Total marks – 120
Attempt Question 1-8
All questions are of equal value
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find

(i) $\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **2**

(ii) $\int \frac{dx}{x^2+2x+2}$ **2**

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{dx}{5+4\cos x+3\sin x}$ **3**

(c) Use the substitution $u = -x$ to evaluate $\int_{-1}^{1} \frac{dx}{e^x+1}$ **3**

(d) Evaluate the following definite integrals:

(i) $\int_0^1 \cos^{-1} x\, dx$ **2**

(ii) $\int_1^2 x(\ln x)^2\, dx$ **3**

Question 2 (15 marks) Start a new booklet

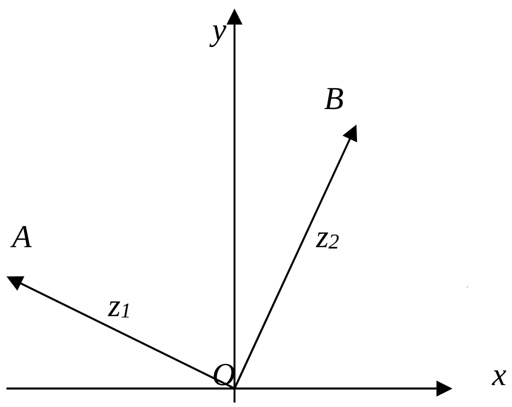
(a) If $z = 3 - 2i$, mark clearly on an Argand diagram the points represented by

(i) $2z$ **1**

(ii) $-2iz$ **2**

(b) If $|z_1 + z_2| = |z_1 - z_2|$, find the possible values of $\arg\left(\dfrac{z_1}{z_2}\right)$. **3**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\dfrac{4\pi}{5} + i\sin\dfrac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\dfrac{7\pi}{15} + i\sin\dfrac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral **2**

(ii) Express $z_2 - z_1$ in modulus-argument form. **3**

(d) z is a complex number such that $\arg z = \dfrac{\pi}{3}$ and $|z| \le 2$.

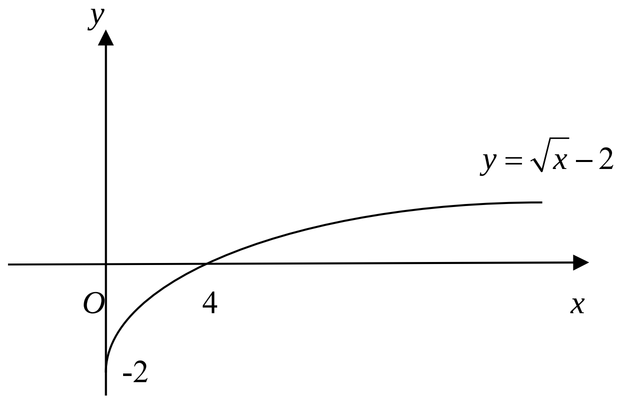
(i) Show the locus of the point P representing z in the Argand diagram. **2**

(ii) Find the possible values of the principal argument of $z - 1$ for z on this locus. **2**

Question 3 **(**15 marks) Start a new booklet

(a) Twelve different books are made into four parcels of three each. How many different sets of parcels could be made? **3**

(b)



The diagram shows the graph of the function $f(x) = \sqrt{x} - 2$. On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = |f(x)|$ **1**

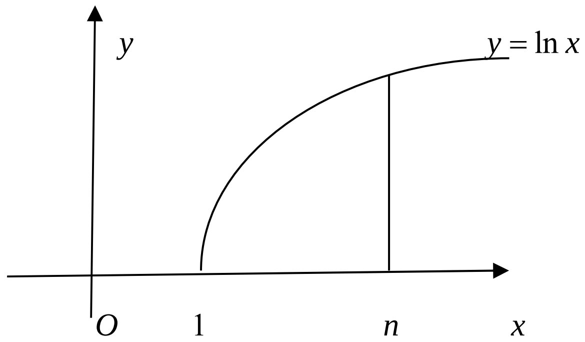
(ii) $y = [f(x)]^2$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \ln f(x)$ **2**

Question 3 Continued.

(c)



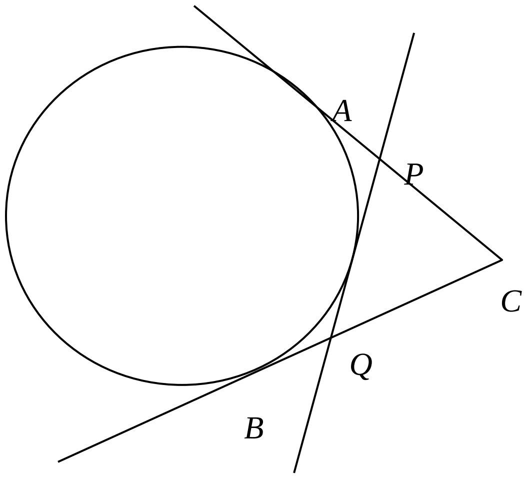
(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x \, dx$ **2**

(ii) Show that $\frac{d}{dx}(x\ln x - x) = \ln x$ and hence find the exact value of $\int_1^n \ln x \, dx$. **2**

(iii) Deduce that $\ln n! < \left(n+\frac{1}{2}\right)\ln n - n + 1$ **2**

Question 4 (15 marks) Start a new booklet

(a) A and B are two points on a circle. Tangents at A and B meet at C. A third tangent cuts CA and CB in P and Q respectively, as shown in the diagram. Show that the perimeter of ΔCPQ is constant and independent of PQ. **3**



(b) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **4**

(c) $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + z^{-n} = 2\cos n\theta$ for $n = 1, 2, 3, ...$ **2**

(ii) Hence show that $4\cos\theta\cos 2\theta\cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$. **3**

(iii) Hence, solve $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta = 1$, giving general solutions. **3**

Question 5 (15 marks) Start a new booklet

(a) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

(i) Find the equation of the tangent at *P*. **2**

(ii) Find the point of intersection, *T*, of the tangents at *P* and *Q*. **2**

(iii) If the chord *PQ* passes through the point $(0, 2)$, find the locus of *T*, **3**

(iv) Find the restriction on the locus of *T*. **1**

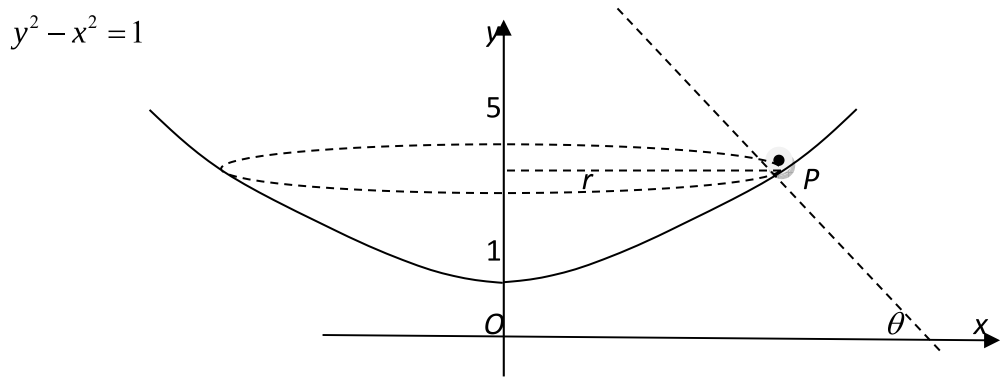
(b) The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved around the line $x = 3$. Express the volume of the resulting solid as a definite integral. Do not calculate the value of this integral. **3**

(c) A solid has, as its base, the circular region in the *xy*-plane bounded by the graph of $x^2 + y^2 = a^2$, where $a > 0$. Find the volume of the solid if every cross-section by a plane perpendicular to the *x*-axis is an equilateral triangle with one side in the base. **4**

Question 6 (15 marks) Start a new booklet

(a) A particle of mass m moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t = 0$, $x = 1$ and $v = 0$. Subsequently, the only force acting on the particle is one of magnitude $m\frac{k}{x^2}$, where k is a positive constant in a direction away from O. Show that v cannot exceed $\sqrt{(2k)}$. **4**

(b)



A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ about the y axis. A particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal where $\tan\theta = \frac{\sqrt{1+r^2}}{r}$. **4**

(ii) Draw a diagram showing the forces acting on P. **1**

(iii) Find expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. **3**

(iv) Find the values of ω for which the described motion of P is possible. **3**

Question 7 (15 marks) Start a new booklet

(a) The ellipse $\mathcal{E}$: $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse $\mathcal{E}$ indicating its foci *S, S'* and its directrices. **1**

(ii) Show that the tangent at $P(x_1, y_1)$ on the ellipse $\mathcal{E}$ has equation **1**

$9xx_1 + 25yy_1 = 225$.

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through *S*. Show that **2**

$4(y_2 - y_1) = x_1y_2 - x_2y_1$.

(iv) It is also known that $Q(x_2, y_2)$ lies on $\mathcal{E}$. Show that the tangents at *P* and *Q* on the ellipse intersect on the directrix corresponding to *S*. **2**

(v) Find the equation of the normal to $\mathcal{E}$ at *P* and decide under what circumstances, if any, it passes through *S* or *S'*. **1**

(b) $I_n = \int_1^e (1 - \ln x)^n \, dx$, $n = 1, 2, 3, \ldots$

(i) Show $I_n = -1 + nI_{n-1}$, $n = 1, 2, 3, \ldots$ **2**

(ii) Hence evaluate $\int_1^e (1 - \ln x)^3 \, dx$. **1**

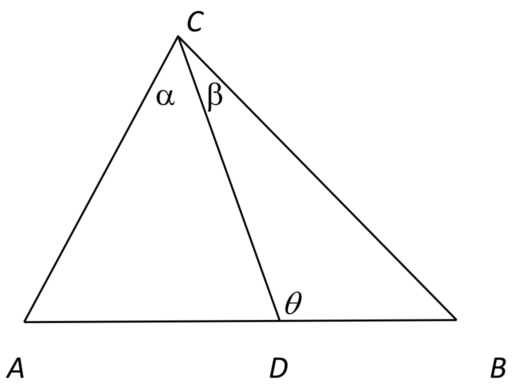
(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$, $n = 1, 2, 3, \ldots$ **2**

(iv) Show that $0 \le I_n \le e - 1$. **1**

(v) Deduce that $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e$. **2**

Question 8 (15 marks) Start a new booklet

(a)



In ΔABC, D is the point on AB that divides AB internally in the ratio $m : n$.
$\angle ACD = \alpha$, $\angle BCD = \beta$ and $\angle CDB = \theta$.

(i) By using the sine rule in each of ΔCAD and ΔCDB, show that **4**

$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{m}{n}.$$

(ii) Hence show that $\tan\theta = \dfrac{(m+n)\tan\alpha\tan\beta}{m\tan\beta - n\tan\alpha}$. **3**

(b) Let $f(x)$ be a function which satisfies the equation:

$f(xy) = f(x) + f(y)$ for all $x, y \neq 0$.

(i) Show that $f(1) = 0 = f(-1)$ and that $f(x)$ is an even function. **2**

(ii) Prove that $f(x+y) - f(x) = f\left(1+\dfrac{y}{x}\right)$ for $x, y, x+y \neq 0$ **2**

(iii) Suppose $f(x)$ is differentiable at $x=1$ and $f'(1)=1$. Deduce that $f(x)$ is differentiable at any $x \neq 0$ and $f'(x) = \dfrac{1}{x}$. **4**

End of Examination

Solutions

Question 1

(a) (i) $u=\sin\theta,\ du=\cos\theta d\theta$

$$\int\frac{du}{u^5}=-\frac{1}{4}u^{-4}+c$$

$$=-\frac{1}{4}\sin^4\theta+c$$

(ii) $$\int\frac{dx}{(x+1)^2+1}=\tan^{-1}(x+1)+c$$

(b) $t=\tan\frac{x}{2},\ \therefore\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}(1+t^2)$

$$\sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$$

$$\int\frac{\frac{2dt}{1+t^2}}{5+\frac{4(1-t^2)}{1+t^2}+\frac{6t}{1+t^2}}$$

$$=\int\frac{2dt}{t^2+6t+9}=\int\frac{2dt}{(t+3)^2}$$

$$=2(t+3)^{-1}+c$$

$$=\frac{-2}{\tan\frac{x}{2}+3}+c$$

(c) $u=-x,-du=dx$

When $x=-1,u=1$ and when $x=1,u=-1$

$$\int_1^{-1}\frac{-du}{e^{-u}+1}=\int_{-1}^{1}\frac{du}{e^{-u}+1}=\int_{-1}^{1}\frac{du}{\frac{1}{e^u}+1}=\int_{-1}^{1}\frac{e^u du}{1+e^u}$$

$$=\left[\ln\left(e^{u}+1\right)\right]_{-1}^{1}=\ln\frac{e+1}{1/e+1}=\ln e=1$$

(d) (i) $$\int_0^{\pi/2}\sin y\;dy=\left[-\cos y\right]_0^{\pi/2}=1$$

(ii) $$\left[\frac{1}{2}x^2\left(\ln x\right)^2\right]_1^2-\int_1^2\frac{1}{2}x^2.2.\frac{1}{x}\ln x\;dx$$

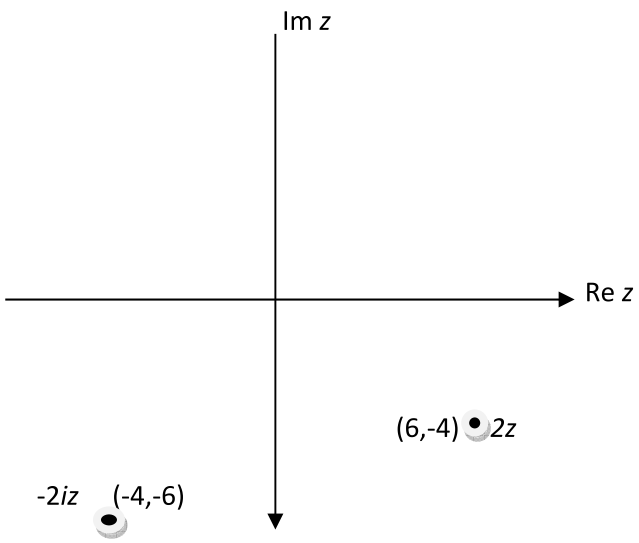
$$=2\left(\ln 2\right)^2-\int_1^2 x\ln x\;dx$$

$$=2\left(\ln 2\right)^2-\left[\frac{1}{2}x^2.\ln x\right]_1^2+\int_1^2 x.\frac{1}{x}dx$$

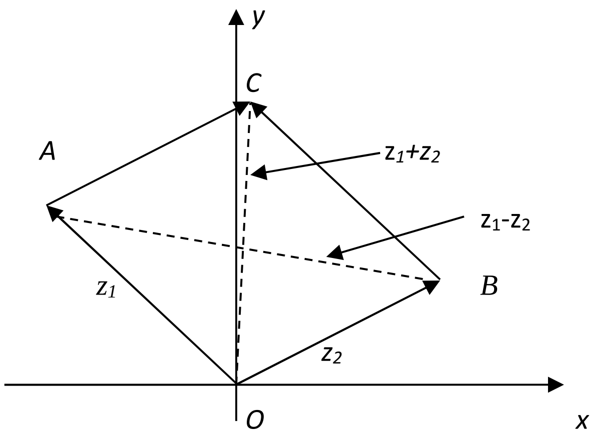
$$=2\left(\ln 2\right)^2-2\ln 2+1$$

Question 2

(a)



(b)



Let $\overrightarrow{OA}, \overrightarrow{OB}$ represent z_1, z_2 respectively. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}, \overrightarrow{BA}$ represent z_1+z_2 and z_1-z_2, respectively. Since $|z_1 + z_2| = |z_1 - z_2|$, $OC + AB$. Hence, $OACB$ is a rectangle.

$$\therefore \angle AOB = \frac{\pi}{2}.$$

But, $\angle AOB = \arg z_1 - \arg z_2$ or $\angle AOB = \arg z_2 - \arg z_1$ (if z_1, z_2 swapped).

$$\therefore \arg \frac{z_1}{z_2} = \pm \frac{\pi}{2}$$

(c) (i) $\therefore \arg \angle AOB = \arg z_1 - \arg z_2 = \frac{4\pi}{5} - \frac{7\pi}{15} = \frac{\pi}{3}$

$$OA = OB = 2$$

$$AB = \sqrt{2^2 + 2^2 - 2.2.2.\cos\frac{\pi}{3}} = 2$$

$\therefore \Delta OAB$ is equilateral.

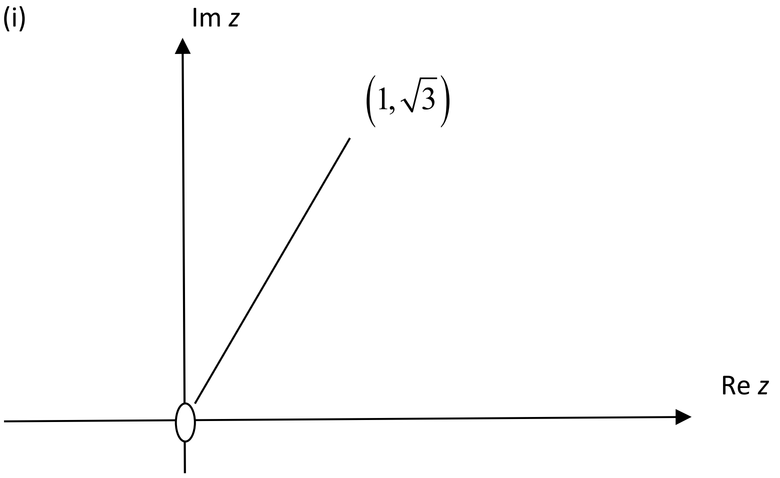
(ii) The vector $\overrightarrow{AB}$ represents $z_2 - z_1$. Now, $\overrightarrow{AB}$ is a clockwise rotation of $\overrightarrow{OB}$ by $\frac{\pi}{3}$.

$$\therefore\ z_2 - z_1 = z_2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$$

$$= 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= 2\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)$$

(d) (i)

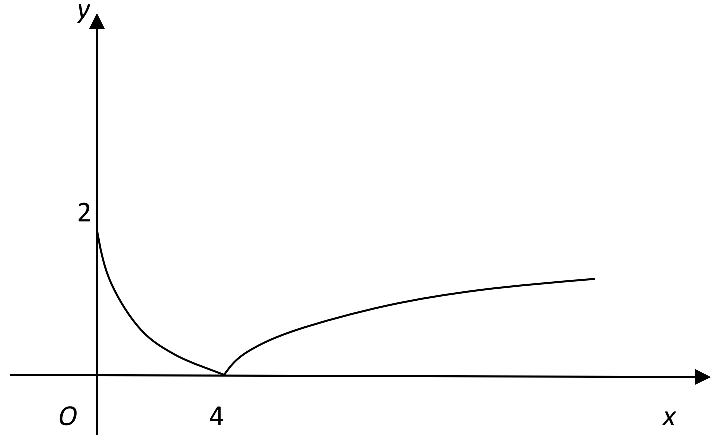


(ii) $\frac{\pi}{2} \le \arg(z-1) \le \pi$

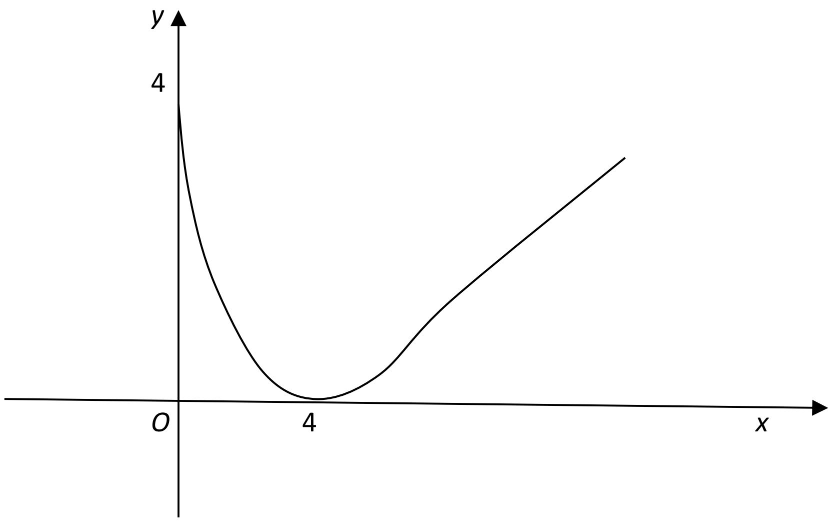
Question 3

(a) $\frac{{}^{12}C_3 . {}^{9}C_3 . {}^{6}C_3 . {}^{3}C_3}{4!} = 15400$

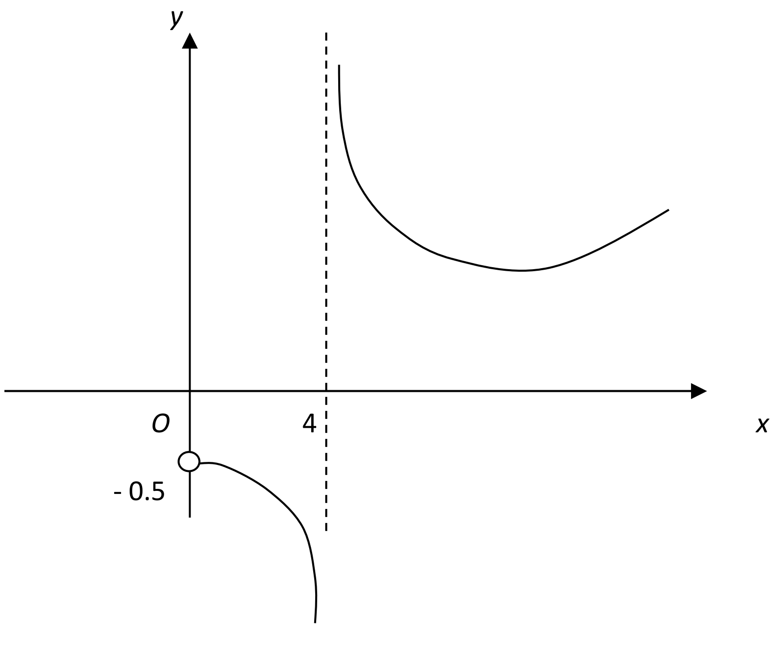
(b) (i) $y = |f(x)|$



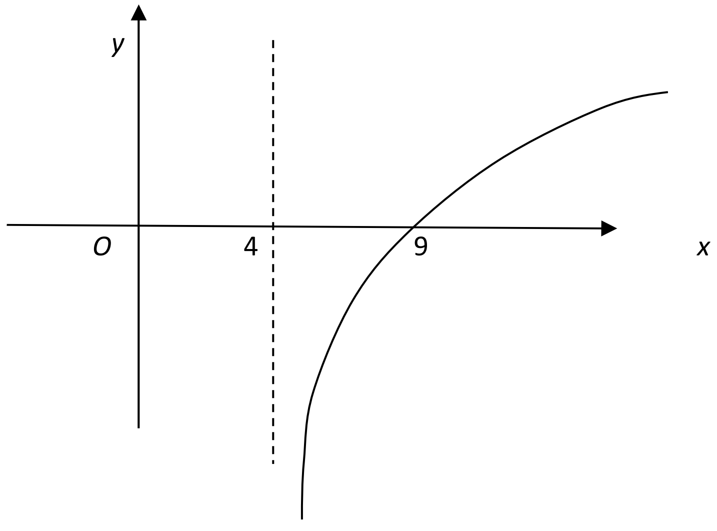
(ii) $y=\left[f(x)\right]^2$



(iii) $y=\dfrac{1}{f(x)}$



(iv) $y=\ln f(x)$



(c) (i) $$\int_1^n \ln x\, dx \approx \frac{1}{2}\left[\ln 1+\ln n+2\left(\ln 2.3.4....(n-1)\right)\right]$$

$$=\frac{1}{2}\ln n+\ln\left(1.2.3.4....(n-1)\right)$$

$$=\frac{1}{2}\ln n+\ln(n-1)!$$

$$=\frac{1}{2}\ln n-\ln n+\ln n+\ln(n-1)!$$

$$=\ln n!-\frac{1}{2}\ln n$$

(ii) $$\frac{d}{dx}(x\ln x-x)=x.\frac{1}{x}+\ln x-1=\ln x$$

$$\therefore \int_1^n \ln x\, dx=\left[x\ln x-x\right]_1^n=n\ln n-n+1$$

(iii) The trapezia lie below the curve.

$$\therefore\ n\ln n-n+1>\ln n!-\frac{1}{2}\ln n$$

$$\ln n!<n\ln n-n+1+\frac{1}{2}\ln n$$

$$\ln n!<\ln n\left(n+\frac{1}{2}\right)-n+1$$

Question 4

(a) Perimeter of $\Delta CPQ=CP+CQ+PQ$

But $PQ=AP+BQ$ (tangents drawn from *P* are of equal length and (tangents drawn from *Q* are of equal length)

Perimeter of $\Delta CPQ=CP+AP+CQ+BQ=CA+CB$

Which is constant and independent of *PQ.*

(b) $P(2)=9, P(3)=4$

$P(x)=Q(x).(x-2)(x-3)+R(x)$, where $R(x)=ax+b$

$P(2)=0+2a+b=9$

$P(3)=0+3a+b=4$

$a=-5,\ b=19$

Remainder is $19-5x$

(c) (i) $z^n+z^{-n}=\cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)$

$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$ (note cosine even and sine odd)

$=2\cos n\theta$

(ii) $z+z^{-1}=2\cos\theta,\ z^2+z^{-2}=2\cos 2\theta,\ z^3+z^{-3}=2\cos 3\theta$

$8\cos\theta\cos 2\theta\cos 3\theta=\left(z+z^{-1}\right)\left(z^2+z^{-2}\right)\left(z^3+z^{-3}\right)$

$=z^6+1+z^2+z^4+z^{-4}+z^{-2}+1+z^{-6}$

$=2+z^2+z^{-2}+z^4+z^{-4}+z^6+z^{-6}$

$=2+2\cos 2\theta+2\cos 4\theta+2\cos 6\theta$

$\therefore 4\cos\theta\cos 2\theta\cos 3\theta=1+\cos 2\theta+\cos 4\theta+\cos 6\theta$

(iii) $2\cos^2\theta+2\cos^2 2\theta+2\cos^2 3\theta=2$

$1+\cos 2\theta+\cos 4\theta+\cos 6\theta=0$

$4\cos\theta\cos 2\theta\cos 3\theta=0$

$\therefore \theta=2k\pi\pm\frac{\pi}{2},\ \text{or } \theta=2k\pi\pm\frac{\pi}{4},\ \text{or } \theta=2k\pi\pm\frac{\pi}{6}$

Question 5

(a) (i) $x\dfrac{dy}{dx}+y=0,\ \therefore \dfrac{dy}{dx}=\dfrac{-y}{x}$

At *P*, $\dfrac{dy}{dx}=\dfrac{-3/p}{3p}=-\dfrac{1}{p^2}$

Required equation: $y-\dfrac{3}{p}=-\dfrac{1}{p}(x-3p)$

Which gives $x+p^2y=6p$

(ii) tangent at *P* $x+p^2y=6p$

Tangent at *Q* $x+q^2y=6q$

When solved simultaneously, we get the coordinates of T:
$\left(\dfrac{6pq}{p+q},\dfrac{6}{p+q}\right)$

(iii) $m_{PQ}=\dfrac{3/p-3/q}{3p-3q}=-\dfrac{1}{pq}$

Equation *PQ*: $y-\dfrac{3}{p}=-\dfrac{1}{pq}(x-3p)$

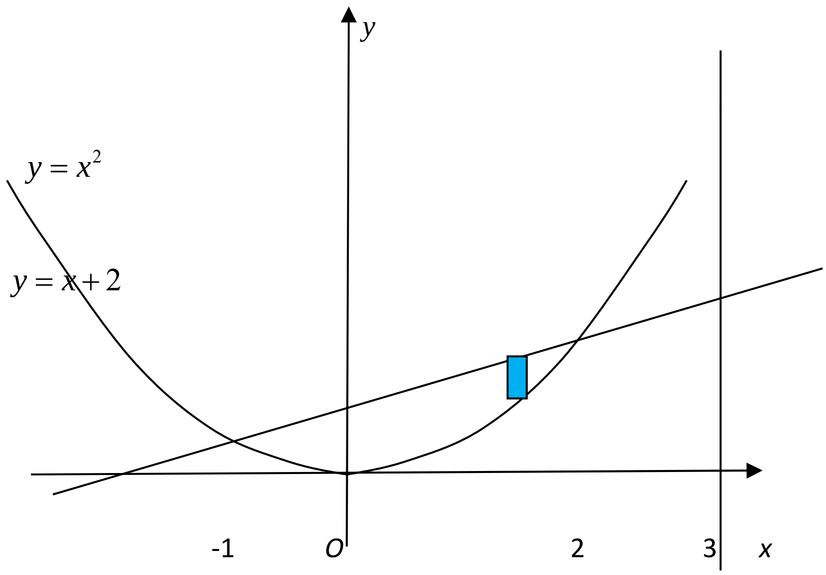
Now, when $x=0, y=2$

$\therefore\ \dfrac{p+q}{pq}=\dfrac{2}{3}$ or $p+q=\dfrac{2pq}{3}$

At *T*, $x=\dfrac{6pq}{p+q}=\dfrac{6pq}{2pq/3}=9$

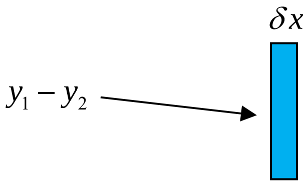
(iv) Locus of *T* is $x=9$, with the restriction that $y<0$. i.e. *T* is in the 4th quadrant only.

(b)



To find points of intersection: $x^2 = x + 2, \quad \therefore x = -1, 2$

Consider a typical strip



Rotate the strip to form a shell. The volume of the shell is given by

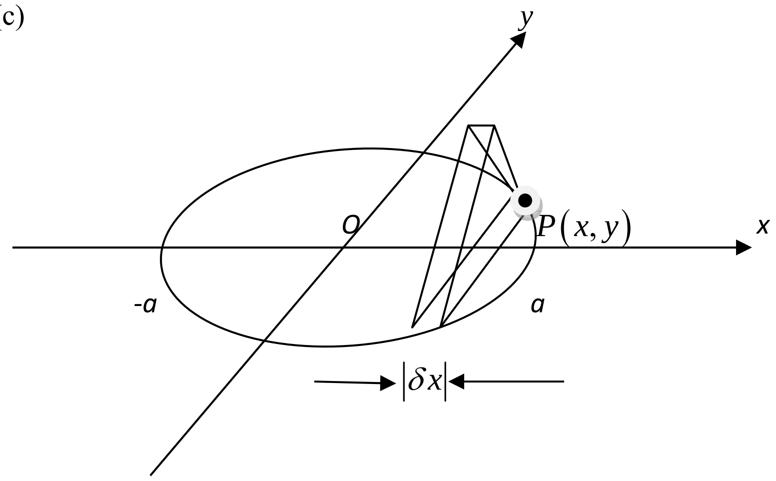
$\delta V = 2\pi rh.\delta x$, where $r = 3 - x$ and $h = y_1 - y_2 = x + 2 - x^2$

$$V \approx \sum_{x=-1}^{x=2} 2\pi (3 - x)\left(x + 2 - x^2\right)\delta x$$

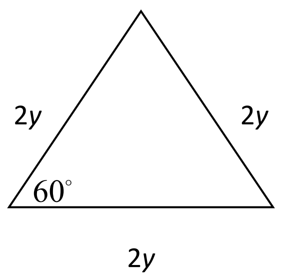
$$V = \lim_{\delta x \to 0} \sum_{x=-1}^{x=2} 2\pi (3 - x)\left(x + 2 - x^2\right)\delta x$$

$$= \int_{-1}^{2} 2\pi (3 - x)\left(x + 2 - x^2\right)dx$$

(c)



Consider a typical slice of width δx.



$$\delta V = \frac{1}{2} 2y.2y.\sin 60^\circ.\delta x = y^2\sqrt{3}.\delta x$$

$$V \approx \sum_{x=-a}^{x=a} y^2\sqrt{3}.\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} y^2\sqrt{3}.\delta x = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} \left(a^2 - x^2\right)\sqrt{3}.\delta x$$

$$= \sqrt{3}\int_{-a}^{a} \left(a^2 - x^2\right) dx$$

$$= 2\sqrt{3}\int_{0}^{a} \left(a^2 - x^2\right) dx$$

$$= 2\sqrt{3}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{4\sqrt{3}}{3} a^3$$

Question 6

(a) Choose the initial direction as positive

$$\ddot{x} = \frac{k}{x^2},\ k > 0$$

$$v\frac{dv}{dx} = \frac{k}{x^2} \Rightarrow vdv = \frac{k}{x^2}dx$$

$$\frac{1}{2}v^2 = -\frac{k}{x} + c\text{, where } c \text{ is constant}$$

Now, when $x = 1, v = 0 \Rightarrow c = k$

$$\therefore v^2 = 2k\left(1 - \frac{1}{x}\right)$$

Now, $x \geq 1$ $\quad \therefore 0 \leq 1 - \frac{1}{x} < 1$

$$\therefore 0 \leq v^2 < 2k$$

Hence, v cannot exceed $\sqrt{2k}$

(b) (i) $\quad y^2 - x^2 = 1 \Rightarrow 2y\frac{dy}{dx} - 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

At *P*, $\frac{dy}{dx} = \frac{r}{\sqrt{1+r^2}}$

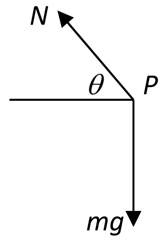
Hence, the gradient of the normal at *P* is $\frac{-\sqrt{1+r^2}}{r}$

Now, the gradient of the normal is the tangent of the angle made with the 'positive' *x* axis.

$$\therefore \tan\left(180^\circ - \theta\right) = \frac{-\sqrt{1+r^2}}{r}$$

$$\therefore \tan\theta = \frac{\sqrt{1+r^2}}{r}$$

(ii)



(iii) Resolve forces

Horizontally

$$mr\omega^2 = N\cos\theta$$

Vertically

$$mg = N\sin\theta$$

$$\therefore \tan\theta = \frac{g}{r\omega^2} = \frac{\sqrt{1+r^2}}{r} \quad \text{from part (i)}$$

$$1+r^2 = \frac{g^2}{\omega^4} \Rightarrow r = \frac{\sqrt{g^2-\omega^4}}{\omega^2}$$

Now, $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{mr\omega^2}{N}\right)^2 + \left(\frac{mg}{N}\right)^2 = 1$$

$$N^2 = m^2r^2\omega^4 + m^2g^2 = m^2\,\frac{g^2-\omega^4}{\omega^4}\,\omega^4 + m^2g^2$$

$$= m^2\left(2g^2-\omega^4\right)$$

$$\therefore N = m\sqrt{\left(2g^2-\omega^4\right)}$$

(iv) $r = \frac{\sqrt{g^2 - \omega^4}}{\omega^2}$ and $r > 0$

$\therefore g^2 > \omega^4$

But $N > 0$ $\therefore 2g^2 > \omega^4$

Both these conditions exist if $g > \omega^2$

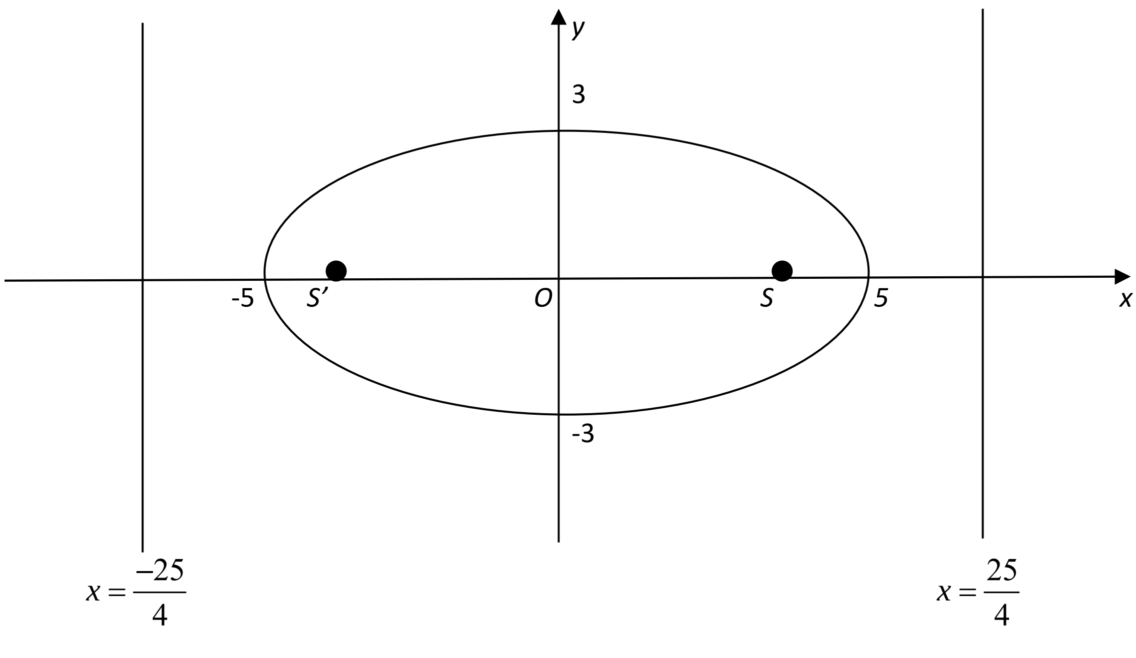
Note: $y \le 5 \Rightarrow y^2 \le 25 \Rightarrow 1 + r^2 \le 25$

$\therefore \frac{g^2}{\omega^4} > 25 \Rightarrow \omega \ge \sqrt{\frac{g}{5}}$

$\therefore \sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$

Question 7

(a) (i)



$ae = \pm 4,\ a = 5,\ \therefore e = \frac{4}{5}$

Hence, the directrices are $x = \frac{\pm 25}{4}$

(ii) $\frac{x^2}{25}+\frac{y^2}{9}=1$

$$\frac{2x}{25}+\frac{2y\,{}^{dy}\!/_{dx}}{9}=1$$

At $P(x_1,y_1)$, $\frac{dy}{dx}=\frac{-9x_1}{25y_1}$

Required equation:

$$y-y_1=\frac{-9x_1}{25y_1}(x-x_1)$$

$$9xx_1+25yy_1=9x_1{}^2+25y_1{}^2=225$$

Note: $P(x_1,y_1)$ lies on the curve $9x_1{}^2+25y_1{}^2=225$

(iii) $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

At $x=4, y=0$

$$-y_1(x_2-x_1)=(y_2-y_1)(4-x_1)$$

$$4(y_2-y_1)=x_1(y_2-y_1)-y_1(x_2-x_1)=x_1y_2-x_2y_1$$

(iv) Two equations are: $9xx_1+25yy_1=225$ and
$9xx_2+25yy_2=225$

$$25y=\frac{9xx_1-225}{y_1}=\frac{9xx_2-225}{y_2}$$

$$\therefore 9x(x_1y_2-x_2y_1)=225(y_2-y_1)$$

$$\therefore 9x.4(y_2-y_1)=225(y_2-y_1)$$

$$x=\frac{225}{36}=\frac{25}{4}$$

(v) $y-y_1=\frac{25y_1}{9x_1}(x-x_1)$

If the normal passes through $S(4,0)$ then

$$-y_1 = \frac{25y_1}{9x_1}(4 - x_1)$$

$$100y_1 = 16x_1y_1$$

Hence, either $x_1 = 6\frac{1}{4}$ or $y_1 = 0$

But $-5 \le x_1 \le 5, \quad \therefore y_1 = 0$

Similarly, if the normal passes through $S'(-4,0)$, $y_1 = 0$

The normals that pass through *S* and *S'* are at $(\pm 5, 0)$. The equation of each normal is $y = 0$.

(b) (i)

$$I_n = \int_1^e (1 - \ln x)^n \, dx$$

$$= \left[x(1 - \ln x)^n \right]_1^e - \int_1^e nx(1 - \ln x)^{n-1} \left(-\frac{1}{x} \right) dx$$

$$= -1 + nI_{n-1}$$

(ii)

$$I_3 = -1 + 3I_2 = -1 + 3(-1 + 2I_1) = -4 - 6(-1 + I_0)$$

$$= -10 + 6\int_1^e dx = -10 + 6(e - 1) = -16 + 6e$$

(iii)

$$I_r = -1 + rI_{r-1}$$

$$\frac{I_r}{r!} = \frac{-1}{r!} + \frac{rI_{r-1}}{r!}$$

$$\frac{I_r}{r!} = \frac{-1}{r!} + \frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^{n} \frac{I_r}{r!} = \sum_{r=1}^{n} \frac{-1}{r!} + \sum_{r=1}^{n} \frac{I_{r-1}}{(r-1)!}$$

$$\frac{I_n}{n!} + \sum_{r=1}^{n-1} \frac{I_r}{r!} = \sum_{r=1}^{n} \frac{-1}{r!} + \sum_{r=0}^{n-1} \frac{I_r}{r!}$$

$$\frac{I_n}{n!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{I_0}{0!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{\int_1^e dx}{0!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{e}{0!} + \frac{-1}{0!} = e - \sum_{r=0}^{n} \frac{1}{r!}$$

(iv) The domain for the original integrand is $1 \le x \le e$

$\ln 1 \le \ln x \le \ln e$

$0 \le \ln x \le 1$

$0 \ge -\ln x \ge -1$

$1 \ge 1 - \ln x \ge 1 - 1$

$\therefore 0 \le (1 - \ln x)^n \le 1$

$$0 \le \int_1^e (1 - \ln x)^n \, dx \le \int_1^e dx$$

$0 \le I_n \le e - 1$

(v) $0 \le \frac{I_n}{n!} \le \frac{e-1}{n!}$

$$0 \le \lim_{n \to 0} \frac{I_n}{n!} \le \lim_{n \to 0} \frac{e-1}{n!}$$

Now, $\lim_{n \to \infty} \frac{e-1}{n!} = 0$, $\therefore \lim_{n \to 0} \frac{I_n}{n!} = 0$

$\therefore \lim_{n \to 0} \left(e - \sum_{r=0}^{n} \frac{1}{r!} \right) = 0$, from part (iii)

$$e - \lim_{n \to 0} \left(\sum_{r=0}^{n} \frac{1}{r!} \right) = 0, \quad \therefore \; e = \lim_{n \to 0} \left(\sum_{r=0}^{n} \frac{1}{r!} \right)$$

Question 8

(a) (i) $\angle CAD = \theta - \alpha$ (ext $\angle$ of ΔCAD)

$$\therefore \frac{CD}{\sin(\theta-\alpha)} = \frac{AD}{\sin\alpha} \qquad (1)$$

$\angle CBD = 180^\circ - (\theta+\beta)$ ($\angle$ sum of ΔCBD)

$$\therefore \frac{CD}{\sin(180^\circ - (\theta+\beta))} = \frac{DB}{\sin\beta} \qquad (2)$$

$$(1)\div(2) \Rightarrow \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} = \frac{AD\sin\beta}{DB\sin\alpha}$$

$$\therefore \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{AD}{DB} = \frac{m}{n}$$

(ii) $n\sin(\theta+\beta)\sin\alpha = m\sin(\theta-\alpha)\sin\beta$

$$n\sin\alpha(\sin\theta\cos\beta+\cos\theta\sin\beta) = m\sin\beta(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$$

Divide both sides by $\cos\alpha\cos\beta\cos\theta$

$$n\tan\alpha(\tan\theta+\tan\beta) = m\tan\beta(\tan\theta-\tan\alpha)$$

$$(n+m)\tan\alpha\tan\beta = \tan\theta(m\tan\beta-n\tan\alpha)$$

$$\therefore \tan\theta = \frac{(n+m)\tan\alpha\tan\beta}{(m\tan\beta-n\tan\alpha)}$$

(b) (i) $f(x) = f(x\times 1) = f(x)+f(1)$

$\therefore f(1) = 0$

$f(1) = f(-1\times -1) = f(-1)+f(-1)$

$2f(-1) = 0, \therefore f(-1) = 0$

$f(-x) = f(-1)+f(x) = f(x)$

$\therefore f(x)$ is even.

(ii)

$$f(x+y)-f(x)=f\left(x\left(1+\frac{y}{x}\right)\right)-f(x)=f(x)+f\left(1+\frac{y}{x}\right)-f(x)=f\left(1+\frac{y}{x}\right)$$

for $x, y, x+y \neq 0$

(iii) $f'(1)=1, \therefore \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=1$

$$\therefore \lim_{h\to 0}\frac{f(1+h)}{h}=1 \qquad (1)$$

Now,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h} \quad \text{from (ii)}$$

$$\therefore f'(x)=\frac{1}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h/x}$$

Now, as $h\to 0, \frac{h}{x}\to 0$ for all $x\neq 0$

Let $u=\frac{h}{x}$.

$\therefore f'(x)=\frac{1}{x}\lim_{u\to 0}\frac{f(1+u)}{u}=\frac{1}{x}$ from **(1)**

$2a+3b+c$