ and the line x=6. Every cross-section of this solid perpendicular to the x-axis is an isosceles triangle of altitude 3.

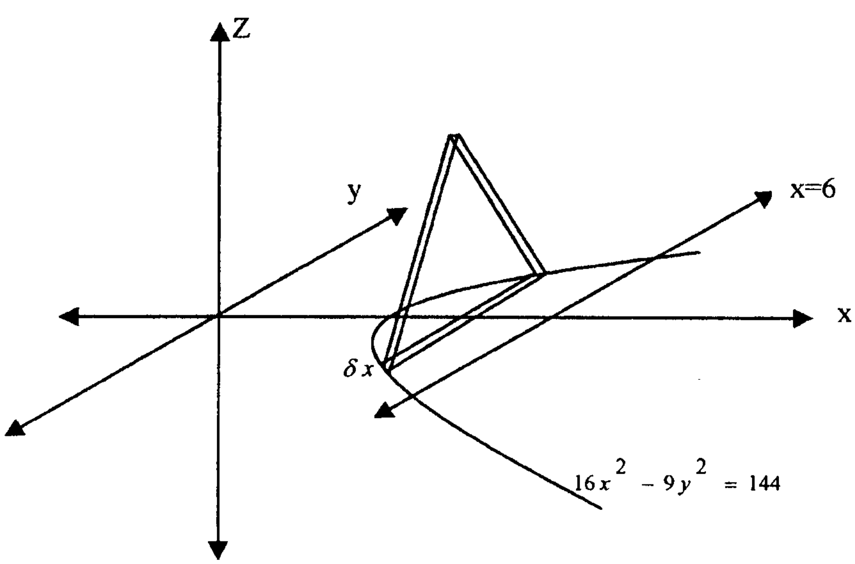
(7marks)

(i) Show that the volume V of the resulting solid is given by:

$$V = 4\int_3^6 \sqrt{x^2 - 9}\, dx$$

(ii) Hence, show that:

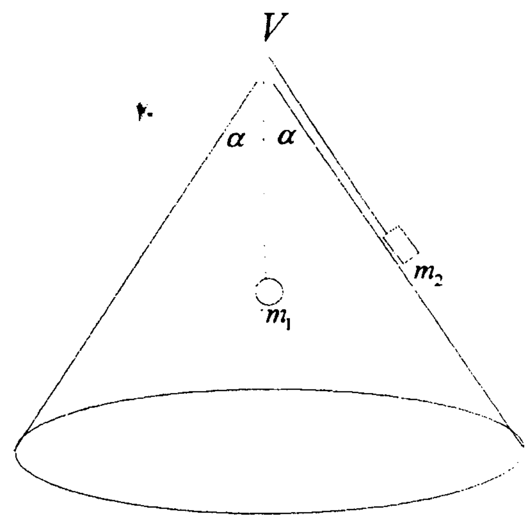
$$V = 36\sqrt{3} - 18\log(2 + \sqrt{3}).$$



/ Q6...page 5

QUESTION 6 (Start a new page.) (15 Marks)

(a) A hollow cone whose vertical angle is 2α is fixed with its axis vertical and with vertex V uppermost. A light inextensible string passes without friction through a small hole at V and carries a particle P_1 of mass m_1 kg at one end so that P_1 hangs vertically at rest inside the cone. The other end of the string carries a particle P_2 of mass m_2 kg, which moves in a horizontal circle at constant angular velocity ω on the smooth outer surface of the cone, at a vertical depth h metres below V.



(7 marks)

(i) Prove $m_2\left(h\omega^2 \sin^2\alpha + g\cos^2\alpha\right) = m_1 g\cos\alpha$.

(ii) Find the magnitude of the force exerted by the surface of the cone on P_2, and hence deduce that $h\omega^2 < g$.

(b) Two particles move in the same vertical line in a medium whose resistance, per unit mass, varies as the velocity. One particle is projected vertically upwards from the ground with initial velocity u, and starting at the same instant, the other particle falls from a height, h metres.

(8 marks)

(i) For the particle which is projected vertically upwards from the ground, show that the expression for its height(x) metres after a time (t) seconds is given by:

$$x = \frac{g+ku}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k},$$

where g is the acceleration due to gravity and k is a constant.

(ii) Assuming that the height of the falling particle is given by

$$h - \frac{gt}{k} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2},$$

prove that the particles meet after a time (T), where:

$$T = \frac{1}{k}\log\left(\frac{u}{u-kh}\right).$$

/ Q7...page 6

QUESTION 7 (Start a new page.) (15 Marks)

(a) A group of men and women is seated randomly around a circular table. What is the probability that none of the men are sitting next to each other if there are:

(4 marks)

(i) 3 men and 4 women?

(ii) n men and $n+1$ women?

(b) If a, b, c and d are positive real numbers, prove that:

(6 marks)

(i) $\frac{a+b}{2} \geq \sqrt{ab}$,

(ii) $(a+b+c+d)^2 \geq 4(ac+bc+bd+ad)$,

(iii) $(a+b+c+d)^2 \geq \frac{8}{3}(ab++ad+bc+cd+bd+ac)$.

(c) A sequence is defined by the relationship $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ where $a_1 = 1$ and n is a positive integer.

(5 marks)

(i) Show, using mathematical induction, that $\frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$.

(ii) Hence find the limiting value of a_n as n becomes large.

/ Q8...page 7

QUESTION 8 (Start a new page.) (15 Marks)

(a) Find all real x such that (3 marks)

$$3\sqrt{x(1-x)} < |x-2|$$

(b) If a curve is given by y = f(x), where f(x) has a continuous derivative in the open interval between x = a and x = b then the length is given by

$$\int_a^b \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \ .$$ (4 marks)

Use this result to prove that the circumference of a circle, with radius r, is equal to $2\pi r$.

(c) (i) Prove that for $t \neq -1$,

$$1-t+t^2-t^3+\dots\dots+t^{2n} = \frac{1}{1+t}+\frac{t^{2n+1}}{1+t}.$$

(8 marks)

(ii) Hence deduce that for $x > -1$

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots\dots\dots+\frac{x^{2n+1}}{2n+1}-\int_0^x \frac{t^{2n+1}}{1+t}\,dt\,.$$

(iii) For $0 \le x \le 1$, find

$$\lim_{n\to\infty}\int_0^x \frac{t^{2n+1}}{1+t}\,dt$$, giving reasons for your answer.

(iv) Hence find an infinite series converging to ln2.

END OF PAPER

A. Lait

Q1 a) i) $\displaystyle\int_0^4 \frac{1}{\sqrt{x^2+9}}\,dx = \left[\ln\left[x+\sqrt{x^2+9}\right]\right]_0^4$

$= \ln[4+5] - \ln 3$

$= \underline{\ln 3}$

ii) $\displaystyle\int_2^4 \frac{dx}{x^2-2x+4}$

$\displaystyle = \int_2^4 \frac{dx}{(x-1)^2+3} = \frac{1}{\sqrt3}\tan^{-1}\frac{(x-1)}{\sqrt3}\Big]_2^4$

$\displaystyle = \frac{1}{\sqrt3}\tan^{-1}\sqrt3 - \frac{1}{\sqrt3}\tan^{-1}\frac{1}{\sqrt3}$

$\displaystyle = \frac{1}{\sqrt3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt3}$

b) (i) $\displaystyle\int \frac{(\sqrt{x}-1)^6}{\sqrt{x}}\,dx$ let $u = \sqrt{x} = x^{\frac12}$

$\displaystyle du = \frac12 x^{-\frac12}dx = \frac{dx}{2\sqrt{x}}$

$\displaystyle = \int (u-1)^6 . 2du = \frac{2(u-1)^7}{7} + C$

$\displaystyle = \frac{2(\sqrt{x}-1)^7}{7} + C$

ii) $\displaystyle\int e^{-x}\cos\frac{x}{2}\,dx = e^{-x}.2\sin\frac{x}{2} - \int 2\sin\frac{x}{2}.-e^{-x}\,dx$

$\displaystyle = 2e^{-x}\sin\frac{x}{2} + 2\left[e^{-x}.-2\cos\frac{x}{2} - \int -2\cos\frac{x}{2}.-e^{-x}\right.$

$\displaystyle = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4\int e^{-\frac{x}{2}}\cos\frac{x}{2}\,dx.$

Q1 c) (i) $I_n = \int_0^{\pi/4} \tan^{2n} x \, dx = \int_0^{\pi/4} \tan^{2n-2} x \cdot \tan^2 x \, dx$

$$I_n = \int_0^{\pi/4} \tan^{2n-2} x \sec^2 x - I_{n-1}$$

$$I_n = \left.\frac{\tan^{2n-1} x}{2n-1}\right]_0^{\frac{\pi}{4}} - I_{n-1} = \frac{1}{2n-1} - I_{n-1}$$

ii) $I_3 = \int_0^{\pi/4} \tan^6 x \, dx$

$$= \frac{1}{5} - I_2$$

$$= \frac{1}{5} - \left[\frac{1}{3} - I_1\right]$$

$$= \frac{1}{5} - \frac{1}{3} + \frac{1}{1} - I_0$$

Now $I_0 = \int_0^{\pi/4} dx = \frac{\pi}{4}$

$\therefore I_3 = \frac{13}{15} - \frac{\pi}{4}$

2 a)(i) $Z = -1 + \sqrt{3}\,i$

$|Z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\arg Z = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ 2nd quad

$= \frac{2\pi}{3}$

(ii) $(-1 + i\sqrt{3})^9 = \left(2 \operatorname{cis} \frac{2\pi}{3}\right)^9 = 2^9 \operatorname{cis} 6\pi$

$= 512$

b) (i) $(-1+\sqrt{3}i)(1+i) = -1 - \sqrt{3} + i\sqrt{3} - i$

$= -(\sqrt{3}+1) + i(\sqrt{3}-1)$

(ii) $-1+\sqrt{3}i = 2 \operatorname{cis} \frac{2\pi}{3}$ $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

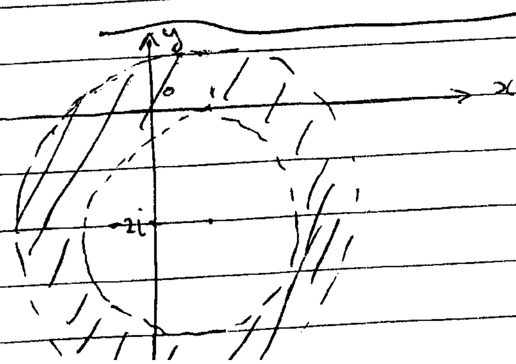
$(-1+\sqrt{3}i)(1+i) = 2\sqrt{2} \operatorname{cis} \frac{2\pi}{3} \operatorname{cis} \frac{\pi}{4}$

$= 2\sqrt{2} \operatorname{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$= 2\sqrt{2} \operatorname{cis} \frac{11\pi}{12}$

$\therefore \cos \frac{11\pi}{12} = -\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ from ↗

c)



Q2 d) $|z| < \frac{1}{2}$

$$|(1+i)z^3 + iz| \leq |(1+i)||z^3| + |i||z| \quad \text{from } \triangle$$

$$\leq \sqrt{2} \cdot \frac{1}{8} + \frac{1}{2}$$

Now $\frac{1}{4} > \frac{\sqrt{2}}{8}$

$$\therefore LHS \leq \frac{\sqrt{2}}{8} + \frac{1}{2} < \frac{3}{4}$$

Q3 (i) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

zeros $a + bi$, $a - 2bi$ $\quad a, b$ real.

Since coeffs are real then $a - bi$ & $a + 2bi$ are the conjugate roots

$\therefore$ roots are

$$a + ib, \quad a - ib, \quad a + 2ib, \quad a - 2ib$$

$$a + ib + a - ib + a + 2bi + a - 2bi = 4$$

$$\therefore a = 1$$

$$(a+ib)(a-ib)(a+2bi)(a-2bi) = (a^2 - b^2)(a^2 - 4b^2) = 10$$

$$\therefore (1 - b^2)(1 - 4b^2) = 10$$

$$4b^4 - 5b^2 - 9 = 0$$

$$(4b^2 - 9)(b^2 + 1) = 0$$

$$\therefore b = \pm \frac{3}{2} \qquad b^2 + 1 \neq 0, \ b \text{ real}.$$

$$a = 1, \quad b = \frac{3}{2}$$

3 a) (ii) zeros of $P(x)$ are

$1 \pm i \; , \; 1 \pm 2i$

$$P(x) = (x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$$
$$= [(x-1)^2+1][(x-1)^2+4]$$
$$= [x^2-2x+2][x^2-2x+5]$$

(i) Let $f(x) = (x-\alpha)^2 g(x)$ $\qquad g(x)$ poly

$$f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$$
$$= (x-\alpha)\underbrace{[2g(x) + (x-\alpha)g'(x)]}_{\text{poly}}$$

$\therefore f'(\alpha) = 0$

hence α is a root of $f'(x) = 0$

) Let $x^n + px + q = 0$ have a double root, α.

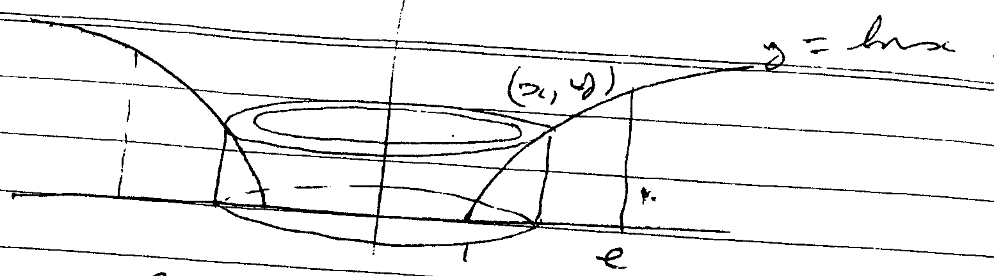
$\therefore nx^{n-1} + p = 0$ has a root α too.

hence $\alpha^n + p\alpha + q = 0$ $\quad$ & $n\alpha^{n-1} + p = 0$

$$\therefore \alpha^{n-1} = \frac{-p}{n}$$

$$\alpha^n + p\alpha + q = \alpha \cdot \alpha^{n-1} + p\alpha + q = 0$$
$$= \alpha \cdot \frac{-p}{n} + p\alpha + q = 0$$
$$p\alpha\left[1 - \frac{1}{n}\right] + q = 0$$
$$\alpha = \frac{qn}{p(1-n)}$$

Q 3 c)



$$\delta V = \pi x^2 y - \pi(x-\delta x)^2 y$$

$$= \pi y\left[x^2 - (x^2 - 2x\,\delta x + \delta x^2)\right]$$

$$\approx \pi y \cdot 2x\,\delta x \qquad \delta x^2 \text{ negligible } \delta x <$$

$$V = \sum_{n=1}^{\infty} \pi y\, 2x\,\delta x = \sum_{n=1}^{\infty} 2\pi(\ln x)\, x\,\delta x$$

$$V = \int_1^e 2\pi y\, x\, dx$$

$$= \int_1^e 2\pi x(\ln x)\, dx$$

$$= 2\pi\left[\ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx\right]$$

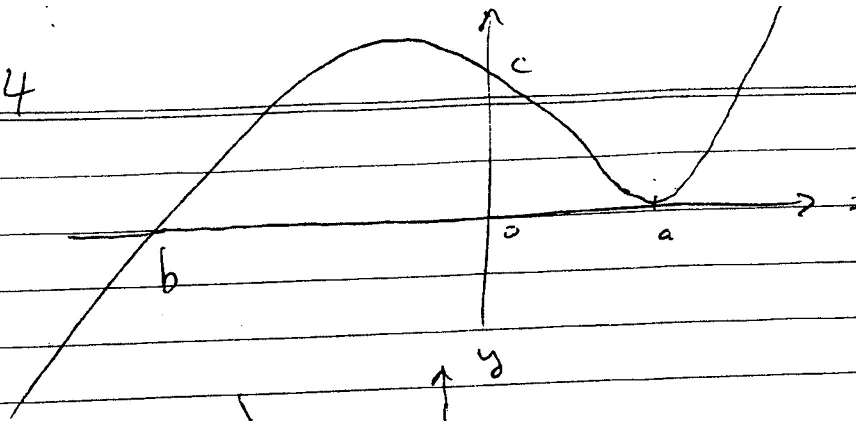
$$= 2\pi\left[\ln x \cdot \frac{x^2}{2} - \frac{x^2}{4}\right]_1^e$$

$$= 2\pi\left[\frac{e^2}{2} - \frac{e^2}{4}\right] - 2\pi\left[0 - \frac{1}{4}\right]$$

$$= 2\pi\left[\frac{e^2}{4} + \frac{1}{4}\right]$$

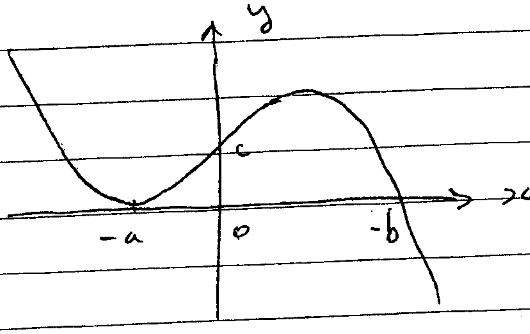
$$= \frac{\pi}{2}(e^2+1)$$

Q4



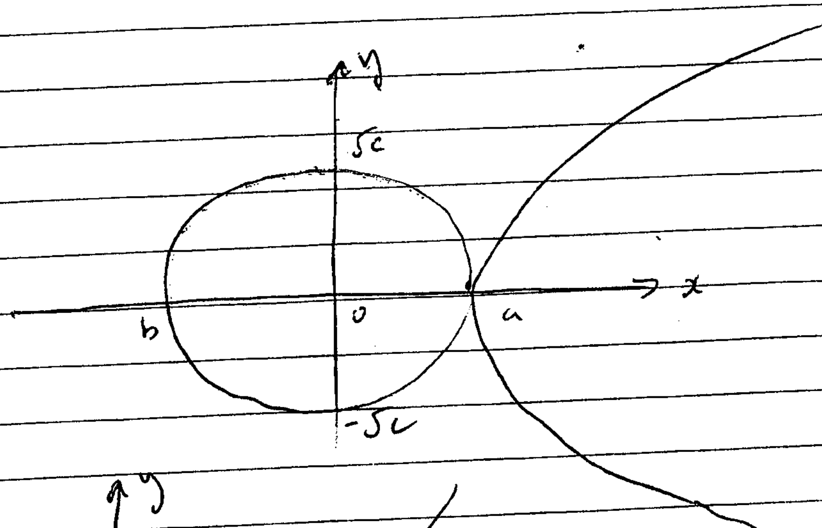
$y = f(x)$

(i)

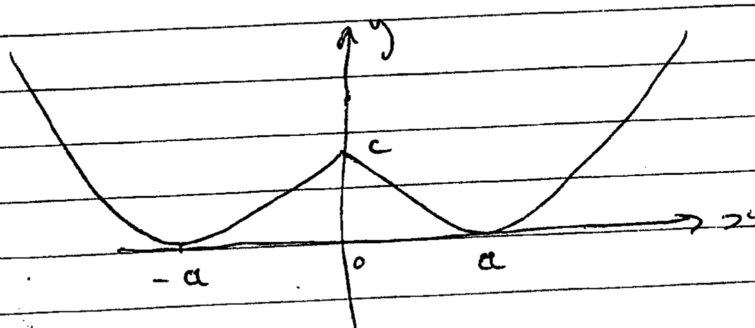


$f(-x)$

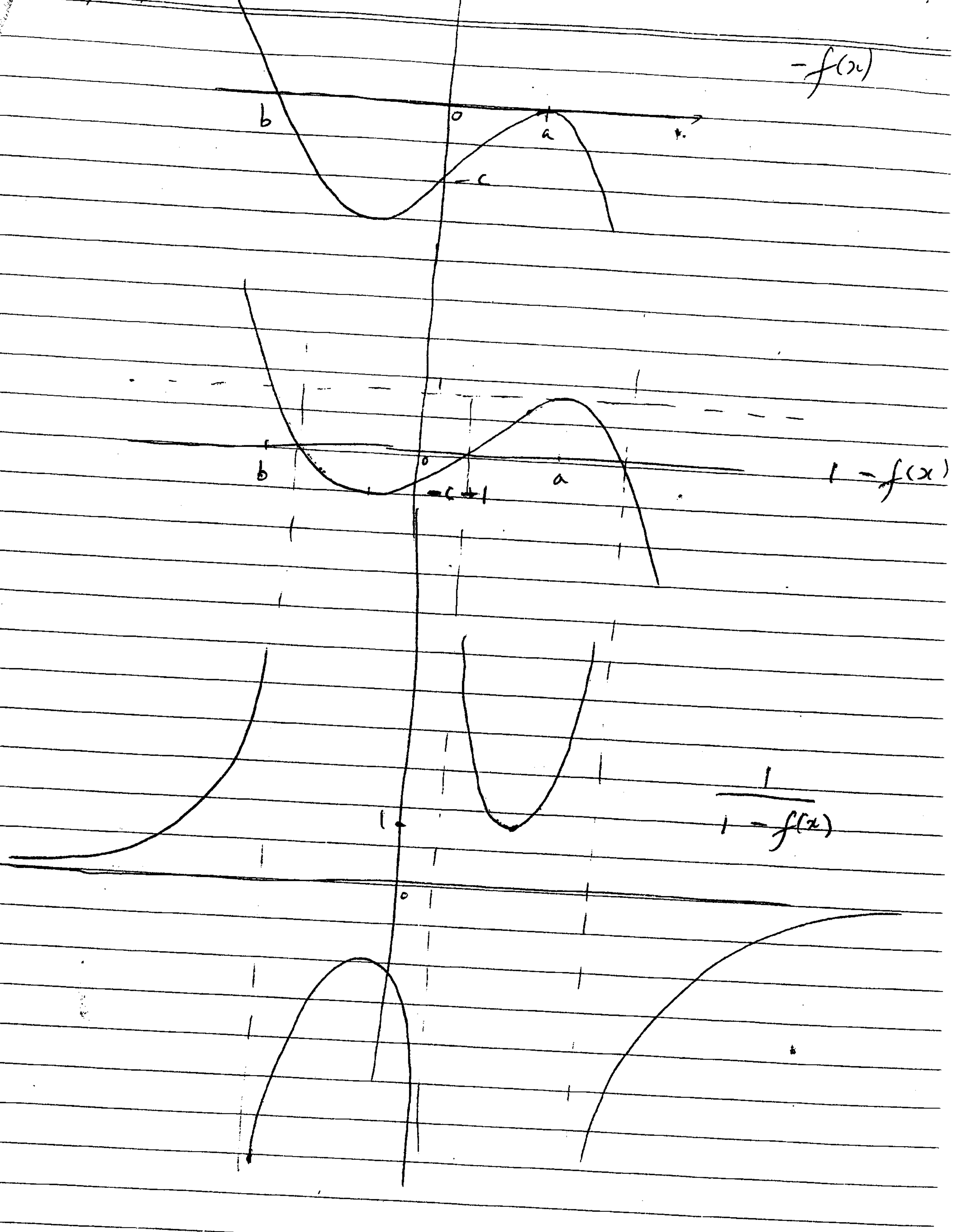
(ii) $y^2 = f(x)$



(iii)



Q4a) iv)



Q4 b) i) $\dfrac{1}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4}$

$$1 \equiv A(x^2+4) + (x+1)(Bx+C)$$

$x=-1$, $\quad 1 = 5A \quad \therefore A = \dfrac{1}{5}$

$x=0$, $\quad 1 = \dfrac{1}{5}.4 + C \quad \therefore C = +\dfrac{1}{5}$

coeff x^2 $\quad 0 = A + B \quad \therefore B = -\dfrac{1}{5}$

$$\therefore \frac{1}{(x+1)(x^2+4)} = \frac{1}{5(x+1)} - \frac{1}{5}\left(\frac{x-1}{x^2+4}\right)$$

$$\therefore \int \frac{1}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int \frac{dx}{x+1} - \frac{1}{5}\int \frac{x-1}{x^2+4}\,dx$$

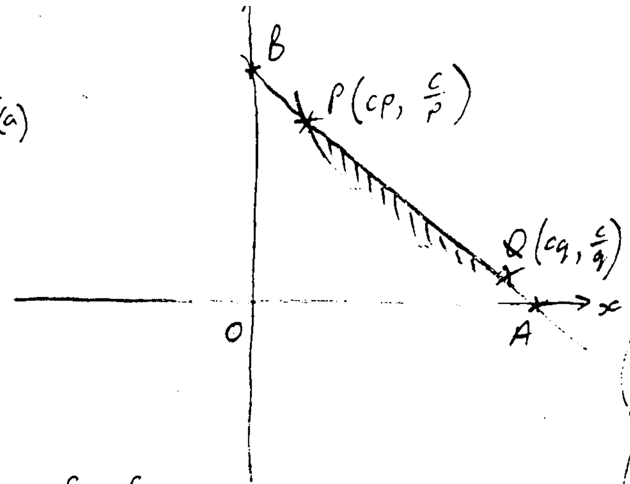
$$= \frac{1}{5}\ln(x+1) - \frac{1}{10}\ln(x^2+4) + \frac{1}{5}.\frac{1}{2}\tan^{-1}$$

$$= \left(\frac{1}{5}\ln 3 - \frac{1}{10}\ln 8 + \frac{1}{10}.\frac{\pi}{4}\right) - \left(0 - \frac{1}{10}\ln 4 - 0\right)$$

$$= \frac{1}{10}\left[\ln 9 - \ln 8 + \ln 4 + \frac{\pi}{4}\right]$$

$$= \frac{1}{10}\left[\ln\frac{9}{2} + \frac{\pi}{4}\right]$$

5/(a)



(i)

$$m_{PQ} = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$$

$$= \frac{-1}{pq}$$

Equation PQ:

$$y - \frac{c}{p} = \frac{-1}{pq}(x - cp)$$

$$pqy - cq = -x + cp$$

$$x + pqy = c(p+q)$$

(ii)

$$A\left(c(p+q), 0\right), \ B\left(0, \frac{c(p+q)}{pq}\right)$$

$$AP = \sqrt{\left(c(p+q) - cp\right)^2 + \left(0 - \frac{c}{p}\right)^2}$$

$$= \sqrt{c^2q^2 + \frac{c^2}{p^2}}$$

$$AQ = \sqrt{(0 - cq)^2 + \left(\frac{c(p+q)}{pq} - \frac{c}{q}\right)^2}$$

$$= \sqrt{c^2q^2 + \frac{c^2}{p^2}}$$

$$\therefore AP = AQ$$

(iii)

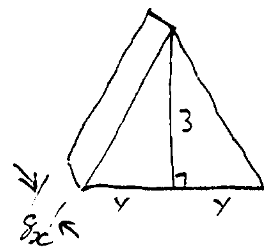
$$\text{Area} = \tfrac{1}{2}(cq - cp)\left(\frac{c}{p} + \frac{c}{q}\right) - \int_{cp}^{cq} \frac{c^2}{x}\,dx$$

$$= \frac{c(q-p).c(q+p)}{2pq} - \left[c^2 \ln x\right]_{cp}^{cq}$$

$$= \frac{c^2(q^2-p^2)}{\quad} - c^2 \ln \frac{q}{\quad}$$

$$= \frac{c^2(q^2-p^2)}{2pq} + c^2 \ln\left(\frac{p}{q}\right)$$

(b) Consider a typical slice



$$16x^2 - 9y^2 = 144$$

$$y = 0, \ x = \pm$$

$$y = \frac{\pm\sqrt{16x^2 - \ldots}}{3}$$

$$\delta V \doteq 3y\,\delta x$$

$$V \doteq \sum_{x=3}^{x=6} 3y\,\delta x$$

$$= \lim_{\delta x \to 0} \sum_{x=3}^{x=6} 3y\,\delta x$$

$$= \int_3^6 3y\,dx$$

$$= \int_3^6 \sqrt{16x^2 - 144}\,dx$$

$$= 4\int_3^6 \sqrt{x^2 - 9}\,dx$$

(ii) Let $x = 3\sec\theta$ $\qquad x = 3, \theta =$

$\therefore dx = 3\sec\theta\tan\theta\,d\theta$ $\qquad x = 6, \theta =$

$$\therefore V = 4\int_0^{\frac{\pi}{3}} \sqrt{9(\sec^2\theta - 1)}\ .\ 3\sec\theta\tan\theta\,d\ldots$$

$$= 36\int_0^{\frac{\pi}{3}} \tan\theta(\sec\theta\tan\theta)\,d\theta$$

<u>Now</u> $\displaystyle\int_0^{\frac{\pi}{3}} \tan\theta(\sec\theta\tan\theta)\,d\theta$

$$= \Big[\tan\theta.(\sec\theta)\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sec\theta.\sec^2\theta\,d\theta$$

5/(b) cont.

$$\int_0^{\frac{\pi}{3}} \sec\theta\tan^2\theta = \left(\sqrt{3}.2-0\right) - \int_0^{\frac{\pi}{3}} \sec\theta(1+\tan^2\theta)\,d\theta$$

$$= 2\sqrt{3} - \int_0^{\frac{\pi}{3}} \sec\theta\,d\theta - \int_0^{\frac{\pi}{3}} \sec\theta\tan^2\theta\,d\theta$$

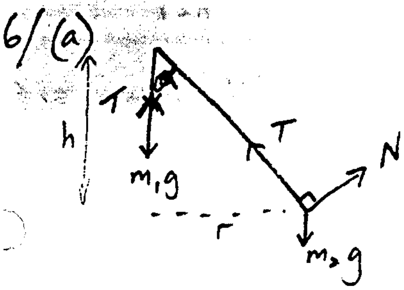
$$\therefore 2\int_0^{\frac{\pi}{3}} \sec\theta\tan^2\theta\,d\theta = 2\sqrt{3} - \Big[\log_e(\sec\theta+\tan\theta)\Big]_0^{\frac{\pi}{3}}$$

$$\therefore \int_0^{\frac{\pi}{3}} \sec\theta\tan^2\theta\,d\theta = \sqrt{3} - \frac{1}{2}\log_e\left(\frac{2+\sqrt{3}}{1+0}\right)$$

$$\therefore V = 36\left(\sqrt{3} - \frac{1}{2}\log_e(2+\sqrt{3})\right)$$

$$= 36\sqrt{3} - 18\log_e(2+\sqrt{3})$$

6/(a)



At P_1 $\quad T = m_1 g$ ——— ①

At P_2 Resolve Horizontally

$T\sin\alpha - N\cos\alpha = m_2 r\omega^2$ ——— ②

Resolve Vertically

$T\cos\alpha + N\sin\alpha = m_2 g$ ——— ③

Note $\quad \tan\alpha = \frac{r}{h} \quad \therefore r = h\tan\alpha$ ——— ④

② $\times \sin\alpha$ + ③ $\times \cos\alpha$

$T(\sin^2\alpha + \cos^2\alpha) = m_2 r\omega^2\sin\alpha + m_2 g\cos\alpha$,

sub in ①, ④

$m_1 g = m_2 h\tan\alpha\,\omega^2\sin\alpha + m_2 g\cos\alpha$

$\therefore m_1 g\cos\alpha = m_2 h\omega^2\sin^2\alpha + m_2 g\cos^2\alpha$

$= m_2(h\omega^2\sin^2\alpha + g\cos^2\alpha$

(ii) ② $\times \cos\alpha$ − ③ $\times \sin\alpha$

$-N(\cos^2\alpha + \sin^2\alpha) = m_2 h\tan\alpha\,\omega^2\cos\alpha - m_2 g$

$\therefore N = m_2 g\sin\alpha - m_2 h\omega^2\sin\alpha$

$= m_2\sin\alpha(g - h\omega^2)$

For N to exist,

$m_2\sin\alpha(g - h\omega^2) > 0$

$\therefore g - h\omega^2 > 0$

$g > h\omega^2$

(b)(i)

$\frac{R}{m} \propto v \quad \therefore \frac{R}{m} = kv$

$\therefore m\ddot{x} = -mg - mkv$

$\ddot{x} = -g - kv$

$\frac{dt}{dv} = \frac{-1}{g+kv}$

$t = -\int_U^v \frac{1}{g+kv}\,dv$

$= -\frac{1}{k}\Big[\log_e(g+kv)\Big]_U^v$

$= \frac{1}{k}\log_e\left(\frac{g+kU}{g+kv}\right)$

Make v the subject.

$e^{kt} = \frac{g+kU}{g+kv}$

$g+kv = e^{-kt}(g+ku)$

$\frac{dx}{dt} = v = \frac{1}{k}\left(e^{-kt}(g+ku) - g\right)$

$x = \frac{1}{k}\left(-\frac{1}{k}e^{-kt}(g+ku) - gt\right) + c$

$x=0, t=0 \quad \therefore c = \frac{1}{k^2}(g+ku)$

$\therefore x = -\frac{1}{k^2}e^{-kt}(g+ku) - \frac{gt}{k} + \frac{1}{k^2}(g+ku)$

$= -\frac{gt}{k} + \frac{1}{k^2}(g+ku)(1-e^{-kt})$

ii) Particles meet when

$x = h - \frac{gt}{k} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2}$

$h - \cancel{\frac{gt}{k}} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2} = -\cancel{\frac{gt}{k}} + \frac{1}{k^2}(g+ku)(1-e^{-kt})$

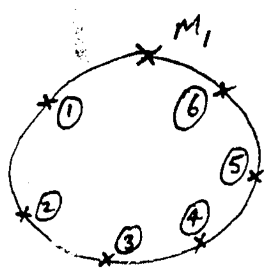
$h = \frac{u}{k}(1-e^{-kt})$

$e^{-kt} = \frac{u-kh}{u}$

$-kt = \ln\left(\frac{u-kh}{u}\right)$

$t = \frac{1}{k}\ln\left(\frac{u}{u-kh}\right)$

1/(a)



No. of ways of seating 2 men of table = $^6C_2$ = 15

No. of ways where they don't sit together: [②,④], [②,⑤], [③,⑤] = 3

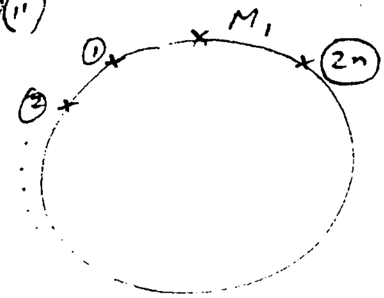
∴ probability = $\frac{3}{\ }$ = $\frac{1}{\ }$

or $N(S) = {}^6P_2 = 30$

$N(E) = 6$ $\left(\begin{matrix}(2,4)(2,5)(3,5)\\(4,2)(5,2)(5,3)\end{matrix}\right)$

$\therefore P(E) = \frac{1}{5}$

(ii)



The remaining (n−1) men can sit at ②, ④, …, 2(n−1) or ②, ④, …, 2n−1, or etc.

i.e. A gap (ie two women) can be n places.

No. of ways of seating (n−1) men = $^{2n}P_{n-1}$

No. of ways which men sit apart = $n(n-1)! = n!$

$\therefore$ Probability $= \dfrac{n!}{\frac{(2n)!}{(2n-(n-1))!}}$

$= \dfrac{n!(n+1)!}{(2n)!}$

(b) (i) $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$\therefore a - 2\sqrt{a}\sqrt{b} + b \geq 0$

$\frac{a+b}{2} \geq \sqrt{ab}$

b) cont.

ii) from (i)

$$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$$

$$\frac{a+b+c+d}{4} \geq \sqrt{\frac{ac+ad+bc+bd}{4}}$$

$$(a+b+c+d)^2 \geq 4(ac+bc+bd+ad) \quad — (1)$$

ii) Similarly

$$\frac{\frac{a+c}{2}+\frac{b+d}{2}}{4} \geq \sqrt{\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)}$$

$$\therefore (a+b+c+d)^2 \geq 4(ab+ad+bc+cd) \quad — (2)$$

$$\frac{\frac{a+d}{2}+\frac{b+c}{2}}{2} \geq \sqrt{\left(\frac{a+d}{2}\right)\left(\frac{b+c}{2}\right)}$$

$$(a+b+c+d)^2 \geq 4(ab+ac+bd+cd) \quad — (3)$$

(1) + (2) + (3)

$$3(a+b+c+d)^2 \geq 4(2ab+2ad+2bc+2cd+2bd+2ac)$$

$$\therefore (a+b+c+d)^2 \geq \frac{8}{3}(ab+ad+bc+cd+bd+ac)$$

c) Step 1 prove true for $n=1$

$$LHS = \frac{a_1-\sqrt{2}}{a_1+\sqrt{2}} = \frac{1-\sqrt{2}}{1+\sqrt{2}} \qquad = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{1-1}} = RHS$$

Step 2 Assume result true for $n=k$, k a positive intege

i.e. $\dfrac{a_k-\sqrt{2}}{a_k+\sqrt{2}} = \left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}$

Step 3 Prove result true for $n=k+1$

i.e. Prove $\dfrac{a_{k+1}-\sqrt{2}}{a_{k+1}+\sqrt{2}} = \left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k+1}}$

$$LHS = \frac{\frac{1}{2}\left(a_k+\frac{2}{a_k}\right)-\sqrt{2}}{\frac{1}{2}\left(a_k+\frac{2}{a_k}\right)+\sqrt{2}}$$

$$= \frac{(a_k)^2+2-2\sqrt{2}a_k}{(a_k)^2+2+2\sqrt{2}a_k}$$

$$= \left(\frac{a_k-\sqrt{2}}{a_k+\sqrt{2}}\right)^2$$

$$= \left(\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}\right)^2 \quad \text{from assumpt}$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}$$

$= RHS.$

Step 4

The result is true for $n=1$ and, if true for $n=k$, it is true for $n=k+1$.

$\therefore$ It is true for $n=1,2,3,\ldots$

8/(a) $3\sqrt{x(1-x)} < |x-2|$

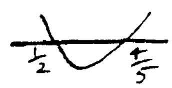
Note: $x(1-x) \geq 0$ 

$\therefore \ 0 \le x \le 1$

$9x(1-x) < (x-2)^2$

$x^2-4x+4-9x+9x^2 > 0$

$10x^2-13x+4 > 0$

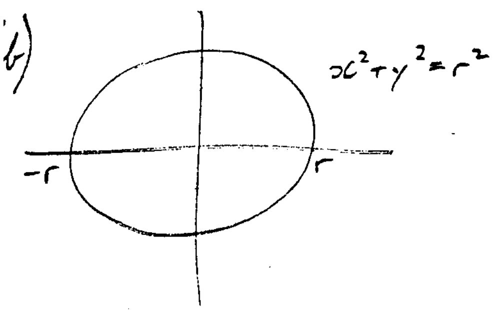
$(5x-4)(2x-1) > 0$ 

$\therefore x > \frac{4}{5}$ or $x < \frac{1}{2}$

but $0 \le x \le 1$

$\therefore$ solution: $0 \le x < \frac{1}{2}$ or $\frac{4}{5} < x \le 1$

(b)



$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$

length of semi-circle

$= \int_{-r}^{r} \left(1+\left(\frac{-x}{y}\right)^2\right)^{\frac{1}{2}} dx$

$= \int_{-r}^{r} \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx$

$= 2\int_{0}^{r} \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\, dx$

$= 2r\int_0^r \frac{dx}{\sqrt{r^2-x^2}}$

$= 2r\left[\sin^{-1}\frac{x}{r}\right]_0^r$

$= 2r . \frac{\pi}{2}$

$= \pi r$

$\therefore$ Circumference of circle $= 2\pi r$

(c) $\underbrace{1-t+t^2-t^3+\dots+t^{2n}}$

G.P. $a=1$, $r=-t$, $2n+1$ terms

$S_{2n+1} = \frac{1\left((-t)^{2n+1}-1\right)}{-t-1}$

$= \frac{1}{1+t} + \frac{\cancel{(-1)}^{2n+1} . t^{2n+1}}{\cancel{(-1)}(1+t)}$

$= \frac{1}{1+t} + \frac{t^{2n+1}}{1+t}$

(ii) $\int_0^x (1-t+t^2-t^3+\dots+t^{2n})\,dt = \int_0^x \left(\frac{1}{1+t} + \frac{t^{2}}{1+t}\right.$

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{x^{2n+1}}{2n+1} = \ln(1+x) + \int_0^x \frac{t^{2}}{1+t}$

$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}$

(iii) $\int_0^x \frac{t^{2n+1}}{1+t}\,dt < \int_0^x \frac{t^{2n+1}}{t}\,dt$

for $x > 0$

$\lim_{n\to\infty} \int_0^x \frac{t^{2n+1}}{t}\,dt$

$= \lim \left(x^{2n+1}\right) = 0$ for $0 < x$

8 (c) cont.

$$\therefore \lim_{n \to \infty} \int_0^x \frac{t^{2n+1}}{1+t} \, dt = 0$$

(iv) Let $x = 1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n+1}$$

for $n = 0, 1, 2, \dots$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$