STUDENT NUMBER ....

Class ....

NORMANHURST BOYS' HIGH SCHOOL
NEW SOUTH WALES

2013
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II Pages 5–11

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 40 minutes for this section

Assessable Outcomes: A student

O1	applies graphical methods to various functions & solves polynomials.
O2	applies a wide variety of techniques involving integration.
O3	applies problem solving techniques with complex numbers.
O4	solves conics & determines volumes by methods of integration.
O5	solves restricted motion problems in mechanics & extension 1 harder topics.

TIE ANSWER SHEET TO THE QUESTION PAPER AND YOUR WRITING PAPER.

HAND UP IN ONE TIED BUNDLE.

Section I

10 marks
Attempt Questions 1–10
Allow about 20 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The equation of the tangent to $xy^3 + 2y = 4$ at the point (2, 1) is

(A) $x + 8y = 10$

(B) $x - 8y = 10$

(C) $x + 8y = -10$

(D) $x - 8y = -10$

2 If $z = 1 - \sqrt{3}i = 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$, then what is the value of z^{21} ?

(A) 2^{21}

(B) -2^{21}

(C) $(2^{21})i$

(D) $-(2^{21})i$

3 When the circle $|z - (3 + 4i)| = 5$ is sketched on the Argand Diagram the maximum value of $|z|$ occurs when z lies at the end of the diameter that passes through the centre and the origin.

What is the maximum value of $|z|$?

(A) $\sqrt{5}$

(B) 5

(C) 10

(D) $\sqrt{10}$

4 One rational root exists for $P(x) = 2x^3 - 3x^2 + 4x + 3$ such that $P(\frac{-1}{2}) = 0$.

When $P(x)$ is fully factorised over the complex field, what is the result?

(A) $(2x+1)(x^2 - 2x + 3)$

(B) $(2x+1)(x - 1 + i\sqrt{2})(x + 1 + i\sqrt{2})$

(C) $(2x+1)(x + 1 - i\sqrt{2})(x + 1 + i\sqrt{2})$

(D) $(2x+1)(x - 1 - i\sqrt{2})(x - 1 + i\sqrt{2})$

5 The cubic equation $2y^3 - 9y^2 + 12y + k = 0$ has two equal roots.

What are the possible values for k?

(A) -4 and -5

(B) -4 and 5

(C) 4 and -5

(D) 4 and 5

6

Which of the following graphs is the locus of the point P representing the complex number z moving in an Argand diagram such that $|z - 2i| = 2 + \text{Im}\, z$?

(A) a circle

(B) a parabola

(C) a hyperbola

(D) a straight line

7 What is the area bounded by the x axis and the curve $y = x(16 + x^2)^{-0.5}$ between $x = 0$ and $x = 3$?

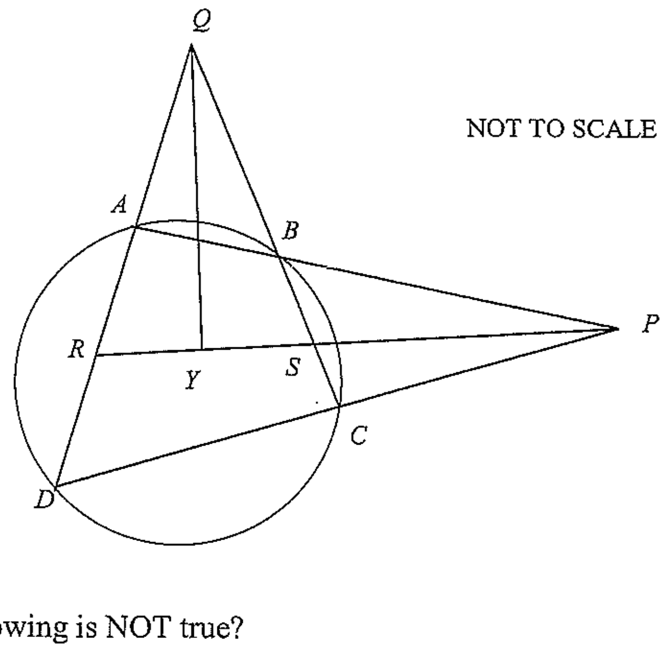
(A) $3\ units^2$
(B) $log_e 3\ units^2$
(C) $log_e e\ units^2$
(D) $log_e 1\ units^2$

8 For constant k, the equation $e^{2x} = k\sqrt{x}$ has exactly one solution when there is a common point as well as a common tangent.

What is the value of k?

(A) 1
(B) $\sqrt{e}$
(C) $2\sqrt{e}$
(D) e

9 $ABCD$ is a cyclic quadrilateral. Q and P are external points such that Y lies on the line PR and S is the intersection of PR and QC. Assume that PR bisects angle APD and QY bisects angle DQC.

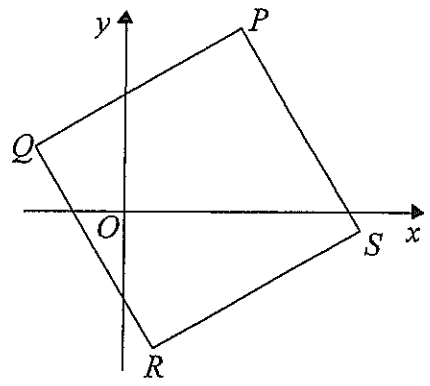


Which of the following is NOT true?

(A) $<QYR$ is a right angle
(B) ΔQRS is always isosceles
(C) $ABCD$ is always a kite
(D) Y is always the midpoint of RS

10

In the Argand diagram below vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$ represent the complex numbers p, q, r, s respectively where $PQRS$ is a square.



The statement $q - s = i(p - r)$ about lengths of the square is

(A) always true

(B) never true

(C) sometimes true

(D) not able to be accurately determined

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours 40 minutes for this section

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing page.

(a) (i) Find a primitive function for each of $\frac{x+1}{x^2+2x+5}$ and $\frac{1}{x^2+2x+5}$. **2**

(ii) Hence, or otherwise, find $\int \frac{x}{x^2+2x+5}\,dx$. **1**

(b) Evaluate $\int_1^3 \frac{dx}{x(x+2)}$. **3**

(c) (i) Express $(\sec x\tan x)^4$ as a product involving $\sec^2 x$. **1**

(ii) Show that $\int_0^{\frac{\pi}{4}} \sec^4 x\tan^4 x\,dx = \frac{12}{35}$. **2**

(d) Use the t-substitution method with $t = \tan\frac{\theta}{2}$ to find the value of **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}.$$

(e) (i) Show that $U_n = \frac{n-1}{n}U_{n-2}$ if $U_n = \int_0^{\frac{\pi}{2}} \sin^n x$. **2**

(ii) Hence, or otherwise, prove that $k = 32$ when $U_4 - U_6 = \frac{\pi}{k}$. **1**

Question 12 (15 marks) Use a SEPARATE writing page.

(a) (i) Write $\frac{3}{x+2} + x - 2$ as a single algebraic fraction. **1**

(ii) Sketch $y = \frac{3}{x+2} + x - 2$. **1**

(iii) Hence, or otherwise, solve the inequality $\frac{x^2-1}{x+2} \leq 0$. **1**

(b) The roots of $x^3 + 2x^2 - 3x - 1 = 0$ are α, β and γ. **4**

Find an equation whose roots are $\frac{\alpha\beta}{\gamma}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\beta\gamma}{\alpha}$.

(c) The points, $P(cp, cp^{-1})$ and $Q(cq, cq^{-1})$, lie on the rectangular hyperbola $xy = c^2$. The chord PQ meets the x axis at C. O is the centre of the hyperbola and R is the midpoint of PQ.

(i) Draw a sketch showing all the information. **1**

(ii) Find the equation of chord PQ. **2**

(iii) Find the co-ordinates of C. **1**

(iv) Find the co-ordinates of R. **1**

(v) Show that OR = RC. **3**

Question 13 (15 marks) Use a SEPARATE writing page.

(a) The hyperbola, H has the Cartesian equation $5x^2 - 4y^2 = 20$.

P is an arbitrary point, $(2\sec\theta, \sqrt{5}\tan\theta)$.

(i) Find the eccentricity of H and state the co-ordinates of its foci, S and S'. **2**

(ii) State the equations of the directrices and both asymptotes for H. **2**

(iii) Sketch the curve, clearly showing all of the above features. **1**

(iv) Demonstrate that $P(2\sec\theta, \sqrt{5}\tan\theta)$ lies on H. **1**

(v) Show that the tangent to H at P is **2**

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1.$$

(vi) The tangent at P cuts the asymptotes at L and M. **3**

Prove that $LP = PM$.

(vii) O is the origin. **2**

Show that the area of ΔOLM is independent of the position P on H.

(b) The function $y = f(x)$ is denoted by $f(x) = x^3 - 6x$.

(i) Sketch the graph of $y = |f(x)| = |x^3 - 6x|$ on a separate set of axes. **1**

(ii) Sketch the graph of $y = \frac{1}{f(x)} = (x^3 - 6x)^{-1}$ on a separate set of axes. **1**

Question 14 (15 marks) Use a SEPARATE writing page.

(a) Consider the region bounded by the two curves $y = 3 - x^2$ and $y = -2x$.

Suppose two vertical lines, one unit apart, intersect the given region.

(i) The vertical lines are $x = x_1$ and $x = x_1 + 1$. **4**

Find the value/s of x_1 so that the area enclosed by the two vertical lines and the two curves is a maximum.

(ii) Show that this enclosed area is $3\frac{11}{12}$ units^2. **2**

Justify that this area is the maximum.

(b) The area bounded by the y axis, the line $y = 1$ and $y = \sin x$ is revolved about the line $y = 1$. **4**

Using a slicing technique, find the volume of the solid of revolution formed between $x = 0$ and $x = \frac{\pi}{2}$.

(c) Use the method of cylindrical shells to find the volume of the solid formed when the area enclosed by $y = (x - 2)^2$ and $y = 4$ is rotated about the y axis. **5**

Question 15 (15 marks) Use a SEPARATE writing page.

(a) (i) Factorise $z^5 + 1$ over the real field. **1**

(ii) List the roots of $z^5 + 1 = 0$ in $rcis\theta$ form. **1**

(iii) Deduce that $2cos\frac{\pi}{5} + 2cos\frac{3\pi}{5} - 1 = 0$. **2**

(b) (i) Using the tan (A – B) expansion, show that if $mx = tan^{-1}Q - tan^{-1}v$

then $mx = tan^{-1}(\frac{Q-v}{1+Qv})$. **1**

(ii) Show that $a = 1$, $b = -1$ and $c = 0$ if $\frac{1}{v+v^3} = \frac{a}{v} + \frac{bv+c}{1+v^2}$. **1**

(c)

A particle moves in a straight line against a resistance numerically equal to $m(v + v^3)$ where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$. Assume $\ddot{x} = -m(v + v^3)$.

(i) Show that the displacement x in terms of v is $x = \frac{1}{m}tan^{-1}(\frac{Q-v}{1+Qv})$. **3**

(ii) Prove that $t = \frac{1}{2m}log_e(\frac{Q^2(1+v^2}{v^2(1+Q^2)})$ where t is the time elapsed. **3**

(iii) Find an expression for the square of the velocity as a function of time. **1**

(iv) By finding the limiting values of velocity and displacement, explain why this particle eventually slows down and show that this occurs near a point where $Q = tan(mx)$. **2**

Question 16 (15 marks) Use a SEPARATE writing page

(a) A sequence of polynomials, called the *Bernoulli Polynomials*, is defined by the three conditions:-

1. $B_0(x) = 1$
2. $B_n'(x) = \frac{d}{dx}(B_n(x)) = nB_{n-1}(x)$
3. $\int_0^1 B_n(x)dx = 0 \quad \text{if } n \geq 1$

(i) Show that $B_1(x) = x - \frac{1}{2}$. **3**

(ii) If $B_n(x+1) - B_n(x) = nx^{n-1}$ and $g(x) = B_{n+1}(x+1) - B_{n+1}(x)$, prove that **2**

$$g'(x) = (n+1)nx^{n-1}.$$

Hence show $g(x) = (n+1)x^n + C$, where C is a constant.

(iii) Use the method of mathematical induction to prove that **5**

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{if } n \geq 1.$$

(b) (i) By squaring, or otherwise, show that for $k \geq 0$, **1**

$$2k + 3 > 2\sqrt{k+2}\,\sqrt{k+1}.$$

(ii) By decomposing $2k + 3$ and factorising $2\sqrt{k+2}\,\sqrt{k+1} - 2(k+1)$ show that for $k \geq 1$, **2**

$$\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1}).$$

(iii) Hence, or otherwise, show for $n \geq 1$, **2**

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots\ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

End of paper

EXTENSION 2 MATHEMATICS TRIAL SOLUTIONS

SECTION I MULTIPLE CHOICE ANSWERS

		OUTCOMES
QUESTION 1	A	1
QUESTION 2	B	3
QUESTION 3	C	3
QUESTION 4	D	1
QUESTION 5	A	1
QUESTION 6	B	3
QUESTION 7	C	5
QUESTION 8	C	1
QUESTION 9	C	5
QUESTION 10	A	3

OUTCOME 1 Q1,4,5,8 /4
OUTCOME 3 Q2,3,6,10 /4
OUTCOME 5 Q7,9 /2
TOTAL /10

FULL NAME ______________ Class: M1 M2

SECTION I: ANSWER SHEET

Mathematics Extension 2

Section I - Answer Sheet
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9
A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.
A ● B ☒ C ○ D ○
- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.
A ● B ☒ (correct) C ○ D ○

1. A ● B ○ C ○ D ○ outcome 1
2. A ○ B ● C ○ D ○ outcome 3
3. A ○ B ○ C ● D ○ outcome 3
4. A ○ B ○ C ○ D ● outcome 1
5. A ● B ○ C ○ D ○ outcome 1
6. A ○ B ● C ○ D ○ outcome 3
7. A ○ B ○ C ● D ○ outcome 5
8. A ○ B ○ C ● D ○ outcome 1
9. A ○ B ○ C ● D ○ outcome 5
10. A ● B ○ C ○ D ○ outcome 3

OFFICE USE ONLY: OUTCOMES G

O1	O3	O5	OUTCO
Q1,4,5,8	Q2,3,6,10	Q7,9	TOTAL
/4	/4	/2	/

2013 EXTENSION II SOLUTIONS MATHEMATICS

MULTIPLE CHOICE SECTION I (ONE MARK EACH)

Q1 $xy^3 + 2y = 4$ at $(2,1)$

$\frac{dy}{dx}$: $x\,3y^2\frac{dy}{dx} + y^3.1 + 2\frac{dy}{dx} = 0$

At $x=2$, $y=1$: $6\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$

$8\frac{dy}{dx} = -1$

$\therefore \frac{dy}{dx} = -\frac{1}{8}$

Slope $-\frac{1}{8}$ Point $(2,1)$

$y - 1 = -\frac{1}{8}(x-2)$

$y - 1 = -\frac{1}{8}x + \frac{1}{4}$

$y = -\frac{1}{8}x + 1\frac{1}{4}$

$\times 8]$ $8y = -x + 10$

$\therefore x + 8y = 10$ (A)

Q2 $z^{21} = \left[2\operatorname{cis}\left(-\frac{\pi}{3}\right)\right]^{21}$

$= 2^{21}\left(\cos\left(\frac{-21\pi}{3}\right) + i\sin\left(\frac{-21\pi}{3}\right)\right)^{21}$

$= 2^{21}\{\cos(-7\pi) + i\sin(-7\pi)\}^{21}$

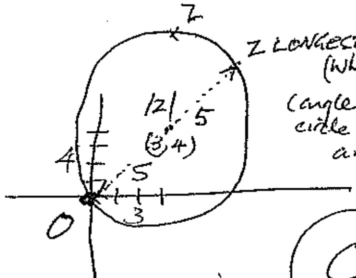
$= 2^{21}(-1 + i.0)^{21}$

$= (-1)^{21}.2^{21} = -2^{21}$ (B)

Q3 $|z - (3+4i)| = 5$

Circle centre $(3,4)$ radius 5

$\therefore |z| = 5 + 5 = 10$ (O lies on the circle) (C)



(Why?) (angle in a semi-circle is a right angle)

Q4 $P(-\frac{1}{2}) = 0 \therefore x = -\frac{1}{2}$

So $2x+1$ is a factor

$$2x+1 \overline{)\,2x^3 - 3x^2 + 4x + 3} = x^2 - 2x + 3$$
$2x^3 + x^2$
$-4x^2 + 4x$
$-4x^2 - 2x$
$6x + 3$
$6x + 3$

$P(x) = (2x+1)(x^2 - 2x + 3)$ over $\mathbb{R}$

$x = \frac{-(-2) \pm \sqrt{4 - 4.1.3}}{2}$

$= \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i$ $\therefore x - 1 - \sqrt{2}i$ and $x - 1 + \sqrt{2}i$ are factors

So $P(x) = (2x+1)(x - 1 - \sqrt{2}i)(x - 1 + \sqrt{2}i)$ (D)

Q5 $2y^3 - 9y^2 + 12y + k = 0$ has two roots

$\therefore \frac{dP}{dy} = 0$ has the same root.

$6y^2 - 18y + 12 = 0$

$y^2 - 3y + 2 = 0$

$(y-2)(y-1) = 0$

So $y = 2$ could be a multiple root or $y = 1$ could be.

If $y = 2$: $2(2)^3 - 9(2)^2 + 12(2) + k = 0$

$16 - 36 + 24 + k = 0$

$k = -4$

If $y = 1$: $2(1)^3 - 9(1)^2 + 12(1) + k = 0$

$2 - 9 + 12 + k = 0$

$k = -5$

NOTE: $(y-2)^2(ay+b) = 0$ gives $k = -4$

$(y-1)^2(Ay+B) = 0$ gives $k = -5$.

(A)

Q6 $|z - 2i| = 2 + \text{Im } z$

Points distant from $2i$ — $y + 2$ Distance from $y = -2$

Equidistant

$\therefore$ LOCUS IS A PARABOLA

$|x + iy - 2i| = 2 + y$

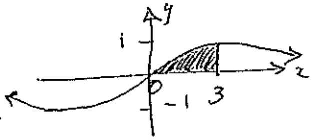
$\sqrt{x^2 + (y-2)^2} = 2 + y$

$x^2 + y^2 - 4y + 4 = 4 + 4y + y^2$

$x^2 = 8y$ A PARABOLA.

(B)

Q7 $y = \frac{x}{\sqrt{16 + x^2}}$



NOTE: $\lim_{x \to \infty} \frac{x}{\sqrt{16+x^2}}$

$= \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{x}{\sqrt{16+x^2}}$

$= \lim_{x \to \infty} \frac{1}{\sqrt{\frac{16}{x^2} + 1}} = \frac{1}{\sqrt{0+1}} = 1$

$\int_0^3 \frac{x}{(16+x^2)^{\frac{1}{2}}} dx = \frac{1}{2} \int_0^3 \frac{2x}{(16+x^2)^{\frac{1}{2}}} dx$

$= \left[\frac{1}{2} \frac{(16+x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^3$

$= \sqrt{16+9} - \sqrt{16+0}$

$= 5 - 4 = 1$ units^2

$1 = \log_e e$ $\therefore$

(C)

Q8 Common tangent

$y = e^{2x}$ $\qquad$ $y = k\sqrt{x}$

$\frac{dy}{dx} = 2e^{2x}$ $\qquad$ $y = kx^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}kx^{-\frac{1}{2}}$

SLOPES EQUAL $2e^{2x} = \frac{1}{2}kx^{-\frac{1}{2}}$

$4\sqrt{x}\, e^{2x} = k$

(C)

common point

So $e^{2x} = k\sqrt{x}$

$e^{2x} = (4\sqrt{x}\, e^{2x})\sqrt{x}$ substitute

$\therefore 1 = 4x$

$x = \frac{1}{4}$

$\therefore e^{2x} = k\sqrt{\frac{1}{4}}$

$2e^{2x} = k$

$2e^{2(\frac{1}{4})} = k$

$2e^{\frac{1}{2}} = k$

$2\sqrt{e} = k$

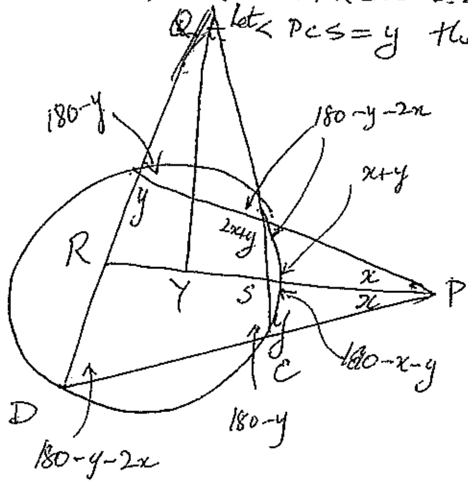
Q9. USING LOGIC, THE ANSWER MUST BE (C) — ABCD IS NOT ALWAYS A KITE.

If $\angle QYR$ is $90°$, then (B) and (D) follow; ISOSCELES + MIDPOINT

(A), (B) and (D) are inter related so (C) is odd one out.

HOWEVER, Let $\angle APR = x$ $\therefore \angle DPR = x$

let $\angle PCS = y$ then $\angle DRB = y$ (cyclic quadrilateral properties)



$\angle DQC = 180 - (180 - y - 2x) - (180 - y)$

$= y + 2x - 180 + y$

$= 2x + 2y - 180$

$\therefore \angle CQY = \frac{1}{2}\angle DQC = x + y - 90$

Now $\angle YQS + \angle YSQ + \angle QYS = 180°$

$(x + y - 90) + (180 - x - y) + \angle QYS = 180$

$-90 + 180 + \angle QYS = 180$

So $\angle QYS = 90°$

$QY \perp RS$.

NOTE: * IF $\angle QYR = 90°$ and $\angle RQY = \angle YQS = x + y - 90°$

THEN $\angle QRY = 180 - 90 - (x + y - 90) = 180 - x - y$

$= \angle QSY$

$\therefore \triangle QRS$ is isosceles

* IF $\triangle QRS$ is isosceles then QY is an axis of symmetry

$\therefore$ Y is midpoint of RS. (NICE QUESTION)

(C)

Q10 DIAGONALS OF A SQUARE ARE PERPENDICULAR

$q - s$ represents diagonal $\vec{QS}$

$p - r$ represents diagonal $\vec{PR}$

So $i(p - r)$ represents a rotation anticlockwise

So $\vec{PR}$ rotates onto $\vec{QS}$

So $q - s = i(p - r)$ will always be true for a square

(A)

Q	1	2	3	4	5	6	7	8	9	10
	A	B	C	D	A	B	C	C	C	A

(1 mark each)

SECTION 2 SOLUTIONS

QUESTION 11

a i $\int \frac{x+1}{x^2+2x+5} = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+5}\,dx$

$= \frac{1}{2}\ln(x^2+2x+5)+C$

$\int \frac{1dx}{x^2+2x+5} = \int \frac{1dx}{x^2+2x+1+4}$

$= \int \frac{1}{(x+1)^2+4}\,dx$

$= \frac{1}{2}tan^{-1}\frac{x+1}{2} + C$

ii $\int \frac{x+1-1}{x^2+2x+5} = \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}tan^{-1}\frac{x+1}{2} + c$ ONE MARK EACH total /3

b $\frac{1}{x(x+2)} = \frac{a}{x} + \frac{b}{(x+2)} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}}{(x+2)}$ ONE MARK

$\therefore \int_1^3 \frac{dx}{x(x+2)} = \int_1^3 \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}}{(x+2)}\,dx = [\frac{1}{2}ln\,x - \frac{1}{2}ln(x+2)]_1^3$ ONE MARK

$= \frac{1}{2}[(ln3 - ln5) - (ln1 - ln3)$

$= \frac{1}{2}(ln9 - ln5) = \frac{1}{2}ln\frac{9}{5}$ ONE MARK

total /3

c i $(secx\,tan\,x)^4 = sec^2x\,.\,sec^2x\,(tan\,x)^4 = sec^2x\,(1+tan^2x)(tan\,x)^4$

ONE MARK $= sec^2x\,((tan\,x)^4 + (tan\,x)^6)$

ii $\int_0^{\frac{\pi}{4}} sec^4x\,tan^4x\,dx = \int_0^{\frac{\pi}{4}} sec^2x\,((tan\,x)^4 + (tan\,x)^6)\,dx$

$= [\,\frac{1}{5}(tan\,x)^5 + \frac{1}{7}(tan\,x)^7\,]_0^{\frac{\pi}{4}}$ ONE MARK

$= (\frac{1}{5}(1)^5 + \frac{1}{7}(1)^7) - (\frac{1}{5}(0)^5 + \frac{1}{7}(0)^7)$

$= \frac{12}{35}$ ONE MARK total /3

d $t = tan\frac{\theta}{2}$ $sin\,\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$ $d\,\theta = \frac{2dt}{1+t^2}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+sin\,\theta+cos\,\theta} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int_0^1 \frac{1dt}{1+t}$ ONE MARK

$= [\,ln(1+t)\,]_1^3$ ONE MARK

$= ln\,4 - ln\,2$

$= ln\,2$ ONE MARK

total /3

e i Show $U_n = \frac{n-1}{n}U_{n-2}$.

$U_n = \int_0^{\frac{\pi}{2}} sin^nx\,dx = \int_0^{\frac{\pi}{2}} sin^{n-1}x.\frac{d}{dx}(-cosx).dx$

integration by parts ONE MARK

$= [(sin^{n-1}x)(-cosx)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(n-1)\,sin^{n-2}x\,.-cos^2x\,dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} sin^{n-2}x.(1 - sin^2x)dx$

$= (n-1)\int_0^{\frac{\pi}{2}}(sin^{n-2}x - sin^nx)\,dx$

$U_n = (n-1)\,U_{n-2} - (n-1)U_n$

$\therefore (n-1)U_n + U_n = (n-1)\,U_{n-2}$

So $U_n = \frac{n-1}{n}U_{n-2}$ ONE MARK

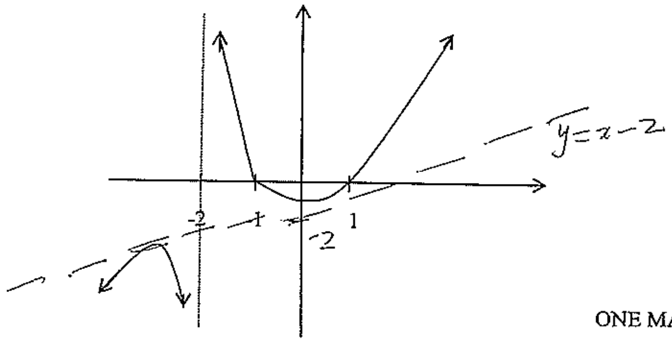
ii $U_4 = \frac{3}{4}.\frac{1}{2}.U_0 = \frac{3}{4}.\frac{1}{2}.\int_0^{\frac{\pi}{2}}dx = \frac{3\pi}{16}$ $U_6 = \frac{5}{6}.U_4 = \frac{5\pi}{32}$

So $U_4 - U_6 = \frac{\pi}{32}$ $\therefore k = 32$ ONE MARK total /3

QUESTION 12

a i $\frac{3}{x+2} + x - 2 = \frac{3(1)+(x+2)(x-2)}{x+2} = \frac{x^2-1}{x+2}$ ONE MARK

ii Sketch $y = \frac{3}{x+2}$ and $y = x-2$ separately and then add ordinates.



ONE MARK

$y = \frac{x^2-1}{x+2}$ has y intercept at $(0, \frac{-1}{2})$ and x intercepts at $x = \pm 1$.

Sketch $y = \frac{3}{x+2} + x - 2 = \frac{x^2-1}{x+2}$

iii $\frac{x^2-1}{x+2} \leq 0$. Graph is negative for $x < -2$ or $-1 \leq x \leq 1$ ONE MARK

total /3

b Find an equation whose roots are $\frac{\alpha\beta}{\gamma} = \frac{\alpha\beta\gamma}{\gamma\gamma}$; $\frac{\alpha\gamma}{\beta} = \frac{\alpha\gamma\beta}{\beta\beta}$ and $\frac{\beta\gamma}{\alpha} = \frac{\beta\gamma\alpha}{\alpha\alpha}$.

Now $\alpha\beta\gamma = 1$ from product $= \frac{-d}{a} = \frac{-(-1)}{1}$. ONE MARK

$\therefore$ new roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

So let $y = \frac{1}{x^2}$ then $x = \frac{1}{\sqrt{y}}$ ONE MARK

Then $x^3 + 2x^2 - 3x - 1 = 0$ becomes $(\frac{1}{\sqrt{y}})^3 + 2(\frac{1}{\sqrt{y}})^2 - 3\left(\frac{1}{\sqrt{y}}\right) - 1 = 0$

$\therefore (\frac{1}{\sqrt{y}})^3 + 2(\frac{1}{\sqrt{y}})^2 - 3\left(\frac{1}{\sqrt{y}}\right) - 1 = 0$ is $1 + 2\sqrt{y} - 3y - (\sqrt{y})^3 = 0$

ONE MARK

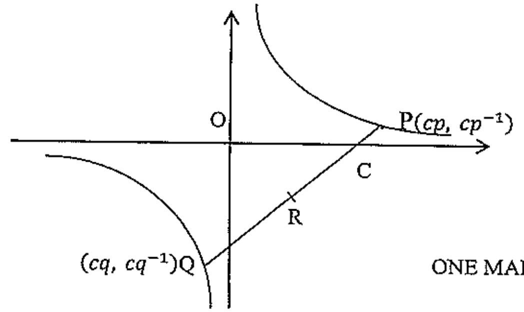
$1 + 2\sqrt{y} - 3y - (\sqrt{y})^3 = 0$ can be written as $2\sqrt{y} - y\sqrt{y} = 3y - 1$

Square both sides $4y - 4y^2 + y^3 = 9y^2 - 6y + 1$

Giving $y^3 - 13y^2 + 10y - 1 = 0$ ONE MARK

total /4

c i



ONE MARK

ii $P(cp, cp^{-1})$ and $Q(cq, cq^{-1})$

So equation is $y - cp^{-1} = \frac{cq^{-1} - cp^{-1}}{cq - cp}(x - cp)$ ONE MARK

$\therefore pq\,(y - cp^{-1}) = -1(x - cp)$ becomes

$x + pq\,y = c\,(p + q)$ ONE MARK

iii C is the x intercept so let $y = 0$ C $(c(p+q), 0)$ ONE MARK

iv $R\,[\frac{cp+cq}{2}, \frac{cp^{-1} - cq^{-1}}{2}]$ gives $R\,[\frac{c}{2}(p+q), \frac{c}{2}\left(\frac{p+q}{pq}\right)]$ ONE MARK

v distance $OR^2 = [\frac{c}{2}\left(\frac{p+q}{pq}\right)]^2 + [\frac{c}{2}(p+q)]^2$ ONE MARK

distance $RC^2 = [\frac{c}{2}\left(\frac{p+q}{pq}\right)]^2 + [\frac{c}{2}(p+q) - c(p+q)]^2$ ONE MARK

$= [\frac{c}{2}(\frac{p+q}{pq})]^2 + [\frac{-c}{2}(p+q)]^2$ ONE MARK

$= [\frac{c}{2}(\frac{p+q}{pq})]^2 + [\frac{c}{2}(p+q)]^2 = OR^2 \quad \therefore OR = RC$ total /3

QUESTION 13

a i $b^2 = a^2(e^2-1)$ $5x^2 - 4y^2 = 20$ becomes $\frac{1}{4}x^2 - \frac{1}{5}y^2 = 1$.

$5 = 4(e^2-1)$ $a = 2$ $b = \sqrt{5}$

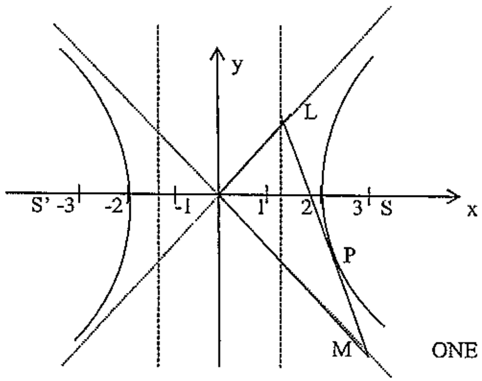
$\frac{9}{4} = e^2$ $\therefore \frac{3}{2} = e$ Foci S and S' $(\pm ae, 0)$ become $(\pm 2.\frac{3}{2}, 0)$

ONE MARK S and S' $(\pm 3, 0)$ ONE MARK

ii directrices $x = \pm\frac{a}{e}$ become $x = \pm\frac{4}{3}$ ONE MARK

asymptotes $y = \pm\frac{b}{a}x$ become $y = \pm\frac{1}{2}\sqrt{5}\,x$ ONE MARK

iii



ONE MARK

iv $x = 2sec\theta$ $5x^2 - 4y^2 = 20$ ONE MARK

$y = \sqrt{5}tan\theta$ LHS $= 5x^2 - 4y^2 = 5(2sec\theta)^2 - 4(\sqrt{5}tan\theta)^2$

$= 20((sec\theta)^2 - (tan\theta)^2) = 20(1) =$ RHS

v $5x^2 - 4y^2 = 20$ So $10x - 8y\frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = \frac{5x}{4y}$.

At $P(2sec\theta, \sqrt{5}tan\theta)$ $\frac{dy}{dx} = \frac{5.2sec\theta}{4.\sqrt{5}tan\theta} = \frac{\sqrt{5}sec\theta}{2tan\theta}$. ONE MARK

Equation of tangent $y - \sqrt{5}tan\theta = \frac{\sqrt{5}sec\theta}{2tan\theta}(x - 2sec\theta)$

$-2ytan\theta + \sqrt{5}\,xsec\theta = 2\sqrt{5}\,((sec\theta)^2 - (tan\theta)^2)$

$-2ytan\theta + \sqrt{5}\,xsec\theta = 2\sqrt{5}\,(1)$

$\frac{1}{2}xsec\theta - \frac{1}{\sqrt{5}}ytan\theta = 1$ ONE MARK

vi Solve $\frac{1}{2}xsec\theta - \frac{1}{\sqrt{5}}ytan\theta = 1$ with $y = \frac{1}{2}\sqrt{5}\,x$

$\frac{1}{2}xsec\theta - \frac{1}{\sqrt{5}}(\frac{1}{2}\sqrt{5}\,x)tan\theta = 1$

$x = \frac{2}{sec\theta - tan\theta}$ So $y = \frac{1}{2}\sqrt{5}\,x$ becomes $y = \frac{\sqrt{5}}{sec\theta - tan\theta}$

L is $(\frac{2}{sec\theta - tan\theta}, \frac{\sqrt{5}}{sec\theta - tan\theta})$ ONE MARK

Similarly solve $\frac{1}{2}xsec\theta - \frac{1}{\sqrt{5}}ytan\theta = 1$ with $y = -\frac{1}{2}\sqrt{5}\,x$ gives

M as $(\frac{2}{sec\theta + tan\theta}, \frac{-\sqrt{5}}{sec\theta + tan\theta})$. ONE MARK

Now if LP = PM, P must be the midpoint of LM.

Midpoint of LM is $(\frac{1}{2}[\frac{2}{sec\theta - tan\theta} + \frac{2}{sec\theta + tan\theta}], \frac{1}{2}[\frac{\sqrt{5}}{sec\theta + tan\theta} + \frac{-\sqrt{5}}{sec\theta + tan\theta}])$

$= (\frac{1}{2}[\frac{4sec\theta}{1}], \frac{1}{2}[\frac{2\sqrt{5}sec\theta}{1})$

$= (2sec\theta, \sqrt{5}tan\theta)$ which is P ONE MARK

So the midpoint of LM is P. $\therefore$ LP = PM vii Area $\Delta OLM = \frac{1}{2}$ x LO x MO x $sin < LOM$ ONE MARK

Now $\frac{1}{2}$ x LO x MO is a constant and independent of θ and so independent of P.

Now $< LOM$ is a combination of the angles that the asymptotes make with the x axis.

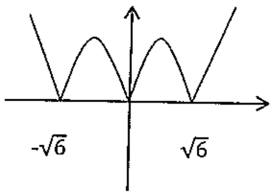
That is $tan^{-1}(\frac{\sqrt{5}}{2})$ and $\pi - tan^{-1}(\frac{-\sqrt{5}}{2})$. So $sin < LOM$ is also a constant and ∴

independent of θ and so independent of P. ONE MARK

total /13

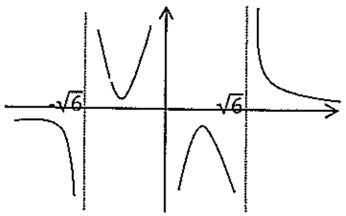
b $f(x) = x^3 - 6x = x(x+\sqrt{6})(x-\sqrt{6})$ ONE MARK EACH total /2

i



$y = |f(x)| = |x^3 - 6x|$

ii

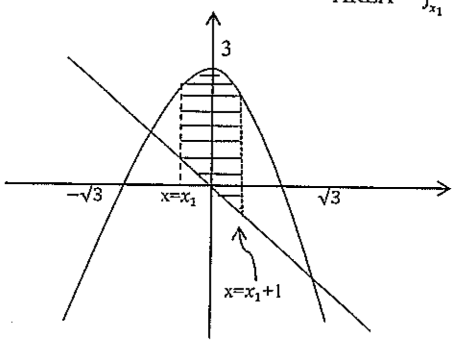


$y = \frac{1}{f(x)} = (x^3 - 6x)^{-1}$

QUESTION 14

ONE MARK

a i AREA $= \int_{x_1}^{x_1+1}[(3 - x^2) - (-2x)]dx$



AREA $= \int_{x_1}^{x_1+1}[(3 - x^2) - (-2x)]dx$ ∴ A $= \int_{x_1}^{x_1+1}[(3 - x^2 + 2x)]dx$

A $= [3x - \frac{1}{3}x^3 + x^2]_{x_1}^{x_1+1}$

$= x_1 + 3\frac{2}{3} - x_1^2$ ONE MARK

Maximum occurs where $\frac{dA}{dx_1} = 0$ ONE MARK

$\frac{dA}{dx_1} = -2x_1 + 1 = 0$

∴ $x_1 = \frac{1}{2}$ ONE MARK

ii A $= x_1 + 3\frac{2}{3} - x_1^2$ So max A $= (\frac{1}{2}) + 3\frac{2}{3} - (\frac{1}{2})^2 = 3\frac{11}{12}$

ONE MARK

Justify the maximum $\frac{dA}{dx_1} = -2x_1 + 1$ ∴ $\frac{d^2A}{dx_1^2} = -2$ concave down

So a maximum occurs at $x_1 = \frac{1}{2}$. ONE MARK

total /6

b The cross-sectional area is the area of a circle. Radius of this circle is $1 - y$, so

$\Delta A = \pi(1-y)^2$. The typical slice volume is $\Delta V = \Delta A \times \Delta x$

∴ $\Delta V = \Delta A \times \Delta x = \pi(1-y)^2 . \Delta x$ ONE MARK

Now $y = sin\,x$ so $\Delta V = \pi(1 - sinx)^2 . \Delta x$

Total volume $= V = \int \Delta V = \int \pi(1 - sinx)^2 . dx$

$= \pi \int_0^{\frac{\pi}{2}} (1 - 2sinx + sin^2x)\,dx$ But $sin^2x = \frac{1}{2}(1 - cos2x)$

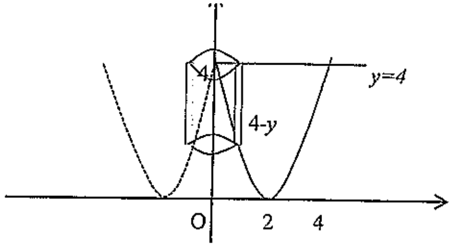
$V = \pi \int_0^{\frac{\pi}{2}} (1\frac{1}{2} - 2sinx - \frac{1}{2}cos2x)\ dx$ ONE MARK

$= \pi[\frac{3}{2}x + 2cosx - \frac{1}{4}sin2x]_0^{\frac{\pi}{2}}$ ONE MARK

$= \pi(\frac{3}{4}\pi - 2)$ cubic units ONE MARK

total /4

c



Typical cylindrical shell has cross-sectional area $\Delta A = 2\pi x.(4 - y)$

But $y = (x-2)^2$ so $\Delta A = 2\pi x.(4 - (x-2)^2)$

$= 2\pi x\,(4x - x^2)$ ONE MARK

$\therefore$ Small shell volume $\Delta V = 2\pi x\,(4x - x^2)\,\Delta x$ ONE MARK

Total volume = $V = \int \Delta V = 2\pi \int x\,(4x - x^2)dx$

$= 2\pi \int_0^4 (4x^2 - x^3)\,dx$ ONE MARK

$= 2\pi\,[\frac{4}{3}x^3 - \frac{1}{4}x^4]_0^4$ ONE MARK

$= \frac{128}{3}\pi$ cubic units ONE MARK

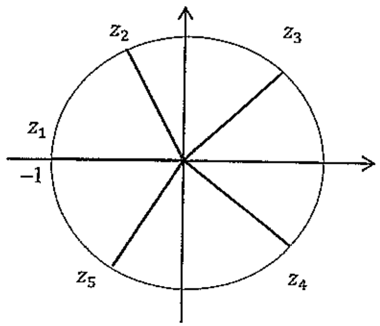
total /5

QUESTION 15

a i A factor is $(z+1)$ so $(z+1)^5 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

TWO MARKS

ii



Each angle is $2\pi/5$

ONE MARK

Angles are at $\frac{\pi}{5}, \frac{3\pi}{5}, \frac{-\pi}{5}, \frac{-3\pi}{5}, \pi$

Question 15

(a)(i) $z^5 + 1 = 0$ One root is $z = -1$

$\therefore (z+1)$ is a factor.

$z+1 \overline{) z^5 \quad\quad +1}$ quotient $z^4 - z^3 + z^2 - z + 1$

$z^5 + z^4$

$-z^4$

$-z^4 - z^3$

z^3

$z^3 + z^2$

$-z^2$

$-z^2 - z$

z

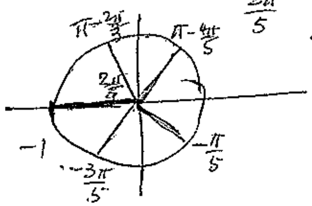
$z + 1$

$\cdot\cdot$

✓

$z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$

(a) (ii) $z^5 = -1$



$\frac{2\pi}{5}$ $r = 1$

$\therefore z_1 = -1 = \text{cis}(-\pi)$

$z_2 = \text{cis}(\frac{3\pi}{5})$

$z_3 = \text{cis}(\frac{\pi}{5})$

$z_4 = \text{cis}(\frac{-3\pi}{5})$

$z_5 = \text{cis}(\frac{-\pi}{5})$ ✓

(a) (iii) Now sum of roots for $z^4 - z^3 + z^2 - z + 1$ is $-\frac{(-1)}{1} = 1$

So $\text{cis}\frac{3\pi}{5} + \text{cis}\frac{\pi}{5} + \text{cis}(\frac{-3\pi}{5}) + \text{cis}(\frac{-\pi}{5}) =$

$\sin\frac{3\pi}{5} = -\sin\frac{-3\pi}{5}$ $\sin\frac{-\pi}{5} = -\sin\frac{\pi}{5}$ etc

$\cos(\frac{3\pi}{5}) = \cos(\frac{-3\pi}{5})$ $\cos(\frac{\pi}{5}) = \cos(\frac{-\pi}{5})$

So $\text{cis}\frac{3\pi}{5} + \text{cis}(-\frac{3\pi}{5}) = 2\cos\frac{3\pi}{5}$

$\text{cis}\frac{\pi}{5} + \text{cis}(\frac{-\pi}{5}) = 2\cos\frac{\pi}{5}$

$\therefore 2\cos\frac{3\pi}{5} + 2\cos\frac{\pi}{5} = 1$ ✓

$2\cos\frac{3\pi}{5} + 2\cos\frac{\pi}{5} - 1 = 0$

(b) $mx = \tan^{-1}Q - \tan^{-1}v$

$\therefore \tan(mx) = \tan(\tan^{-1}Q - \tan^{-1}v)$

$= \frac{\tan\tan^{-1}Q - \tan\tan^{-1}v}{1 + \tan\tan^{-1}Q.\tan\tan^{-1}v}$

$\tan(mx) = \frac{Q - v}{1 + Qv}$

$\therefore mx = \tan^{-1}(\frac{Q-v}{1+Qv})$

(c) $\frac{a}{v} + \frac{bv + c}{1+v^2} = \frac{1}{v} + \frac{-v+0}{1+v^2}$

$= \frac{1+v^2}{v(1+v^2)} + \frac{-v^2}{v(1+v^2)}$

$= \frac{1+v^2-v^2}{v(1+v^2)}$

$= \frac{1}{v(1+v^2)}$ as required.

Q15(c) (i)

$\ddot{x} = -m(v+v^3)$

$v\frac{dv}{dx} = -m(v+v^3)$ ✓

$\frac{dv}{dx} = -m(1+v^2)$

$\int \frac{dv}{1+v^2} = \int -m\,dx$

$\tan^{-1} v = -mx + c$ ✓

$mx = c - \tan^{-1} v$

$\left.\begin{matrix} x=0 \\ v=Q \end{matrix}\right\}$ $0 = c - \tan^{-1} Q$

$\tan^{-1} Q = c$ ✓

$mx = \tan^{-1} Q - \tan^{-1} v$

$x = \frac{1}{m}(\tan^{-1} Q - \tan^{-1} v)$

from (b) (i) $x = \frac{1}{m}\tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$

(c) (ii) $\ddot{x} = -m(v+v^3)$

$\frac{dv}{dt} = -m(v+v^3)$

$\int \frac{dv}{v+v^3} = \int -m\,dt$

$\int \frac{a}{v} + \frac{bv}{1+v^2} = \int -m\,dt$

$\int \frac{1}{v} - \frac{1}{2}\frac{2v}{1+v^2}\,dv = -mt + c$ from (b)(ii) above

$\ln v - \frac{1}{2}\ln(1+v^2) = -mt + c$ ✓

$2\ln v - \ln(1+v^2) = -2mt + K$

$\ln \frac{v^2}{1+v^2} = -2mt + K$

$\begin{matrix} t=0 \\ v=Q. \end{matrix}$ $\ln \frac{Q^2}{1+Q^2} = K$ ✓

So $\ln\left(\frac{v^2}{1+v^2}\right) = -2mt + \ln\frac{Q^2}{1+Q^2}$

$\therefore 2mt = \ln\left(\frac{Q^2}{1+Q^2} \times \frac{1+v^2}{v^2}\right)$

$t = \frac{1}{2m}\ln\left\{\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right\}$ ✓

Q15(c) (iii) $t = \frac{1}{2m}\ln\left\{\frac{Q^2}{v^2}\frac{(1+v^2)}{(1+Q^2)}\right\}$

$\times 2m]$ $2mt = \ln\left\{\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right\}$

$e^?]$ $e^{2mt} = \frac{Q^2(1+v^2)}{v^2(1+Q^2)}$

$v^2(1+Q^2)e^{2mt} = Q^2(1+v^2)$

$v^2(1+Q^2)e^{2mt} - Q^2v^2 = Q^2$

$v^2 = \frac{Q^2}{(1+Q^2)e^{2mt} - Q^2}$ ✓

(c) (iv) Limiting value of velocity $t \to \infty$ $e^{2mt} \to \infty$

$\therefore v^2 = \frac{Q^2}{\infty} \to 0$ ✓

$v = 0$ in the limit

$\therefore$ particle is slowing down

When $v = 0$ $x = ?$

$x = \frac{1}{m}\tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$

$v \to 0$ $x \to \frac{1}{m}\tan^{-1} Q$

$mx \to \tan^{-1} Q$ ✓

$\tan(mx) \to Q.$

QUESTION 16

a. i Noting $B_n'(x) = \frac{d}{dx}(B_n(x)) = nB_{n-1}(x)$ and $B_0(x) = 1$

$B_1'(x) = \frac{d}{dx}(B_1(x)) = 1.B_0(x)$

$\therefore\ B_1'(x) = 1$

So $B_1(x) = x + C$ ONE MARK

Now $\int_0^1 B_1(x)dx = \int_0^1 (x + C)\,dx = \int_0^1 x\,dx + \int_0^1 C\,dx$

Noting $\int_0^1 B_1(x)dx = 0$ then $0 = \int_0^1 x\,dx + \int_0^1 C\,dx$ ONE MARK

$0 = \int_0^1 x\,dx + C$

$\therefore C = -\int_0^1 x\,dx$

$= -[\frac{1}{2}x^2]_0^1 = -\frac{1}{2}$ ONE MARK

So $B_1(x) = x + C$ becomes $B_1(x) = x - \frac{1}{2}$.

ii $g(x) = B_{n+1}(x+1) - B_{n+1}(x)$

So $g'(x) = B_{n+1}'(x+1) - B_{n+1}'(x)$

Noting that $B_n'(x) = \frac{d}{dx}(B_n(x)) = nB_{n-1}(x)$

$\therefore\ g'(x) = B_{n+1}'(x+1) - B_{n+1}'(x)$

$g'(x) = (n+1)B_n(x+1) - (n+1)B_n(x)$

$= (n+1)[B_n(x+1) - B_n(x)]$

$g'(x) = (n+1)[\,nx^{n-1}\,]$ (given data) ONE MARK

Integrate both sides gives $g(x) = (n+1)[\,nx^{n-1+1}.\frac{1}{n}] + c$

Also $g(x) = (n+1)\,x^n + c$ ONE MARK

iii Prove that $B_n(x+1) - B_n(x) = nx^{n-1}$ if $n \geq 1$.

<u>STEP 1</u> Show true for $n = 1$ ONE MARK

LHS $= 1.x^{1-1} = 1$ RHS $= B_1(x+1) - B_1(x)$

$=$ RHS $= (x+1-\frac{1}{2}) - (x - \frac{1}{2}) = 1$

<u>STEP 2</u> Assume true $n = k$ ONE MARK

$B_k(x+1) - B_k(x) = kx^{k-1}$ ....(*)

<u>STEP 3</u> Prove true $n = k + 1$

Aim: To prove $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^{k+1-1} = (k+1)x^k$

Proof: Now $g(x) = B_{k+1}(x+1) - B_{k+1}(x)$

So $g'(x) = B_{k+1}'(x+1) - B_{k+1}'(x)$

$= (k+1)B_k(x+1) - (k+1)B_k(x)$

$= (k+1)[B_k(x+1) - B_k(x)]$

$= (k+1)[\,kx^{k-1}]$ from (*) above

ONE MARK

$\therefore\ g(x) = (n+1)\,x^n + c$

Hence $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)\,x^k + c$

$B_{k+1}(1) - B_{k+1}(0) = (k+1).0 + c$

$0 = c$ ONE MARK

So $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)\,x^k$ as required.

<u>STEP 4</u> The proposition is true for $n = 1$ and since it is true for $n = k+1$ it is true for $n = 1+1 = 2$, and for $n = 2+1 = 3$ and so on for all values of $n \geq 1$.

Hence, by mathematical induction, $B_n(x+1) - B_n(x) = nx^{n-1}$, $n \geq 1$.

ONE MARK

total /10

b i $2k+3 > 2\sqrt{k+2}\,\sqrt{k+1}$

Squaring LHS $= 4k^2 + 12k + 9$ RHS $= 4(k^2 + 3k + 2)$

$= 4k^2 + 12k + 8$

LHS > RHS $\therefore 2k+3 > 2\sqrt{k+2}\,\sqrt{k+1}$ ONE MARK

ii $2k+3 > 2\sqrt{k+2}\,\sqrt{k+1}$

So $2k+3 = 2k+2+1 > 2\sqrt{k+2}\,\sqrt{k+1}$

$\therefore 1 > 2\sqrt{k+2}\,\sqrt{k+1} - 2(k+1)$ ONE MARK

$1 > 2\sqrt{k+1}\,(\sqrt{k+2} - \sqrt{k+1})$

So $\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1})$. ONE MARK

iii $\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1})$

$k = 0 \quad \frac{1}{\sqrt{1}} = 1 > 2(\sqrt{2} - 1)$

$k = 1 \quad \frac{1}{\sqrt{2}} > 2(\sqrt{3} - \sqrt{2})$

$k = 2 \quad \frac{1}{\sqrt{3}} > 2(\sqrt{4} - \sqrt{3})$

....

$k = n-1 \quad \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - \sqrt{n})$. ONE MARK

Now $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + \ldots\ldots + 2(\sqrt{n+1} - \sqrt{n})$

$> 2\sqrt{2} - 2 + 2\sqrt{3} - 2\sqrt{2} + \ldots\ldots 2\sqrt{n} + 2\sqrt{n+1} - 2\sqrt{n}$

$> 2\sqrt{n+1} - 2$ ONE MARK

So $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$ total /5

OR, OTHERWISE, BY MATHS INDUCTION

PROVE $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$ where n:

<u>Step 1</u> Prove true for $n=1$ (HSC) $n=2$ $n=3$

$n=1$

LHS $= 1$ or $\frac{1}{\sqrt{1}}$

RHS $= 2(\sqrt{2} - 1)$

$\doteqdot 2(0.414)$

$= 0.828$

LHS > RHS

TRUE

$n=2$

LHS $= 1 + \frac{1}{\sqrt{2}}$

$\doteqdot 1 + 0.7071$

$\doteqdot 1.7071$

RHS $= 2(\sqrt{3} - 1)$

$\doteqdot 2(0.732)$

$\doteqdot 1.464$

LHS > RHS

$n=3$

LHS $= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3}$

$\doteqdot 1 + 0.7071 + 0.574 \doteqdot 2.305$

RHS $= 2(\sqrt{4} - 1)$

$= 2(1) = 2$

LHS > RHS

<u>Step 2</u> Assume true for $n = k$

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1)$ ———— (*)

<u>Step 3</u> Prove true for $n = k+1$

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$ ✓

LHS $= 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

From (*) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}$

From (b)(ii) above $\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1})$

So $2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + 2(\sqrt{k+2} - \sqrt{k+1})$

$> 2\sqrt{k+1} - 2 + 2\sqrt{k+2} - 2\sqrt{k+1}$

$> 2\sqrt{k+2} - 2$

$> 2(\sqrt{k+2} - 1)$ ✓

That is $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$

So proven that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$

<u>Step 4</u> Since the statement is true for $n=1$ (and $n=2$ and $n=3$) and because it is true for $n=k+1$ it must be true for $n = 1+1 = 2$, (and $n = 2+1 = 3$, and $n = 3+1 = 4$) and so on for all values of integer $n > 0$.