**Question 1 12 Marks** **Marks**

(a) Evaluate the following definite integral $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4}$ **2**

.

(b) Solve $\dfrac{5}{x-3} \geq 2$. **2**

(c) Show that $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + C$, where $C$ is constant. **2**

(d) Find the general solution for $\cos 2\theta = \dfrac{\sqrt{3}}{2}$ **2**

(e) (i) Show that the derivative of $\dfrac{1+\sin x}{\cos x}$ is $\dfrac{1}{1-\sin x}$. **4**

(ii) Hence, deduce that $\displaystyle\int_{0}^{\pi/4} \frac{dx}{1-\sin x} = \sqrt{2}$

**Question 2 12 Marks Start a new booklet**

(a) Let $f(x) = x^3 + 5x^2 + 17x - 10$. The equation $f(x) = 0$ has only one real root. **4**

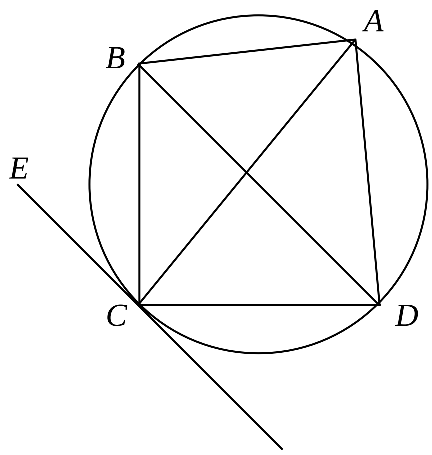
(i) Show that the root lies between 0 and 2.

(ii) Use one application of the 'halving the interval' method to find a smaller interval containing the root.

(iii) Which end of the smaller interval found in part (ii) is closer to the root? Briefly justify your answer.

(b) Evaluate $\displaystyle\int_{-1}^{2} \frac{x\,dx}{\sqrt{3-x}}$ using the substitution $x = 3-u$. **4**

(c) ABCD is a cyclic quadrilateral in which AC bisects $\angle DAB$. CE is the tangent to the circle at C. Prove $CE \parallel DB$. **4**



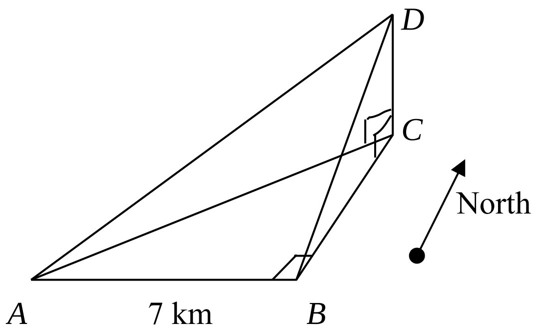

*Question 3 page 2.*

**Question 3 12 Marks Start a new booklet** **Marks**

(a) (i) Express $\sin\theta + \sqrt{3}\cos\theta$ in the form $R\sin(\theta + \phi)$. **4**

(ii) Hence, or otherwise, solve the equation $\sin\theta + \sqrt{3}\cos\theta = 1$ for values of $\theta$ between 0 and $2\pi$.

(b) Cadel notices that the angle of elevation of the top of a mountain due north is 14°. Upon riding 7 kilometres due west, he finds that the angle of elevation of the top of the mountain is 10°. How high is the mountain? Give your answer correct to the nearest metre. **4**




$\angle DBC = 14^\circ$, $\angle DAC = 10^\circ$

(c) (i) Show that $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$ **4**

(ii) Hence, show that $\dfrac{1}{1+\sec x} = 1 - \dfrac{1}{2}\sec^2\dfrac{x}{2}$.

(iii) Use part (ii) to deduce that $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sec x} = \frac{\pi}{2} - 1$.

*Question 4 page 3.*

**Question 4 12 Marks Start a new booklet**

(a) The sides of a cube are increasing at a rate of 2 $\text{cms}^{-1}$. Find at what rate the surface area is increasing when the sides are each 10 cm. **2**

(b) Prove by induction $9^{n+2} - 4^n$ is divisible by 5, for $n \geq 1$ **4**

(c) 100 grams of cane sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. That is, if *m* grams are converted in *t* minutes, then $\frac{dm}{dt} = k(100 - m)$, where *k* is constant. **6**

(i) Show that $m = 100 + Ae^{-kt}$, where *A* is a constant, satisfies this equation.

(ii) Find the value of *A*.

(iii) If 40 grams are converted in the first 10 minutes, find how many grams are converted in the first 30 minutes.

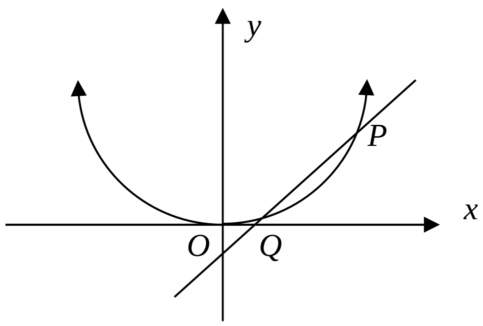
(iv) What is the limiting value of *m* as *t* increases indefinitely?

**Question 5 12 Marks Start a new booklet**

(a) Solve the equation $6x^3 - 17x^2 - 5x + 6 = 0$, given that two of its roots have a product of $-2$. **3**

(b) Find the values of *a* and *b* so that $x^4 + 4x^3 - x^2 + ax + b$ is divisible by $(x-2)(x+1)$. **3**

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. **6**




(i) Show that the equation of the chord *PQ* is $\frac{(p+q)x}{2} - y = apq$

(ii) The line *PQ* passes through the point $(0, -a)$. Show that $pq = 1$.

(iv) Hence, or otherwise, if *S* is the focus of the parabola, show that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.

*Question 6 page 4.*

**Question 6 12 Marks Start a new booklet**

(a) By considering the expansion of $\left(1+x\right)^{2n}$ in ascending powers of $x$ show that ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \ ... \ + {}^{2n}C_{2n} = 4^n$ **2**

(b) (i) Write the expansion of $\left(2+3x\right)^{20}$ in the form $\sum_{r=0}^{20} c_r x^r$ , where $c_r$ is the coefficient of $x^r$ in the expansion. **4**

(ii) Show that $\dfrac{c_{r+1}}{c_r} = \dfrac{60-3r}{2r+2}$

(iii) Hence, or otherwise, find the greatest coefficient in the expansion of $\left(2+3x\right)^{20}$ . Leave your answer in index form.

(c) Of a set of otherwise similar cards, ten are white, six are red and four are yellow. Three cards are taken at random. What is the probability that: **3**

(i) They are all different colours;

(ii) They are the same colour?

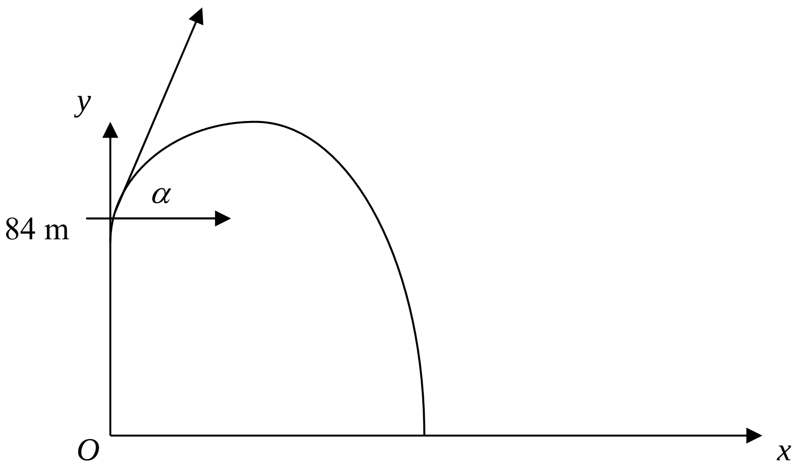
(d) (i) In how many ways can 2 boys and 1 girl be arranged in a row if the selection is made from 4 boys and 3 girls? **3**

(ii) In how many of these arrangements does a girl occupy the middle position?

*Question 7 page 5.*

**Question 7 12 Marks Start a new booklet**

(a) A particle is projected with speed 40 $\text{ms}^{-1}$ from the top of a cliff 84 metres high at an angle of elevation $\alpha = \tan^{-1}\frac{4}{3}$. Assume that the equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -10\ \text{ms}^{-2}$. **6**




(i) Derive the equations $x = 24t$ and $y = 84 + 32t - 5t^2$.

(ii) Hence or otherwise find the range on the horizontal plane through the foot of the cliff.

(iii) Find the speed of the body when it reaches this plane. Answer correct to two significant figures.

(b) A particle moves in simple harmonic motion about $x = 0$ and its displacement $x$ metres, at time $t$ seconds, is given by $x = a\sin n(t+\alpha)$. The particle moves with a period of 16 seconds. It passes through the centre of motion when $t = 2$ seconds. Its velocity is $4\,\text{ms}^{-1}$ when $t = 4$ seconds. **6**

(i) Show $\ddot{x} = -\frac{\pi^2}{64}x$.

(ii) Find the maximum displacement.

(iii) Find the speed of the particle when $t = 10$ seconds.

**END OF EXAM**

## Question 1

(a) $\int x \sin(x^2)\,dx$

$= -\dfrac{\cos(x^2)}{2} + C$

(b) $\int_0^{\frac{1}{2}} \cos^{-1}x\,dx = \int_0^{\frac{1}{2}} \cos^{-1}x \dfrac{d[x]}{dx} \cdot dx$

$= \left[x \cos^{-1}x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \dfrac{-x}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2} \times \frac{\pi}{3} - \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \dfrac{\pi}{6} - \dfrac{\sqrt{3}}{2} + 1$

(c) (i) let $x \equiv a(x+3) + b(x-2)$

let $x = 2 \quad \therefore a = \frac{2}{5}$

$x = -3 \quad \therefore b = \frac{3}{5}$

$\therefore \dfrac{x}{(x-2)(x+3)} \equiv \dfrac{\frac{2}{5}}{x-2} + \dfrac{\frac{3}{5}}{x+3}$

(ii) $\therefore \int \dfrac{x}{(x-2)(x+3)}\,dx = \frac{2}{5}\int \dfrac{1}{x-2}\,dx + \frac{3}{5}\int \dfrac{1}{x+3}\,dx$

$= \frac{2}{5}\ln|x-2| + \frac{3}{5}\ln|x+3| + C$

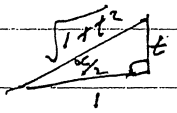
(d) $\int \dfrac{x+4}{x^2-4x+13}\,dx = \frac{1}{2}\int \dfrac{2x-4}{x^2-4x+13}\,dx + \int \dfrac{6}{(x-2)^2+9}\,dx$

$= \frac{1}{2}\ln|x^2-4x+13| + 2\tan^{-1}\dfrac{x-2}{3} + C$

(e) $t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1}t$

$\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$




$\therefore \cos x = 2\cos^2\frac{x}{2} - 1$

$= \dfrac{1-t^2}{1+t^2}$

$\therefore \int_0^{\frac{\pi}{2}} \dfrac{dx}{1+\cos x} = \int_0^1 \dfrac{\frac{2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,dt$

$= \int_0^1 \dfrac{2}{1+t^2+1-t^2}\,dt$

$= \int_0^1 1\,dt$

$= [t]_0^1$

$= 1$

## Question 2

(a) $w = 1 + i$ , $z = 1 - i\sqrt{3}$

(i) $w\bar{z} = (1+i)(1+i\sqrt{3})$

$= (1-\sqrt{3}) + (1+\sqrt{3})i$ ✓

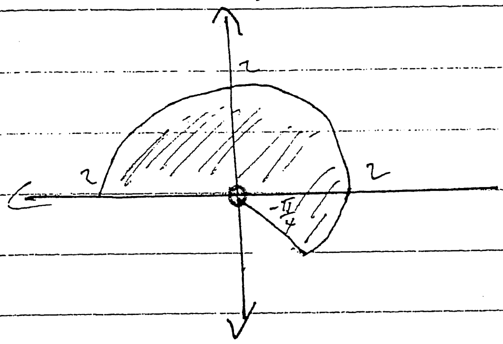
(ii) $\frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2}$

$= \frac{1}{2} - \frac{1}{2}i$ ✓

(iii) $\frac{i\left[Re(z) - z\right]}{Im(z)} = \frac{i\left[1 - 1 + i\sqrt{3}\right]}{-\sqrt{3}}$ ✓

$= 1$ ✓

(b)




(c) $x^2 - 4x + (1-4i) = 0$

$x = \frac{4 \pm \sqrt{16 - 4(1-4i)}}{2}$ ✓

$= \frac{4 \pm \sqrt{12 + 16i}}{2}$

$= 2 \pm \sqrt{3 + 4i}$

$\therefore x = 2 + (2 + i)$

$= 4 + i$ from ✱

or $x = 2 - 2 - i$

$= -i$ ✓

Now let $\sqrt{3+4i} = x + iy$

$\therefore x^2 - y^2 = 3$

$2xy = 4 \quad \therefore y = \frac{2}{x}$

$\therefore x^2 - \frac{4}{x^2} = 3$

$(x^2)^2 - 3(x^2) - 4 = 0$ ✓

$(x^2 - 4)(x^2 + 1) = 0$ ✱

$\therefore x = \pm 2 \quad \therefore$ when $x = 2, y = 1$

when $x = -2, y = -1$

(d)(i) $z = 1 + 2i + t(3 - 4i)$

$= 1 + 2i + 3t - 4ti$

$= (1+3t) + (2-4t)i$

$\therefore x = 1 + 3t \quad \ldots (i)$ ✓

$y = 2 - 4t \quad \ldots (ii)$

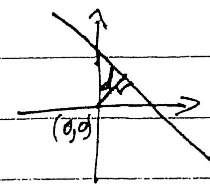
from (i) $t = \frac{x-1}{3}$

Sub into (ii) $y = 2 - \frac{4(x-1)}{3}$

$\therefore 3y = 6 - 4x + 4$

$\therefore 4x + 3y = 10$ ✓

(ii) From




✓

$|z|$ minimum at

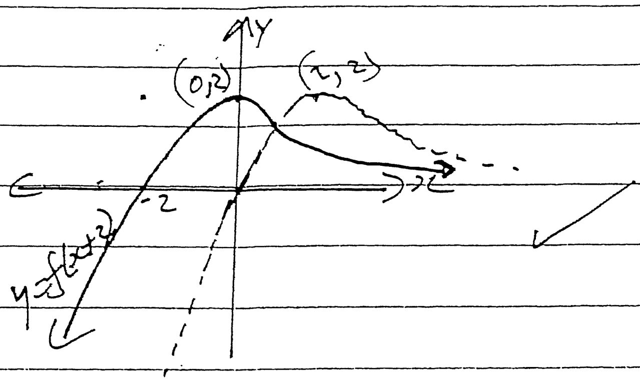
$d = \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}}$

$= \frac{|-10|}{5}$

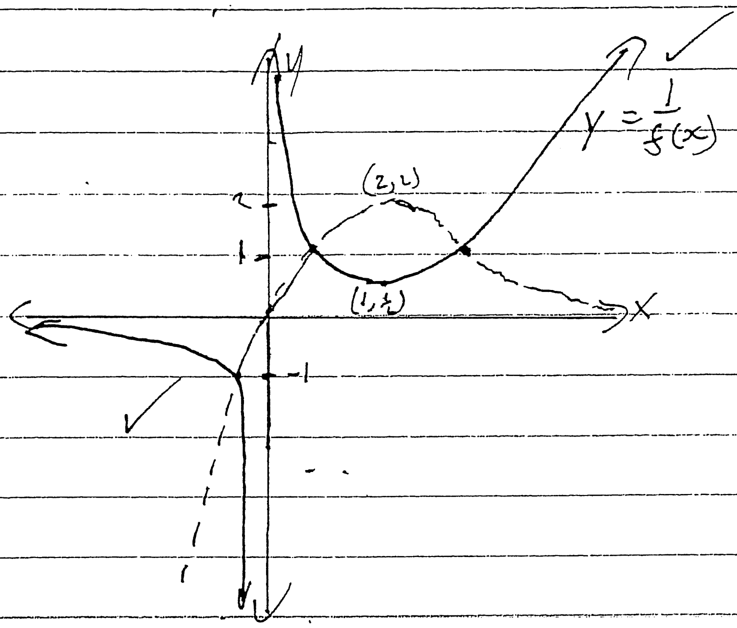
$= 2$ ✓

## Question 3

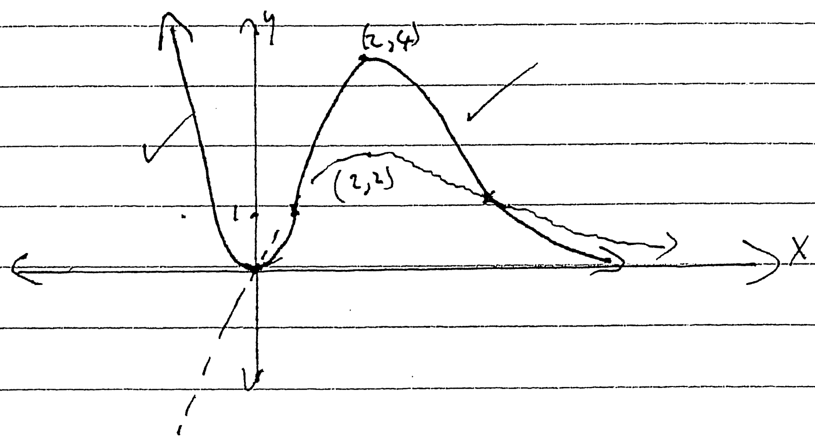
a) (i)



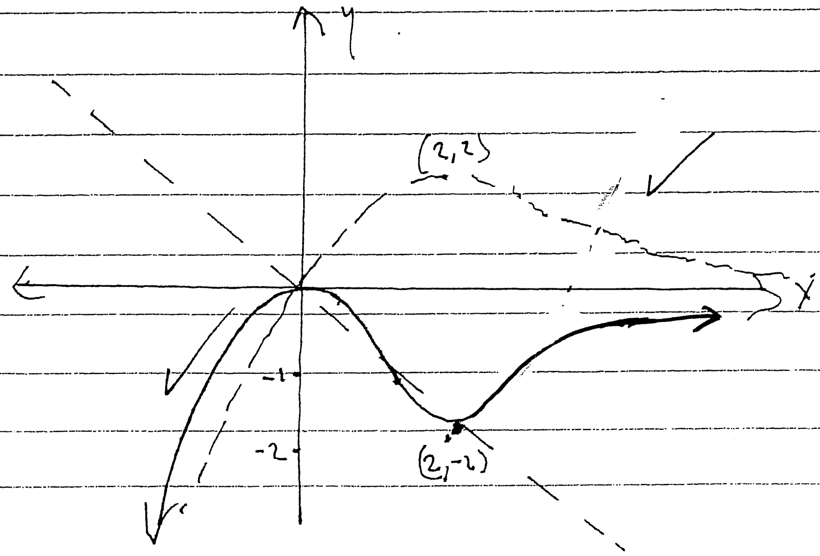

(ii)




(iii)




(iv)



Question 3 (Continued)

3 (b) (i) $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$

$P'(x) = 5x^4 - 12x^3 + 12x^2 - 8x + 3$ $\quad P'(1) = 5 - 12 + 12 - 8 + 3$

$P''(x) = 20x^3 - 36x^2 + 24x - 8$ $\quad = 0$ ✓

$P''(1) = 20 - 36 + 24 - 8 = 0$

Now $P(1) = 1 - 3 + 4 - 4 + 3 - 1$

$= 0$ $\quad \therefore$ as $P(1) = P'(1) = P''(1) = 0$ ✓

$\therefore$ triple root at $x = 1$

[1 Mark for finding $P'(x)$ & $P''(x)$]
1 Mark for conclusion

(ii) $P(i) = i^5 - 3i^4 + 4i^3 + 4i^2 + 3i - 1$

as $i^2 = -1$ $\quad = i - 3 + 4i + 4 + 3i - 1$ ✓

$= 0$

(iii) if $x = i$ is a root $\therefore P(-i) = 0$ ✓

$\therefore$ Roots are $1, i, -i$ ✓

(c) $x^3 - 2x^2 - 5x - 1 = 0$

Let new equation have roots in the form

$X = \frac{1}{\alpha^2}$

$\therefore \alpha^2 = \frac{1}{X}$

$\alpha = \pm \frac{1}{\sqrt{x}}$ ✓

as $\alpha$ is a root of $x^3 - 2x^2 - 5x - 1 = 0$

$\therefore \left(\pm\frac{1}{\sqrt{x}}\right)^3 - 2\left(\pm\frac{1}{\sqrt{x}}\right)^2 - 5\left(\pm\frac{1}{\sqrt{x}}\right) - 1 = 0$

$\pm\frac{1}{\sqrt{x}}\left(\frac{1}{x} - 5\right) = \frac{2}{x} + 1$

$\therefore \frac{1}{x}\left(\frac{1}{x} - 5\right)^2 = \left(\frac{2}{x} + 1\right)^2$ ✓

$\therefore \frac{1}{x}\left(\frac{1}{x^2} - \frac{10}{x} + 25\right) = \frac{4}{x^2} + \frac{4}{x} + 1$

$\frac{1}{x^3} - \frac{10}{x^2} + \frac{25}{x} = \frac{4}{x^2} + \frac{4}{x} + 1$

$\therefore 1 - 10x + 25x^2 = 4x + 4x^2 + x^3$ ✓

$\therefore x^3 - 21x^2 + 14x - 1 = 0$ has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$.

Note students may get the same result using: $x^3 - (\text{Sum})x^2 + (\text{Sum} \times 2)x - \text{Product} = 0$ #

Question 4

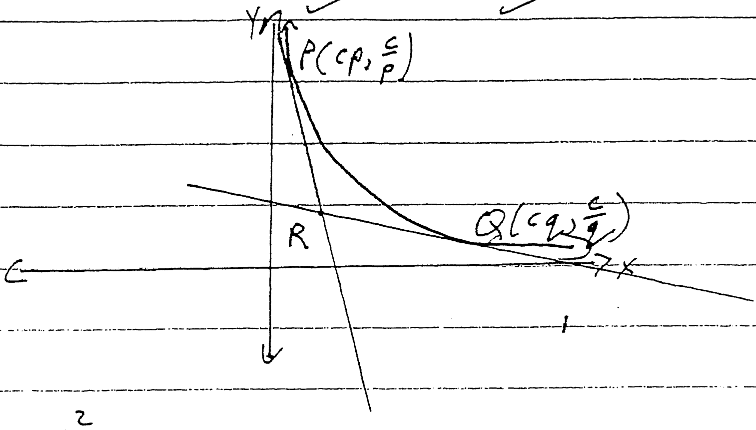
(a) (i) Circle $5-k = k-3$ ✓

$k = 4$ ✓

(ii) Hyperbola $5-k<0$ or $k-3<0$, but not both

$\therefore k>5$ or $k<3$ ✓ ✓

(b)




(i) $xy = c^2$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

at $T(ct, \frac{c}{t})$ $\frac{dy}{dx} = -\frac{c/t}{ct}$

$= -\frac{1}{t^2}$

$\therefore y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ ✓

$t^2y - ct = -x + ct$

$\therefore x + t^2y - 2ct = 0$

(ii) at P the tangent is $x + p^2y - 2cp = 0$ --- (1)

$x + q^2y - 2cq = 0$ --- (2)

$\therefore$ (1)-(2) gives $(p^2-q^2)y = 2c(p-q)$

$y = \frac{2c}{p+q}$ ✓

Sub into (1)

$\therefore x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$x = 2cp - \frac{2cp^2}{p+q}$

$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$

$= \frac{2cpq}{p+q}$ ✓

$\therefore R\left[\frac{2cpq}{p+q}, \frac{2c}{p+q}\right]$

Question 4 (Continued)

Q4(b)

(iii) If $xy + y^2 = 2c$ $\therefore R\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ satisfies it.

$$LHS = \frac{2cpq}{p+q} \cdot \frac{2c}{p+q} + \left(\frac{2c}{p+q}\right)^2$$ ✓

[ALTERNATE] SOLUTION ~~ATTACHED~~

$$= \frac{4c^2pq}{(p+q)^2} + \frac{4c^2}{(p+q)^2}$$

$$= \frac{4c^2(pq+1)}{p^2+q^2+2pq}$$

$$= \frac{4c^2(pq+1)}{2+2pq} \quad \text{given } p^2+q^2=2$$ ✓

$$= \frac{4c^2(pq+1)}{2(pq+1)}$$

$$= 2c^2$$ ✓

$= RHS$ #

4(c) $P(a\cos\theta, b\sin\theta)$, $Q(a\cos\phi, b\sin\phi)$ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $$M_{PQ} = \frac{b(\sin\phi - \sin\theta)}{a(\cos\phi - \cos\theta)}$$ ✓

$\therefore$ Equation of chord: $y - b\sin\theta = \frac{b(\sin\phi - \sin\theta)}{a(\cos\phi - \cos\theta)}(x - a\cos\theta)$ ✓

$\therefore ay(\cos\phi - \cos\theta) - ab\sin\theta(\cos\phi - \cos\theta) = b(\sin\phi - \sin\theta)x - ab\cos\theta(\sin\phi - \sin\theta)$

(ii) Focal chord passes through $(ae, 0)$ ✓

$\therefore abe(\sin\phi - \sin\theta) = ab\cos\theta(\sin\phi - \sin\theta) - ab\sin\theta(\cos\phi - \cos\theta)$

$e(\sin\phi - \sin\theta) = \cos\theta\sin\phi - \cos\theta\sin\theta - \sin\theta\cos\phi + \sin\theta\cos\theta$ ✓

$= \sin(\phi - \theta)$

$$\therefore e = \frac{\sin(\phi - \theta)}{\sin\phi - \sin\theta}$$ ✓

Q 4(b)(iii) ALTERNATE SOLUTION

Given $p^2+q^2=2$

$\therefore (p+q)^2-2pq=2$

$\therefore pq=\frac{(p+q)^2-2}{2}$ *

Now R $\left[\frac{2cpq}{p+q}, \frac{2c}{p+q}\right]$

$\therefore y=\frac{2c}{p+q}$

$\therefore p+q=\frac{2c}{y}$

from * $pq=\frac{\left(\frac{2c}{y}\right)^2-2}{2}$

$=\frac{4c^2-2y^2}{2y^2}$

$=\frac{2c^2-y^2}{y^2}$

Now $x=\frac{2cpq}{p+q}$

$\therefore x=\frac{2c\times\frac{2c^2-y^2}{y^2}}{\frac{2c}{y}}$

$\therefore x=\frac{2c^2-y^2}{y}$

$xy=2c^2-y^2$

$\therefore xy+y^2=2c^2$

Question 5

(a) $N(\text{Sample}) = \dfrac{{}^{12}C_5 \cdot {}^{7}C_5}{2}$

$= 8316$ ✓

$N(\text{Event}) = \dfrac{{}^{3}C_3 \times {}^{9}C_2 \times {}^{7}C_2}{8316}$ ✓

${}^{3}C_3$ – triplets together
${}^{9}C_2$ – other two members of their team.
${}^{7}C_2$ – second team.

$= \dfrac{72}{792}$

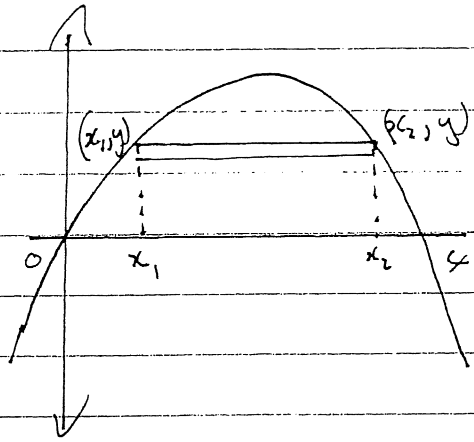
$= \dfrac{1}{11}$ ✓

(b) $y = 4x - x^2$

(i) $\therefore x^2 - 4x + y = 0$

$x = \dfrac{4 \pm \sqrt{16 - 4y}}{2}$ ✓

$= 2 \pm \sqrt{4 - y}$



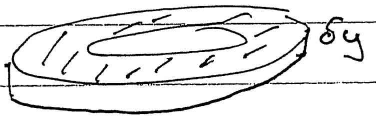

$\therefore x_2 = 2 + \sqrt{4 - y}$ ✓

$x_1 = 2 - \sqrt{4 - y}$

(i) Max value of $y = 4$ ✓ (By inspection)

(ii) Using slices perpendicular to $y$-axis

$\therefore$ Disc




✓

Volume of disc $\doteq \pi (x_2^2 - x_1^2)\,\delta y$ ✓

$\text{Volume} = \lim_{\delta y \to 0} \sum_{y=0}^{y=4} \pi (x_2^2 - x_1^2)\,\delta y$ ✓

$= \pi \int_0^4 (x_2 - x_1)(x_2 + x_1)\,dy$

$= 8\pi \int_0^4 \sqrt{4 - y}\,dy$ ✓

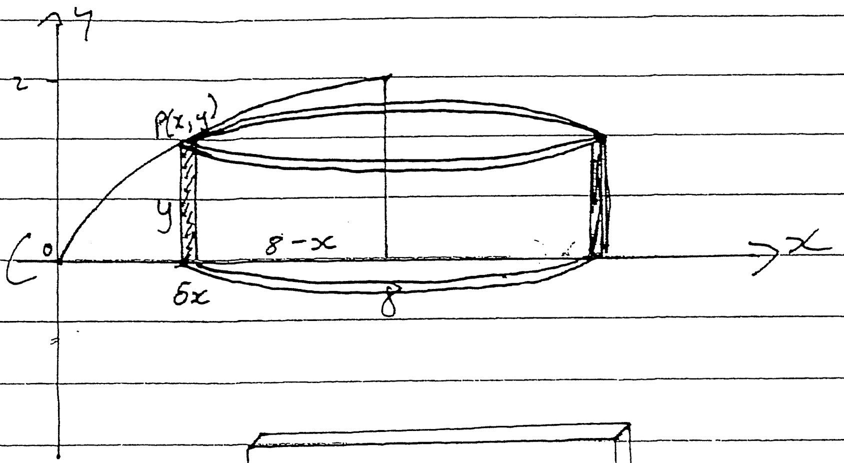
$= 8\pi \left[ -\tfrac{2}{3} (4 - y)^{3/2} \right]_0^4$

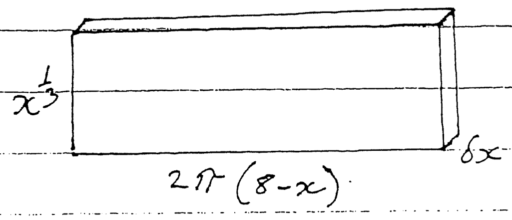
$= \dfrac{8\pi}{3} \cdot -\dfrac{2}{3} (0 - 8)$

$= \dfrac{128\pi}{3}$ units$^3$ ✓

Question 5 (continued)

(c)




∴ Volume of shell $\delta V = 2\pi(8-x)x^{\frac{1}{3}}.\delta x$

∴ Volume $= \lim_{\delta x \to 0} \sum_{x=0}^{8} 2\pi(8-x).x^{\frac{1}{3}}\delta x$

$= 2\pi \int_0^8 \left(8x^{\frac{1}{3}} - x^{\frac{4}{3}}\right) dx$

$= 2\pi \left[6x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}}\right]_0^8$

$= 2\pi \left(6 \times 16 - \frac{3}{7} \times 128\right)$

$= 2\pi \left(96 - \frac{384}{7}\right)$

$= \frac{576\pi}{7}$ units$^3$

Question 6

(a) Given $z^n - \frac{1}{z^n} = 2i\sin n\theta$ and $(z-\frac{1}{z})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

(i) $z^1 - \frac{1}{z^1} = 2i\sin\theta$

Now $(z-\frac{1}{z})^5 = z^5 - \frac{1}{z^5} - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z})$ ✓

$\therefore (2i\sin\theta)^5 = 2i\sin5\theta - 10i\sin3\theta + 20i\sin\theta$ ✓

$32i\sin^5\theta = 2i(\sin5\theta - 5\sin3\theta + 10\sin\theta)$

$\therefore \sin^5\theta = \frac{1}{16}(\sin5\theta - 5\sin3\theta + 10\sin\theta)$ ✓

(ii) Now from (i)

$16\sin^5\theta = \sin5\theta - 5\sin3\theta + 10\sin\theta$

$\therefore$ if $16\sin^5\theta = \sin5\theta$

$\therefore -5\sin3\theta + 10\sin\theta = 0$ ✓

$\therefore 2\sin\theta = \sin3\theta$

$= \sin(2\theta+\theta)$

$= \sin2\theta\cos\theta + \cos2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\therefore 4\sin^3\theta - \sin\theta = 0$ ✓

$\therefore \sin\theta(4\sin^2\theta - 1) = 0$

$\therefore \sin\theta(2\sin\theta - 1)(2\sin\theta + 1) = 0$

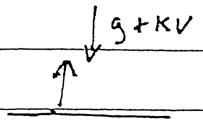
$\therefore \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{2}$ ✓

$\therefore \theta = n\pi$ or $n\pi + (-1)^n\frac{\pi}{6}$ or $n\pi + (-1)^n(-\frac{\pi}{6})$ ✓

$2k\pi + \frac{\pi}{6}$ $\quad (2k+1)\frac{-\pi}{6}$

Question 6 (continued)

(b) (i)




$$\ddot{x} = -g - kv$$

$$\therefore \frac{dv}{dt} = -g - kv \quad \checkmark$$

$$\therefore \frac{dt}{dv} = \frac{-1}{g+kv}$$

$$t = -\frac{1}{k}\int_u^v \frac{k}{g+kv}\,dv$$

$$= -\frac{1}{k}\Big[\ln(g+kv)\Big]_u^v$$

$$= -\frac{1}{k}\ln\left(\frac{g+kv}{g+ku}\right) \quad \checkmark$$

$$\therefore -kt = \ln\left(\frac{g+kv}{g+ku}\right)$$

$$\therefore g+kv = e^{-kt}(g+ku)$$

$$\therefore v = \frac{g+ku}{k}e^{-kt} - \frac{g}{k} \quad \checkmark$$

$$\therefore x = \int_0^t \left[\frac{g+ku}{k}e^{-kt} - \frac{g}{k}\right]dt \quad \checkmark$$

$$= -\frac{(g+ku)}{k^2}e^{-kt} - \frac{g}{k}t + \frac{g+ku}{k^2}$$

$$= \frac{g+ku}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k} \quad \checkmark$$

(ii) Now at $t = T$

$$\frac{g+ku}{k^2}\left(1-e^{-kT}\right) - \frac{gT}{k} = h + \frac{g}{k^2}\left(1-e^{-kT}\right) - \frac{gT}{h} \quad \checkmark$$

$$\therefore \left(1-e^{-kT}\right)\left[\frac{g+ku}{k^2} - \frac{g}{k^2}\right] = h$$

$$1 - e^{-kT} = \frac{hk}{u}$$

$$e^{-kT} = \frac{u-hk}{u} \quad \checkmark$$

$$\therefore -kT = \ln\left(\frac{u-kh}{u}\right)$$

$$T = \frac{1}{k}\ln\left(\frac{u}{u-kh}\right) \quad \checkmark$$

\-

Question 7

(a) (i) $\int_0^a f(x)\,dx$ let $u = a - x$

$x = a - u$ $\quad x = 0,\ u = a$

$\frac{dx}{du} = -1$ ✓ $\quad x = a,\ u = 0$

$= \int_a^0 -f(a-u)\,du$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$ ✓

(ii) $\int_0^{\pi} x\cos^2 x\,dx = \int_0^{\pi} (\pi - x)\cos^2(\pi - x)\,dx$

$\int_0^{\pi} x\cos^2 x\,dx = \int_0^{\pi} (\pi\cos^2 x - x\cos^2 x)\,dx$ ✓

$\therefore\ 2\int_0^{\pi} x\cos^2 x\,dx = \pi\int_0^{\pi} \cos^2 x\,dx$ ✓

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\therefore\ 2\int_0^{\pi} x\cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} (1 + \cos 2x)\,dx$

$\int_0^{\pi} x\cos^2 x\,dx = \frac{\pi}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi}$ ✓

$= \frac{\pi}{4}(\pi - 0)$

$= \frac{\pi^2}{4}$

Question 7 (continued)

(b)(i) $(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$

$= \cos^n\theta + \binom{n}{1}\cos^{n-1}\theta(i\sin\theta) + \binom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2 + \binom{n}{3}\cos^{n-3}\theta(i\sin\theta)^3 + \binom{n}{4}\cos^{n-4}\theta(i\sin\theta)^4 + \ldots$ ✓

$= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \ldots + i\left[\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ldots\right]$ ✓

as $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ....

∴ Equating the real and imaginary parts of both sides ✓

(1) $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \ldots$

(2) $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\sin^3\theta + \ldots$ ✓

(ii) From (1) & (2)

$\cos n\theta = \cos^n\theta\left[1 - \binom{n}{2}\cos^{-2}\sin^2\theta + \binom{n}{4}\cos^{-4}\theta\sin^4\theta - \ldots\right]$ ✓

$= \cos^n\theta\left[1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \ldots\right]$ --- (3)

$\sin n\theta = \cos^n\theta\left[\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \ldots\right]$ --- (4) ✓

∴ (4) ÷ (3)

$$\therefore \tan n\theta = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \ldots}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \ldots}$$ ✓

(c) (i) Number of points of intersection $= {}^6C_2 = 15$ ✓

There are 6 lines with 5 point of intersection on each

(ii) P(four of these pts do not all lie on one of the given lines)

= 1 − P(four of these pts chosen at random all lie on one of the given lines) ✓

$= 1 - \dfrac{6 \times {}^5C_4}{{}^{15}C_4} = \dfrac{89}{91}$ ✓

Question 8

(a) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\therefore \cos(x+y) - \cos(x-y) = -2\sin x \sin y$

Let $S = x+y$

$T = x-y$

$\therefore x = \frac{S+T}{2}, \ y = \frac{S-T}{2} \quad \therefore \cos S - \cos T = -2\sin\left(\frac{S+T}{2}\right).\sin\left(\frac{S-T}{2}\right)$

(b) $I_n = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2nx}{\sin 2x}\,dx$

(i) $I_1 = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{\sin 2x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{2\sin^2 x}{2\sin x \cos x}$ (as $\cos 2x = 1 - 2\sin^2 x$)

$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$ ✓

$= -\left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$= -\ln\frac{1}{\sqrt{2}}$

$= \frac{1}{2}\ln 2$ ✓

(ii) $I_{2r+1} - I_{2r-1} = \int_0^{\frac{\pi}{4}} \frac{1-\cos(4\times r+2x)}{\sin 2x} - \frac{1-\cos(4\times r-2x)}{\sin 2x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{\cos(4\times r-2x) - \cos(4\times r+2x)}{\sin 2x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{-2\sin 4\times r.\sin(-2x)}{\sin 2x}\,dx$ From (a) ✓

$= 2\int_0^{\frac{\pi}{4}} \sin 4\times r\,dx$

$= -\frac{1}{4r}\times 2\left[\cos 4\times r\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2r}\left[\cos r\pi - 1\right]$

$= -\frac{1}{2r}\left\{(-1)^r - 1\right\} = \frac{1-(-r)^r}{2r}$ ✓

Question 8 (continued)

(b) (iii) as $I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}$ *

$\therefore I_9 = I_{2\times4+1}$

$\therefore I_9 - I_7 = 0$ from *

$\therefore I_9 = I_7$

$I_7 = I_{2\times3+1}$

$\therefore I_7 - I_5 = \frac{1+1}{6}$

$\therefore I_7 = \frac{1}{3} + I_5$ ✓

Now $I_5 = I_{2\times2+1}$

$\therefore I_5 - I_3 = 0$

$\therefore I_5 = I_3$

$I_3 = I_{2\times1+1}$

$\therefore I_3 - I_1 = \frac{2}{2}$

$= 1$ ✓

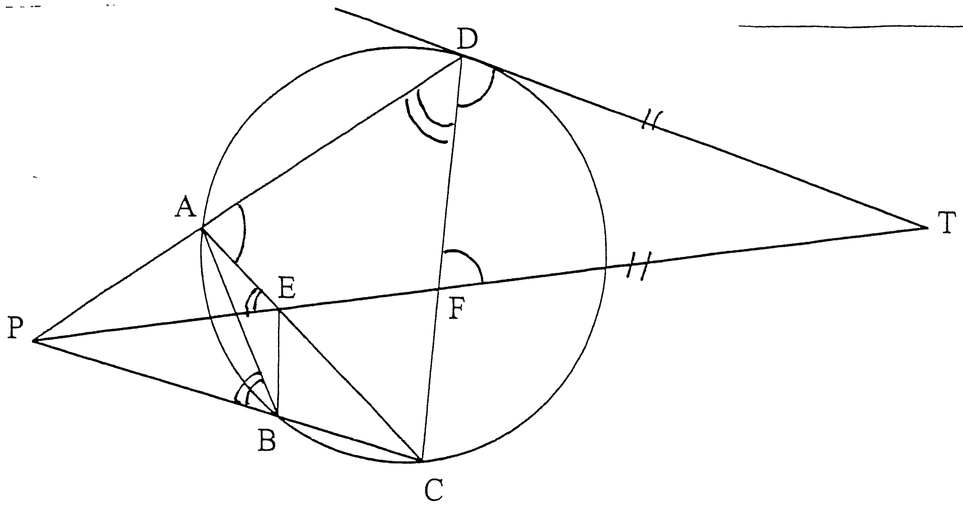
$I_3 = 1 + I_1$

$= 1 + \frac{1}{2}\ln 2$ from (i)

$\therefore I_9 = \frac{1}{3} + 1 + \frac{1}{2}\ln 2$ ✓

Question 8 (continued)

(c)




(i) Prove that AEFD is a cyclic quadrilateral

Proof: As TF = TD (given)

$\angle FDT = \angle DFT$ [base angles of isosceles △ are equal]

✓ $\angle FDT = \angle CAD$ [The angle between a tangent to a circle and chord at point of contact is equal to alt. ∠]

Now $\angle EFD = 180 - \angle DFT$ [Supplementary angles]

$= 180 - \angle CAD$

✓ ∴ AEFD is a cyclic quad [opposite angles are supplementary]

(ii) Prove that PBEA

Now $\angle AEP = \angle ADF$ [Exterior angle to a cyclic quad is equal to the opposite interior angle]

✓ AEFD is concyclic.

✓ $\angle PBA = \angle ADF$ [ " ]

ABCD is concyclic

∴ $\angle AEP = \angle PBA$

✓ ∴ PBEA is a cyclic quadrilateral.

[If two points lie on the same side of an interval, and the angles subtended at these points by the interval are equal, then the two points and the endpoints of the interval are concyclic]