Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) Use Integration by parts to find $\int xe^{-3x}\,dx$ **2**

(b) Use the substitution $x = \frac{2}{3}\sin\alpha$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$ **3**

(c) Use the table of standard integrals to help evaluate $\int \frac{dx}{\sqrt{x^2-4x+29}}$ **2**

(d) Evaluate that $\int_4^6 \frac{2dt}{(t-1)(t-3)}$ **4**

(e) Use the substitution $t = \tan\frac{x}{2}$ to show that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\frac{1}{3}$$ **4**

Question 2 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) For the complex number $z = 1 - \sqrt{3}i$, express each of the following in the form $a + bi$, (a, b are real numbers). **3**

(i) $\overline{z}$

(ii) z^2

(iii) $\dfrac{1}{z}$

(b) If $z_1 = 1 - 2i$; $z_2 = 2 + i$ and $z = \dfrac{z_1}{z_2}$ find :- **3**

i) $|z|$

ii) $\arg(z)$

(c) Prove that $|z|^2 = z\overline{z}$ for all complex numbers z . **2**

(d) If ω is a complex cube root of unity,

(i) Write down the value of $1 + \omega + \omega^2$. **1**

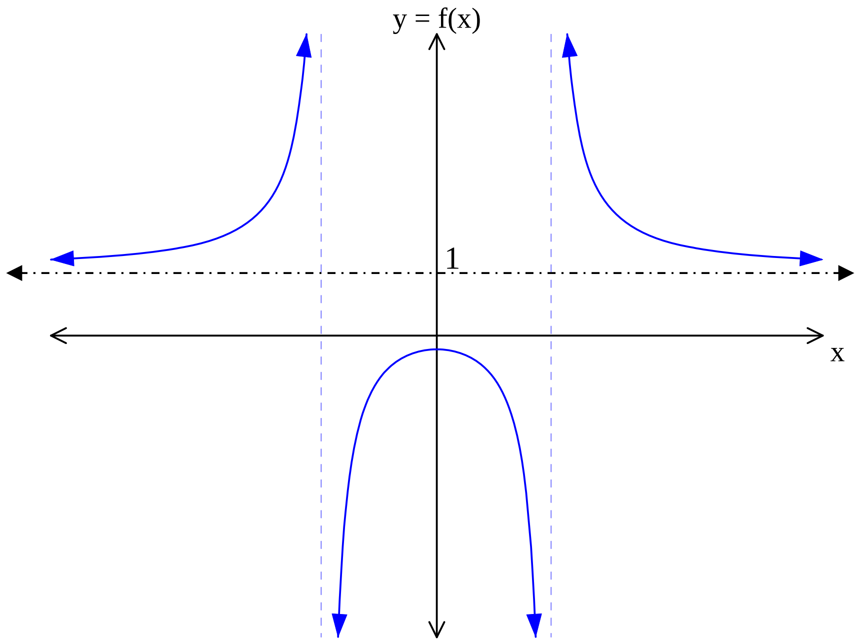
(ii) Simplify $\omega^4 + \omega^5 + \omega^6$. **1**

(e) Sketch on an Argand diagram the region in which z lies, showing all important features where $2 \le |z| \le 3$ and $\dfrac{\pi}{4} < \arg(z - i) < \dfrac{3\pi}{4}$. **2**

(f) P_1 and P_2 are points representing the complex numbers z_1 and z_2 on an Argand diagram. If OP_1P_2 is an isosceles triangle and angle P_1OP_2 is a right-angle, show that $z_1^{\,2} + z_2^{\,2} = 0$. **3**

Question 3 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) The diagram below shows the graph of the function $y = f(x)$ where $f(x) = \dfrac{1 + x^2}{x^2 - 9}$.



Draw a separate sketch of each of the following graphs.
Use about one third of a page for each graph. Show all significant features.

(i) $y = [f(x)]^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

(iv) Draw $y = f(x)$ and $y = \sqrt{f(x)}$ on the same number plane. **2**

(v) $|y| = f(x)$ **2**

(b) For the curve defined by $3x^2 + y^2 - 2xy - 8x + 2 = 0$,

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x - y - 4}{x - y}$. **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$. **3**

Question 4 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) The equation $x^3 + 2x^2 + bx - 16 = 0$ has roots α, β and γ such that $\alpha\beta = 4$.

(i) Show that $b = -20$. **2**

(ii) Find the equation with roots given by α^2, β^2 and γ^2 **2**

(b) Consider the polynomial $P(x) = (x - \alpha)^3 . Q(x)$, where $Q(x)$ is also a polynomial and α is a real zero of $P(x)$.

(i) Show that $P(\alpha) = P'(\alpha) = P''(\alpha) = 0$ **2**

(ii) Hence or otherwise, solve the equation
$8x^4 - 25x^3 + 27x^2 - 11x + 1 = 0$
given that it has a triple root. **2**

(c) (i) Show that the solutions of $z^6 + z^3 + 1 = 0$ are contained in the solutions of $z^9 - 1 = 0$. **2**

(ii) Sketch the nine solutions of $z^9 - 1 = 0$ on an Argand Diagram. (about one third of a page in size) **2**

(iii) Mark clearly on your diagram, the six roots $z_1, z_2, z_{3,} z_4, z_5, z_6$ of $z^6 + z^3 + 1 = 0$. **4**

(iv) Show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ can be given by **2**
$$2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$$

Question 5 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) For the curve with Cartesian equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, **5**

(i) find the eccentricity and the coordinates of a focus
(ii) find the equation of the corresponding directrix
(iii) hence write down the coordinates of a focus and the equation of the corresponding directrix for the curve with the Cartesian equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(b) Let $P(4\sec\theta, 3\tan\theta)$ be any point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

(i) Derive the equations of the tangent and normal at P and show that they are respectively: **4**

$$3x\sec\theta - 4y\tan\theta = 12 \text{ and } 4x\tan\theta + 3y\sec\theta = 25\sec\theta\tan\theta$$

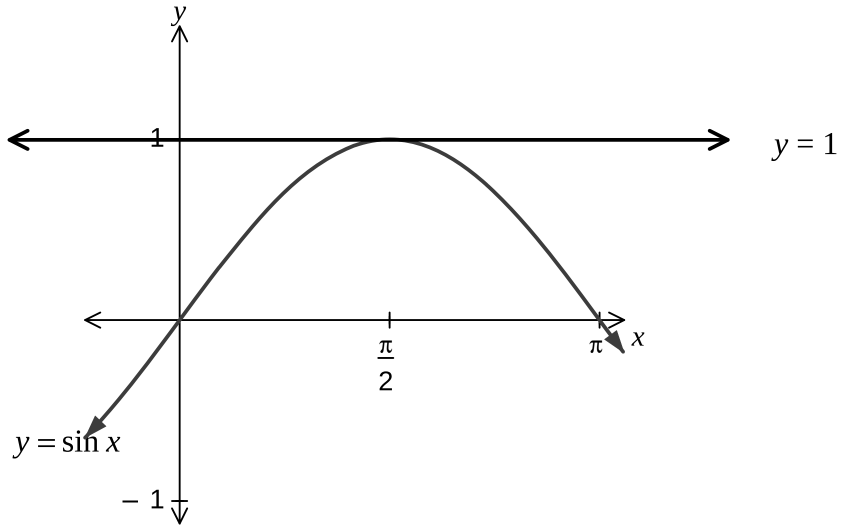
(ii) The tangent and normal at P meet the $y-axis$ at T and N respectively. Show that **2**

$T=(0,-3\cot\theta)$ and $N=(0,\frac{25}{3}\tan\theta)$.

(iii) Show that the circle with diameter NT passes through a focus. **4**

Question 6 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a)



The area defined by $y \geq \sin x$, $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq 1$ is rotated about the straight line $y = 1$.

(i) Copy the diagram above into your writing booklet and shade in the region defined by the simultaneous inequalities $y \geq \sin x$, $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq 1$. **1**

(ii) Find the total volume of the solid formed, by taking slices perpendicular to the axis of rotation. **3**

(b) The horizontal base of a solid is an ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **5**

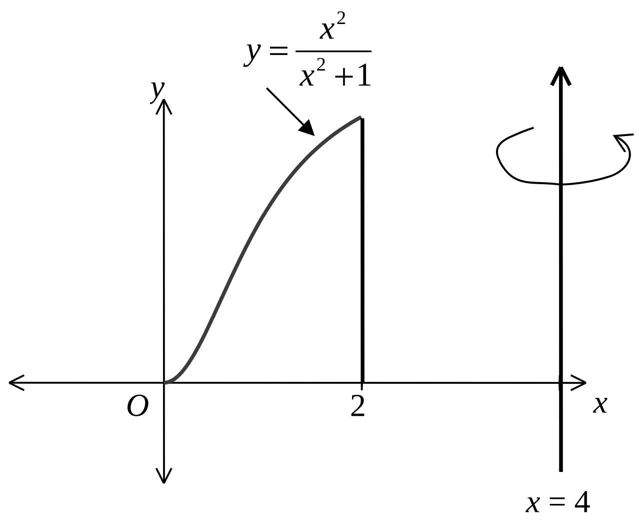
Vertical cross-sections taken perpendicular to the y axis are squares with one side in the horizontal base of the solid.

Find the volume of the solid formed in terms of a and b.

Question 6 continues on the next page …

Question 6 continued … **Marks**

(c) The region bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x axis and $0 \le x \le 2$, is rotated about the line $x = 4$ to form a solid.



(i) Using the method of cylindrical shells, explain why the volume δV of a typical shell distant x units from the origin and with thickness δx is given by **3**

$$\delta V = 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)\delta x .$$

(ii) Hence, find the total volume of the solid formed. **3**

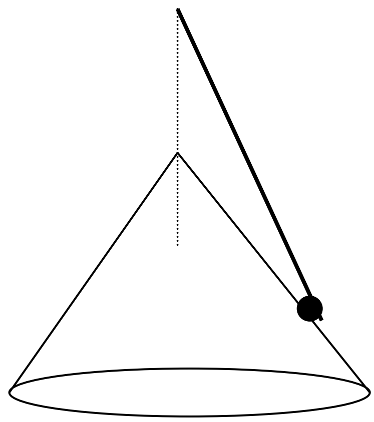
Question 7 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1 + v^2)$ where k is a constant and v is its speed at time t. **5**

(i) Show that the particle is brought to rest after a time $\frac{1}{k}\tan^{-1} u$.

(ii) Find an expression for the distance travelled by the particle in this time.

(b) A smooth circular cone, with its vertex up, and its axis vertical, has a semi-vertex angle of $60°$. A particle of mass 1 kg is attached by a light inelastic string from a point vertically above the vertex of the cone, and moves with constant speed $v\ m/s$ on the outer surface of the cone in a horizontal circle of radius 0.5 m. The string makes an angle of $30°$ with the vertical. Let the magnitude of the tension in the string be T newton, and let the magnitude of the reaction of the cone on the particle be R newton.



(i) Draw a diagram showing the forces acting on the particle, and the magnitude of the angles made by these forces with the vertical. **2**

(ii) By resolving forces in two directions write down equations of motion for the particle. **2**

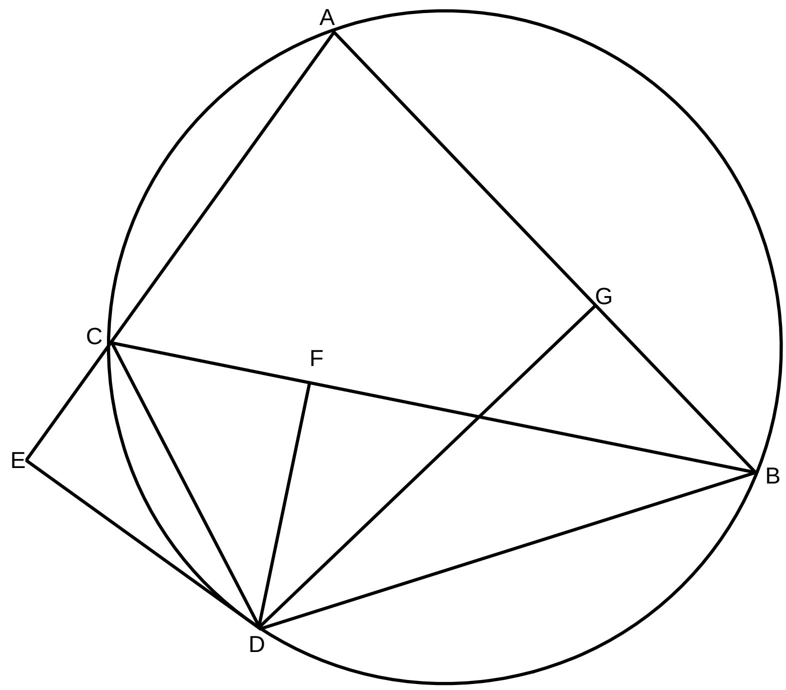
(iii) If $v = 1$, find the value of T and R correct to two decimal places. **2**

(iv) Find the maximum value of v in order that the particle remains in contact with the cone. **2**

(v) Deduce the maximum value that T can take if the particle is to remain in contact with the cone. **2**

Question 8 *(15 marks)* Use a SEPARATE writing booklet. **Marks**

(a)



The above diagram shows a triangle *ABC* inscribed in a circle with *D* a point on the arc *BC*. *DE* is perpendicular to *AC* produced, *DF* is perpendicular to *BC* and *DG* is perpendicular to *AB*.

Copy or trace this diagram into your writing booklet.

(i) Explain why *DECF* and *DFGB* are cyclic quadrilaterals. **2**

(ii) Show that the points *E*, *F* and *G* are collinear. **3**

Question 8 continues on the next page…

Question 8 continued…

Marks

(b) (i) Evaluate $\int_0^{\frac{\pi}{4}} \tan x \quad dx$. **2**

(ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \quad dx$, n = 0, 1, 2, 3, …, show that for n = 2, 3, 4, … **3**

$$I_n = \frac{1}{n-1} - I_{n-2}$$

(iii) Hence, evaluate I_5. **2**

(c) If $P(x) = x^m\left(b^n - c^n\right) + b^m\left(c^n - x^n\right) + c^m\left(x^n - b^n\right)$ where m and n are positive integers, show that $x^2 - (b+c)x + bc$ is a factor of $P(x)$. **3**

End of paper

EXT 2 TRIAL HSC 2005

Q1

a) $\int x e^{-3x}\,dx$

Let $u = x \quad v' = e^{-3x}$

$u' = 1 \quad v = -\frac{1}{3}e^{-3x}$ (1)

OR Let $u = e^{-3x} \quad v' = x$

$\int uv'\,dx = uv - \int u'v\,dx$

/2

$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$ (1)

b) $\int_0^{2/3}\sqrt{4-9x^2}\,dx$

Let $x = \frac{2}{3}\sin\alpha$

$\frac{dx}{d\alpha} = \frac{2}{3}\cos\alpha$.

$dx = \frac{2}{3}\cos\alpha\,d\alpha$ (1)

when $x = 0 \quad \alpha = 0$

$x = 2/3 \quad \alpha = \pi/2$.

$= \int_0^{\pi/2}\sqrt{4-9.\frac{4}{9}\sin^2\alpha}\,.\,\frac{2}{3}\cos\alpha\,d\alpha$

$= \frac{2}{3}\int_0^{\pi/2} 2\sqrt{\cos^2\alpha}\,.\,\cos\alpha\,d\alpha$.

$= \frac{4}{3}\int_0^{\pi/2}\cos^2\alpha\,d\alpha$.

$= \frac{4}{3}\int_0^{\pi/2}\frac{1}{2}(\cos 2\alpha + 1)\,d\alpha$ (1)

/3

$= \frac{2}{3}\left[\frac{1}{2}\sin 2\alpha + \alpha\right]_0^{\pi/2}$

(1)

$= \frac{\pi}{3}$

$\therefore \int_0^{2/3}\sqrt{4-9x^2}\,dx = \frac{\pi}{3}$.

c) $\int \frac{dx}{\sqrt{x^2-4x+29}}$

$= \int \frac{dx}{\sqrt{(x-2)^2+25}}$ (1) $/2$

$= \ln\left(x-2+\sqrt{x^2-4x+29}\right)$ (1)

d) $\int_4^6 \frac{2\,dt}{(t-1)(t-3)}$

$\frac{2}{(t-1)(t-3)} = \frac{A}{t-1} + \frac{B}{t-3}$ (1)

$2 = A(t-3) + B(t-1)$

Let $t=1$ $\quad t=3$ (1)

$2=-2A \quad 2=2B$

$A=-1 \quad B=1$ $/4$

$\therefore \int_4^6 \frac{2\,dt}{(t-1)(t-3)} = \int_4^6 \frac{-1}{t-1} + \frac{1}{t-3}\,dt$

$= \left[\ln\left(\frac{t-3}{t-1}\right)\right]_4^6$ (1)

$= \ln\frac{3}{5} - \ln\frac{1}{3}$

$= \ln\frac{9}{5}$ (1)

e) when $t=\tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$ (1)

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$dt = \frac{1}{2}(1+t^2)\,dx$ (1)

$\therefore dx = 2\,\frac{1}{1+t^2}\,dt$

$\int_0^{\pi/2} \frac{dx}{5+4\cos x} = \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot 2\,\frac{1}{(1+t^2)}\,dt$

$= 2\int_0^1 \frac{1}{5+5t^2+4-4t^2}\,dt$ $/4$

$= 2\int_0^1 \frac{1}{9+t^2}\,dt$ (1)

$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1$ (1)

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

Q2

a) i/ $\bar{z} = 1+\sqrt{3}i$ (1) ans

ii/ $z^2 = (1-\sqrt{3}i)^2$

$= 1-2\sqrt{3}i + 3i^2$

$= -2-2\sqrt{3}i$ (1) ans

iii/ $\frac{1}{z} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{1+\sqrt{3}i}{1+3}$

$= \frac{1}{4} + \frac{\sqrt{3}}{4}i$ (1) ans

b) $|z_1| = \sqrt{5}$ $\quad |z_2| = \sqrt{5}$

$\arg z_1 = \tan^{-1}(2)$ $\quad \arg z_2 = \tan^{-1}\left(\frac{1}{2}\right)$

i/ $|z| = \frac{|z_1|}{|z_2|}$

$= 1$ (1) ans

ii/ 

$\arg(z) = \arg(z_1) - \arg(z_2)$

$= \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$

Let $\alpha = \tan^{-1} 2$ $\quad \beta = \tan^{-1}\frac{1}{2}$

$\therefore 2 = \tan\alpha$ $\quad \tan\beta = \frac{1}{2}$ (1) working

$\tan(\arg(z)) = \frac{2-\frac{1}{2}}{1-2.\frac{1}{2}}$

$= \frac{1\frac{1}{2}}{0}$

$\therefore \arg z = -\pi/2$ (1) ans

c) Let $z = a + ib$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$

LHS

$|z| = \sqrt{a^2 + b^2}$

$\therefore |z|^2 = a^2 + b^2$ ① LHS

RHS

$z\bar{z} = (a + ib)(a - ib)$

$= a^2 + aib - aib - i^2b^2$

$= a^2 + b^2$

$\therefore |z|^2 = z\bar{z} \quad \forall z \in \mathbb{C}$. ① RHS

d) i) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$\therefore (\omega - 1)(1 + \omega + \omega^2) = 0$

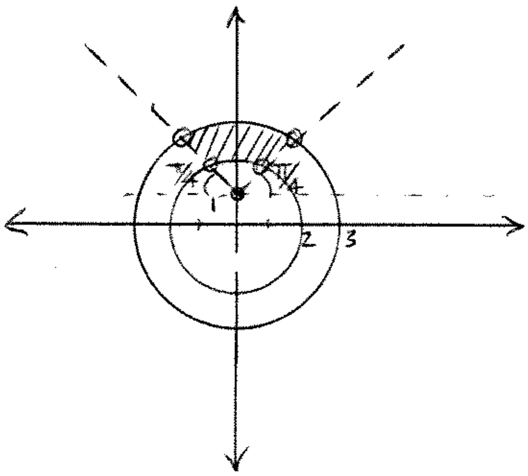
$\therefore 1 + \omega + \omega^2 = 0$ if $\omega \neq 1$. ① statement of condition.

$= 3$ if $\omega = 1$

ii) $\omega^4 + \omega^5 + \omega^6 = \omega^4(1 + \omega + \omega^2)$

$= 0$ or 3ω? ①

e)



① circular region w/ radii + solid lines

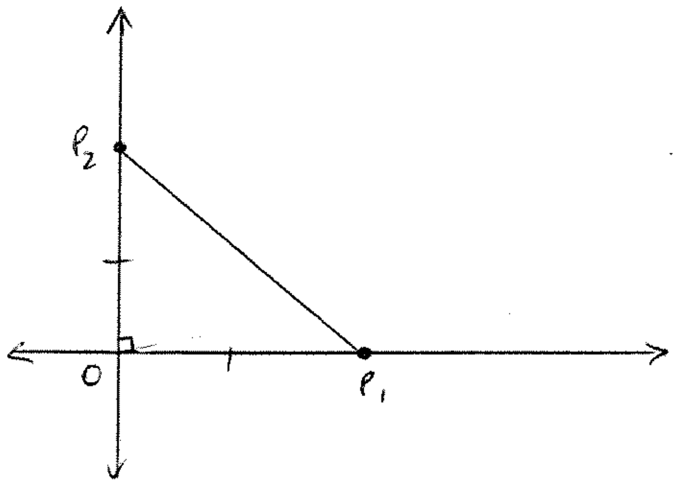
① angular region begins at (0, 1) with correct angles + dotted lines

OR

① Right idea

② Right idea with detail.

f)



① Diagram

$$|z_1| = |z_2|$$
$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \quad \therefore \quad z_2 = iz_1$$
$$\therefore \quad |z_1|^2 + |z_2|^2 = |z_1 - z_2|^2$$

Let $z_1 = a$ and $z_2 = ai$

Then $z_1^2 + z_2^2 = a^2 - a^2$ ① specific case

$= 0$.

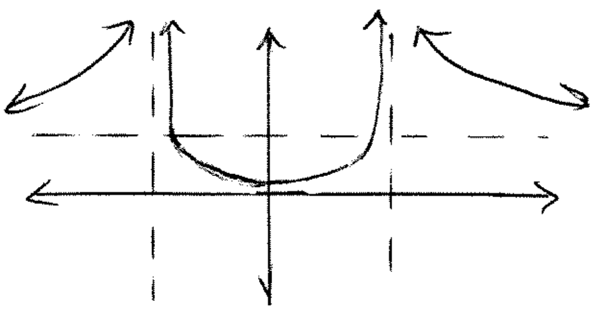
Similarly, in the general case

$$z_2 = iz_1$$
$$\therefore \quad z_1^2 + z_2^2 = z_1^2 + (iz_1)^2$$ ① general case.
$$= z_1^2 - z_1^2$$
$$= 0.$$

Q3

a)

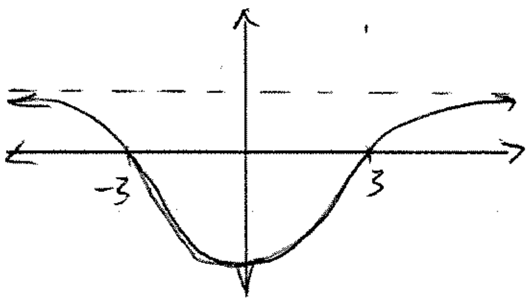
i/



① Branches less steep

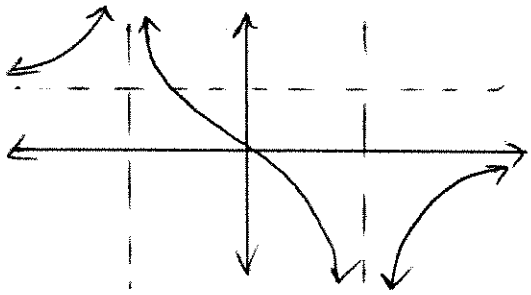
① Centre section reflected
(should also be flatter
but no marks deducted
for this)

ii/



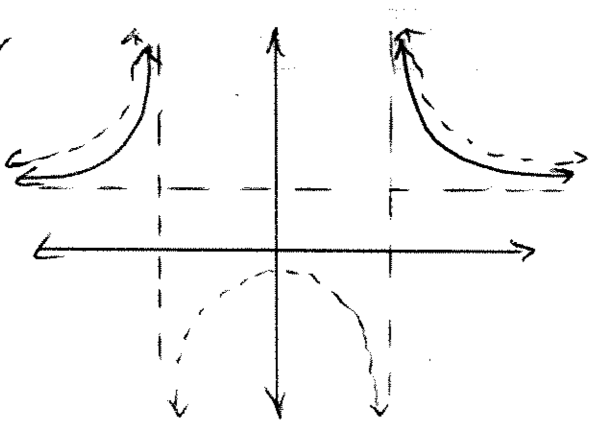
① Intercepts at 3, -3.
① Shape

iii/



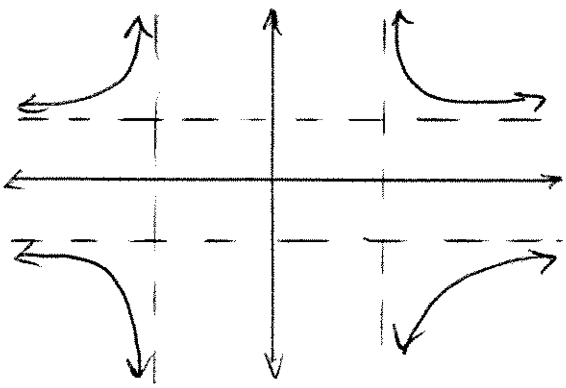
① Signs
① Shape

iv/



① Branches below original
① No $y = \sqrt{f(x)}$ for $f(x) < 0$.

v/



① Reflection over y of some section

① Correctly reflects only +ve y onto -y axis

b) i/ $3x^2 + y^2 - 2xy - 8x + 2 = 0$ ① chain rule

$6x + 2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y - 8 = 0$

$\therefore \frac{dy}{dx} 2(y-x) = -6x + 2y + 8$

① otherwise correct.

$\therefore \frac{dy}{dx} = \frac{3x-y-4}{x-y}.$

ii/ $m = 2$

$\therefore \frac{3x-y-4}{x-y} = 2$

$3x - y - 4 = 2x - 2y$

$x + y - 4 = 0$

$\therefore y = 4 - x$ ① Condition on $x + y$

$\therefore 3x^2 + (4-x)^2 - 2x(4-x) - 8x + 2 = 0$

$3x^2 + 16 - 8x + x^2 - 8x + 2x^2 - 8x + 2 = 0$

$6x^2 - 24x + 18 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$ ① Solution

$\therefore x = 1 \text{ or } 3$

$y = 3 \text{ or } 1$

① Coordinates

$\therefore$ Coords of pts are $(1,3)$ + $(3,1)$.

(Q4)

a)

i/ $\alpha\beta = 4$

$\therefore \gamma = -4$

$b = 4 + \alpha\gamma + \beta\gamma$

$= 4 - 4(\alpha+\beta)$

Now $\alpha + \beta + \gamma = -2$. (1) Reasonable logic

$\therefore \alpha + \beta = -6$

$\therefore b = 4 - 24$

$\therefore b = -20$

ii/ Let ρ represent new roots + x the original roots

$\rho = x^2$

$\therefore x = \sqrt{\rho}$

$\therefore$ New polynomial has

$(\sqrt{\rho})^3 + 2(\sqrt{\rho})^2 - 20\sqrt{\rho} - 16 = 0$ (1) subst $\sqrt{}$

$\sqrt{\rho}(\rho - 20) = -2\rho + 16$.

$\rho(\rho-20)^2 = (16-2\rho)^2$ (1) square + simplify

$\rho(\rho^2 - 40\rho + 400) = 256 - 64\rho + 4\rho^2$

$\rho^3 - 44\rho^2 + 464\rho - 256 = 0$ is new polynomial.

b) i/ $P(\alpha) = 0$ since $P(\alpha) = (\alpha-\alpha)^3 Q(x)$

$= 0$

$P'(x) = 3(x-\alpha)^2 Q(x) + Q'(x)(x-\alpha)^3$

$(x-\alpha)^2\left(3Q(x) + (x-\alpha)Q'(x)\right)$ (1) $P'(\alpha)$

$\therefore P'(\alpha) = 0$

$P''(\alpha) = 2(x-\alpha)\left(3Q(x) + (x-\alpha)Q'(x)\right) + (x-\alpha)^2(\ldots)$

$\therefore P''(\alpha) = 0$ (1) $P''(\alpha)$

$\therefore P(\alpha) = P'(\alpha) = P''(\alpha) = 0$.

ii) $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$

$P'(x) = 32x^3 - 75x^2 + 54x - 11$

$P''(x) = 96x^2 - 150x + 54$

Let $P''(x) = 0$

Then $96x^2 - 150x + 54 = 0$

$16x^2 - 25x + 9 = 0$

$(16x - 9)(x - 1) = 0$

$x = 9/16$ or $x = 1$ ① Identify triple root

Check $P(1) = 0$ $P'(1) =$

$\therefore P(x) = (x-1)^3(8x-1)$ ① Determine fourth root

$\therefore x = 1, 1/8$.

c) i) $z^9 - 1 = 0$

$(z^3 - 1)(z^6 + z^3 + 1) = 0$ ①

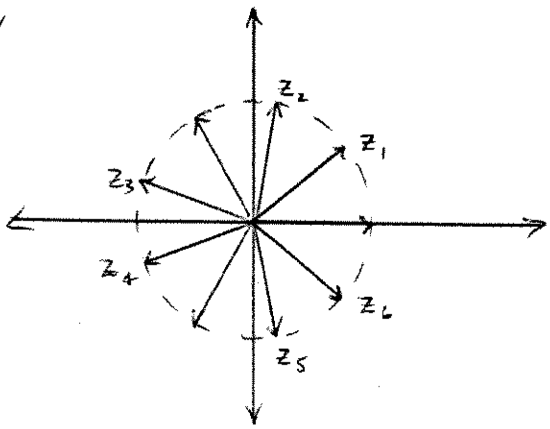
Check

$z^9 + z^6 + z^3 - z^6 - z^3 - 1 = 0$ ① Working

$z^9 - 1 = 0$

$\therefore$ Solns of $z^6 + z^3 + 1 = 0$ are contained in $z^9 - 1 = 0$

ii)
iii)



Roots of $z^3 - 1 = 0$ are $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$, 1.

① Unit circle
① 9 roots marked w/ vectors at equal angles.

① Identify roots of $z^3 - 1$
① Mark all 6 roots of $z^6 + z^3 + 1 = 0$.

iv) Roots are: $cis\left(\pm\frac{2\pi}{9}\right)$, $cis\left(\pm\frac{8\pi}{9}\right)$, $cis\left(\pm\frac{4\pi}{9}\right)$ ① Name roots

$\therefore$ Sum $= 2\cos\frac{2\pi}{9} + 2\cos\frac{8\pi}{9} + 2\cos\frac{4\pi}{9}$

but $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$ ① Identity on cos.

$\therefore$ Sum $= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$

Ext 2 05

(5)

(a)(i) $b^2 = a^2(1-e^2)$ (1)

$4 = 9(1-e^2)$

$e = \frac{\sqrt{5}}{3}$ (1)

$S(\sqrt{5}, 0)$ (1)

(ii) $x = \frac{9}{\sqrt{5}}$ (1)

(iii) $S(0, \sqrt{5})$, $y = \frac{9}{\sqrt{5}}$ (1) both

(b)(i) $\frac{2x}{16} - \frac{2y\frac{dy}{dx}}{9} = 0$ (1)

$\frac{dy}{dx} = \frac{9x}{16y}$

$= \frac{3\sec\theta}{4\tan\theta}$ (1)

tangent: $y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}(x - 4\sec\theta)$ (1)

etc.

normal: $y - 3\tan\theta = \frac{-4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$ (1)

(ii) At T, $x = 0$ $\therefore y = \frac{-3}{\tan\theta} = -3\cot\theta$ (1) correct intermediate step

At N, $x = 0$, $\therefore y = \frac{16\tan\theta}{3} + 3\tan\theta = \frac{25\tan\theta}{3}$ (1) "

(iii) $b^2 = a^2(e^2-1)$

$9 = 16(e^2-1)$

$e = \frac{5}{4}$

$\therefore S = (5, 0)$

$m_{NS} \cdot m_{TS} = \frac{\frac{25}{3}\tan\theta}{-5} \cdot \frac{-3\cot\theta}{-5} = -1$ $\therefore N\hat{S}T = 90^\circ$

$\therefore$ circle with diameter NT passes through S ($N\hat{S}T$ = angle in semicircle)

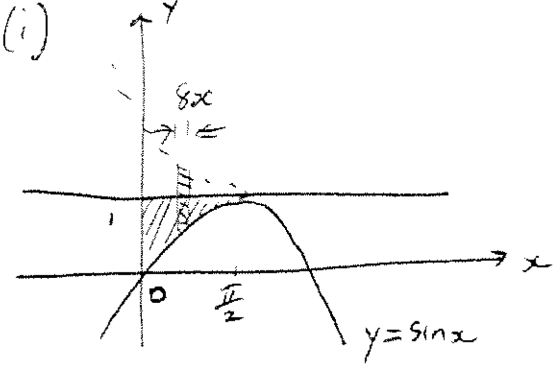
(1) correct focus

(2) 1 for each gradient

(1) stating '∠ in semi-circle'

5) (a)

(i)



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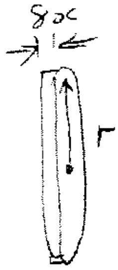
(ii)



$$\delta V \doteqdot \pi r^2 \delta x$$

$$V \doteqdot \sum_{x=0}^{x=\frac{\pi}{2}} \pi(1-y)^2 \delta x$$

$$= \lim_{\delta x \to 0} \sum_{x=0}^{x=\frac{\pi}{2}} \pi(1-y)^2 \delta x$$

$$V = \pi \int_0^{\frac{\pi}{2}} (1-y)^2 dx$$ 1

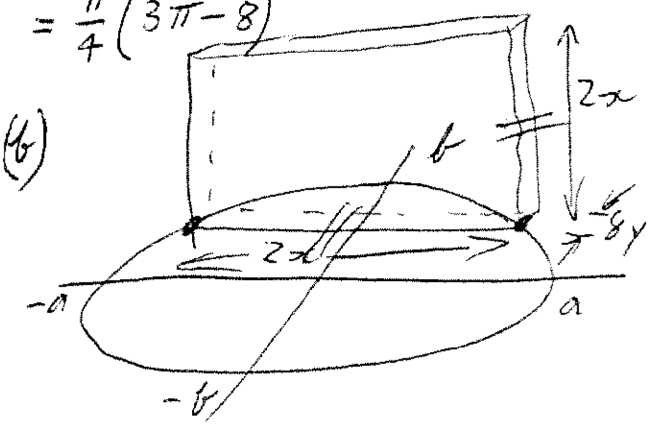
$$= \pi \int_0^{\frac{\pi}{2}} (1-2y+y^2)\, dx, \quad y=\sin x$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(1-2\sin x+\tfrac{1}{2}(1-\cos 2x)\right) dx$$

$$= \pi \left[x+2\cos x+\tfrac{1}{2}\left(x-\tfrac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{2}}$$ 1

$$= \frac{\pi}{4}(3\pi-8)$$ 1

(b)



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$x^2 = a^2\left(1-\frac{y^2}{b^2}\right)$$

$$\delta V \doteqdot 4x^2 \delta y$$

$$V \doteqdot \sum_{y=-b}^{y=b} 4x^2 \delta y$$

$$V = \lim_{\delta y \to 0} \sum_{y=-b}^{y=b} 4x^2 \delta y$$

6(b) ctd

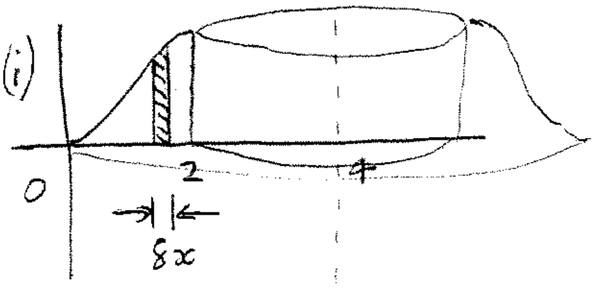
$$V = \int_{-b}^{b} 4a^2\left(1-\frac{y^2}{b^2}\right)dy$$ 1

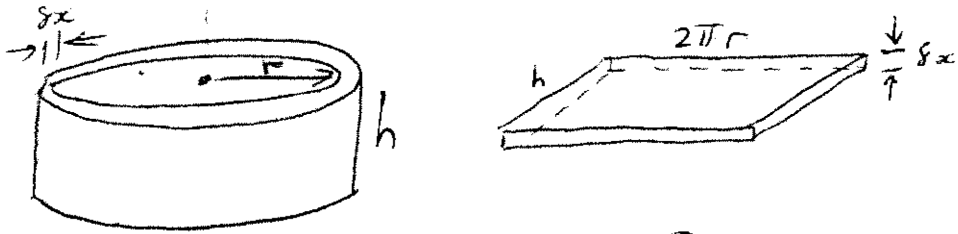
$$= 8a^2\int_{0}^{b}\left(1-\frac{y^2}{b^2}\right)dy$$

$$= 8a^2\left[y-\frac{y^3}{3b^2}\right]_0^b$$ 1

$$= \frac{16a^2b}{3}$$ 1

(c) (i)





$$\delta V \doteqdot 2\pi r h\,\delta x$$ 1

$$= 2\pi(4-x)\left(\frac{x^2}{x^2+1}\right)$$ 1

$$= 2\pi(4-x)\left(\frac{x^2+1-1}{x^2+1}\right)$$ 1

$$= 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)$$

(ii) $$V = \int_0^2 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)dx$$

$$= 2\pi\int_0^2\left(4-\frac{4}{1+x^2}-x+\frac{x}{1+x^2}\right)dx$$

$$= 2\pi\left[4x-4\tan^{-1}x-\frac{x^2}{2}+\frac{1}{2}\log(1+x^2)\right]_0^2$$

$$= 2\pi\left(6-4\tan^{-1}2+\frac{1}{2}\log 5\right)$$

7) (a) (i) $m\ddot{x} = -mk(1+v^2)$

$\ddot{x} = -k(1+v^2)$ ①

$\frac{dv}{dt} = -k(1+v^2)$

$\frac{dt}{dv} = \frac{-1}{k} \cdot \frac{1}{1+v^2}$ ①

$\int_0^T dt = -\frac{1}{k}\int_U^0 \frac{dv}{1+v^2}$

$T = \frac{1}{k}\left[\tan^{-1} v\right]_0^U$ ①

$= \frac{1}{k}\tan^{-1} U$

(ii) $v\frac{dv}{dx} = -k(1+v^2)$ ①

$\frac{dv}{dx} = -k \cdot \frac{1+v^2}{v}$

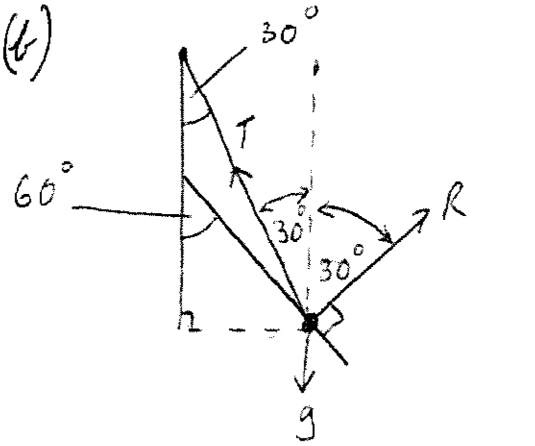
$\frac{dx}{dv} = -\frac{1}{k} \cdot \frac{v}{1+v^2}$

$\int_0^X dx = -\frac{1}{k}\int_U^0 \frac{v\,dv}{1+v^2}$

$X = +\frac{1}{2k}\left[\log(1+v^2)\right]_0^U$

$= \frac{1}{2k}\log(1+U^2)$ ①

(b)



① Forces

① Angles

7 ctd

Horizontal: $T\sin 30^\circ - R\sin 30^\circ = \frac{mv^2}{r}$ $\quad m=1, r=0.5$ $\quad$ 1

$T - R = 4v^2$ —— 1

Vertical: $T\cos 30^\circ + R\cos 30^\circ = mg$ $\quad$ 1

$T + R = \frac{2g}{\sqrt{3}}$ —— 2

iii) $v = 1 \quad \therefore T - R = 4$ —— 3

2 + 3

$2T = \frac{2g}{\sqrt{3}} + 4$

$T = \frac{g}{\sqrt{3}} + 2 \quad , \quad g = 9.8$

≈ 7.66 (2 decimal places) $\quad$ 1

sub in 3

$R \approx 3.66$ (2 decimal places) $\quad$ 1

iv) T_{max} when $R = 0$ $\quad$ 1 $R = 0$ or $R > 0$

$\therefore T = \frac{2g}{\sqrt{3}}$ is the upper limit for T $\quad$ 1 Answer

(or $R = \frac{2g}{\sqrt{3}} - T \quad R > 0 \therefore \frac{2g}{\sqrt{3}} - T > 0$

$T < \frac{2g}{\sqrt{3}}$)

v) $R = 0 \Rightarrow 4v^2 = T = \frac{2g}{\sqrt{3}}$

$v^2 = \frac{g}{2\sqrt{3}}$ $\quad$ 1 v^2

$v = \left(\frac{g}{2\sqrt{3}}\right)^{\frac{1}{2}}$ $\quad$ 1 Answer.

or $R = T - 4v^2 > 0$

$v^2 < \frac{T}{4}, \quad v < \left(\frac{g}{2\sqrt{3}}\right)^{\frac{1}{2}}$

(8) (a) (i) $C\hat{E}F + C\hat{F}D = 90° + 90°$ (Given)

$= 180°$

$\therefore DECF$ cyclic (opp $\angle$s supplementary) (1)

$D\hat{F}B = D\hat{G}B = 90°$ (Given)

$\therefore DFGB$ cyclic (DB diameter, $D\hat{F}B, D\hat{G}B$ $\angle$s in semicircle) (1)

(ii) Let $E\hat{C}D = \alpha$

$\therefore E\hat{F}D = \alpha$ ($\angle$s in same segment of circle ECFD)

But $G\hat{B}D = E\hat{F}D$ (ext $\angle$ of cyclic quad. DFGB = opp. int. $\angle$)

$= \alpha$

$\therefore D\hat{F}G = 180° - \alpha$ (opp. supplementary $\angle$s of cyclic quad DFGB)

$\therefore E\hat{F}D + D\hat{F}G = \alpha + 180° - \alpha$

$= 180°$

$\therefore E, F, G$ collinear

(1) 2 appropriate facts with reasons.

(1) Using 3 cyclic quads

(1) correct answer.

(b) (i) $\int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= -\left[\log(\cos x)\right]_0^{\frac{\pi}{4}}$ (1)

$= \log\sqrt{2}$ (1)

8) ctd

(ii) $$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$$ ①

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} \Big[\tan^{n-1} x \Big]_0^{\frac{\pi}{4}} - I_{n-2}$$ ② 1 each.

$$= \frac{1}{n-1} - I_{n-2}$$

iii) $$I_5 = \frac{1}{4} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$

$$= -\frac{1}{4} + \log\sqrt{2}$$

(c) If $(x-b)$ and $(x-c)$ are factors then $(x-b)(x-c)$ is a factor ①

Note: $(x-b)(x-c) = x^2 - (b+c)x + bc$

$$P(b) = b^m(c^n - b^n) + b^m(b^n - c^n) + c^m(b^n - b^n)$$

$$= 0$$ ①

$$P(c) = c^m(b^n - c^n) + b^m(c^n - c^n) + c^m(c^n - b^n)$$

$$= 0$$ ①

$\therefore (x-b), (x-c)$ are factors.