NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2016 Assessment Task 4 (Trial Examination)
Thursday, 4 August 2016

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

BOSTES NUMBER: .... **# BOOKLETS USED:** ....

Marker's use only

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

---

1 What is the value of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^9$?

(A) -1
(B) 0
(C) 1
(D) 2

2 A particle in a straight line so that its velocity at any particular time is given by $v = k\,(a - x)$, where x is its displacement from a given point O.

The particle is initially at O.

Which of the following gives an expression for x?

(A) $x = a(1 - e^{kt})$
(B) $x = a(1 + e^{kt})$
(C) $x = a(1 - e^{-kt})$
(D) $x = a(1 + e^{-kt})$

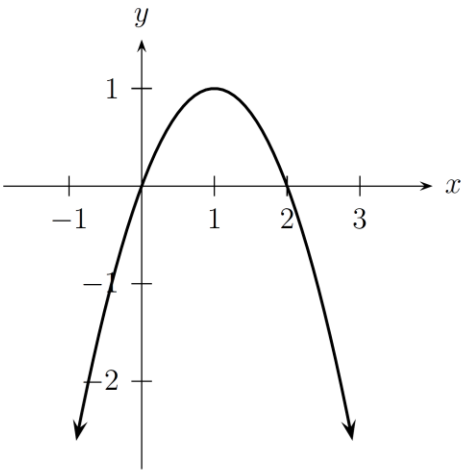
3 Let $z = 2 - 3i$. What is the value of z^{-1}?

(A) $-\frac{1}{5}(2+3i)$
(B) $\frac{1}{13}(2+3i)$
(C) $\frac{1}{5}(2-3i)$
(D) $\frac{1}{13}(2-3i)$

4 The eccentricity of the ellipse $3x^2 + 5y^2 - 15 = 0$

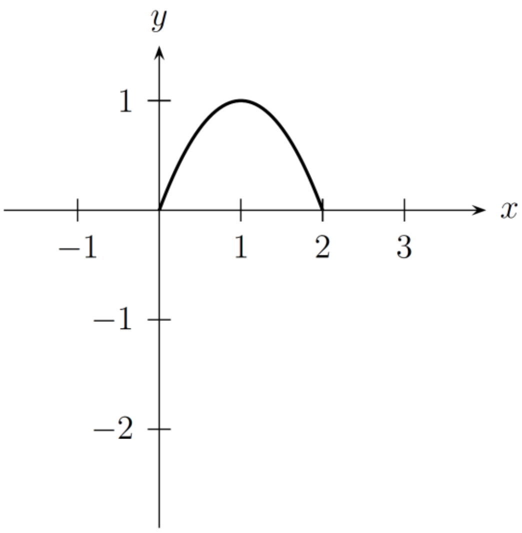
(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{2}{5}}$
(C) $\sqrt{\frac{8}{5}}$ (D) $\sqrt{\frac{5}{8}}$

5 The diagram below shows the graph of the function $y = f(x)$.

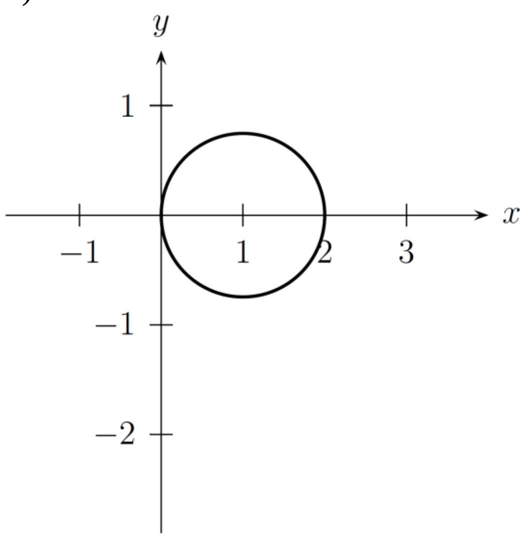


Which of the following is the graph of $y = \sqrt{f(x)}$?

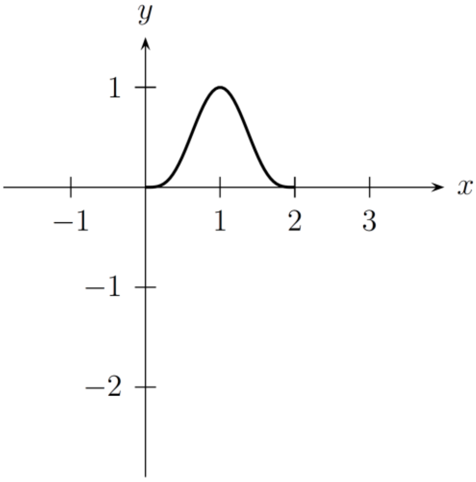
(A)



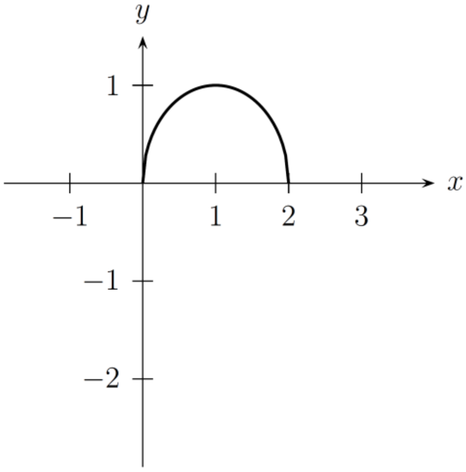
(B)



(C)



(D)



6 What is the value of $\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the substitution $x = 4\sin^2\theta$?

(A) 0.75π
(B) $\pi - 2$
(C) $\pi + 6$
(D) $3\pi - 8$

7 What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3 + pz + q = 0$?

(A) $p = -2$ and $q = -4$
(B) $p = -2$ and $q = 4$
(C) $p = 2$ and $q = -4$
(D) $p = 2$ and $q = 4$

8 The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola

$$xy = c^2 \quad (p \neq q)$$

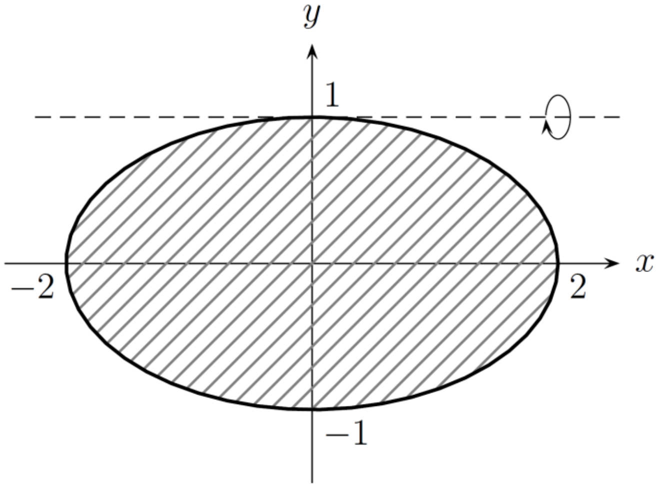
The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x - py + c - cp^4 = 0$
(B) $p^3x - py + c - cp^4 = 0$
(C) $x + p^2y - 2c = 0$
(D) $x + p^2y - 2cp = 0$

9 A car of mass m is travelling at a constant velocity V_o m/s. The brakes are applied, resulting in a constant deceleration until the car comes to a stop in l m. What is the force exerted by the brakes?

(A) $\frac{m{V_o}^2}{2l}$
(B) $\frac{m{V_o}^2}{l}$
(C) $-\frac{m{V_o}^2}{2l}$
(D) $-\frac{m{V_o}^2}{l}$

10 The diagram below shows the ellipse



The ellipse is rotated around the line $y = 1$ to form a solid.

Which integral represents the volume of the torus?

(A) $4\pi \int_{-1}^{1} (1-y)\sqrt{1-y^2}\, dy$

(B) $8\pi \int_{-1}^{1} (1-y)\sqrt{1-y^2}\, dy$

(C) $4\pi \int_{-1}^{1} y\sqrt{1-y^2}\, dy$

(D) $8\pi \int_{-1}^{1} y\sqrt{1-y^2}\, dy$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) (i) Show that $(1-2i)^2 = -3-4i$. **1**

(ii) Hence solve the equation $z^2 - 5z + (7+i) = 0$. **2**

(b) Let $z = 2 + 2i$

(i) What is the exact value of $|z|$ and arg z? **2**

(ii) Find z^3 in $x + iy$ form by .using De Moivre's theorem **2**

(c) Let two complex numbers be $z_1 = 2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ and $z_2 = 2i$.

(i) On an Argand diagram sketch the vectors *OA* and *OB* to represent z_1 and z_2 respectively. **1**

(ii) Draw the vectors $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram. **1**

(iii) What are the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$? **2**

(d) (i) Find the three cubic roots of –1 by solving the equation $z^3 + 1 = 0$. **2**

(ii) Let α be a cubic root of –1, where α is not real. Show that $\alpha^2 = \alpha - 1$. **1**

(iii) Hence simplify $(1-\alpha)^6$. **1**

Question 12 (15 marks) **Marks**

(a) If α, β, γ are the roots of the polynomial equation $3x^3 - 5x^2 - 4x + 3 = 0$, Find the equation with roots $\alpha - 1,\ \beta - 1,\ \gamma - 1$. **1**

(b) The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$. What are the values of *a* and *b*? **2**

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

PQ is a diameter of the ellipse. The tangent to the ellipse at *P* meets the vertical through *Q* at *R*.

(i) Prove the equation of the tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. **2**

(ii) Show the ratio of the area of ΔPQR to the area of the ellipse is $2 : \pi|\tan\theta|$. **3**

You may use the fact that the area of the an ellipse is $A = \pi ab$.

(d) The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the directrix at *Q*. The focus of the parabola is *S*.

(i) Find the equation of the tangent at *P*. **2**

(ii) Find the coordinates of *Q*. **1**

(iii) Show that *PQ* subtends a right angle at *S*. **4**

Question 13 (15 marks) **Marks**

(a) When a submarine of mass m is travelling horizontally underwater at maximum power, its engine delivers a force F on this submarine. The water exerts a resistive force proportional to the square of the submarine's speed v.

(i) Explain why $\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$, where k is a positive constant. **1**

(ii) The submarine increases its speed from v_1 to v_2. Show that the distance travelled during this period is **3**

$$\frac{m}{2k} log_e \frac{F - k{v_1}^2}{F - k{v_2}^2}$$

(b) A particle of mass m is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ Newtons. After t seconds, the particle has fallen x metres, and has the velocity v ms^{-1} and the acceleration a ms^{-2}. **3**

The particle hits the ground $\log_e(1 + \sqrt{2})$ seconds after it is dropped. Take $g = 10\ \text{ms}^{-2}$. Express v as a function of t.

(c) A particle of mass 1 kg is moving horizontally in a straight line. It is initially at the origin and is moving with velocity $U\ ms^{-1}$ $(U > 0)$.

The particle is moving against a resistance $v^2 + v^3$, where v is the velocity.

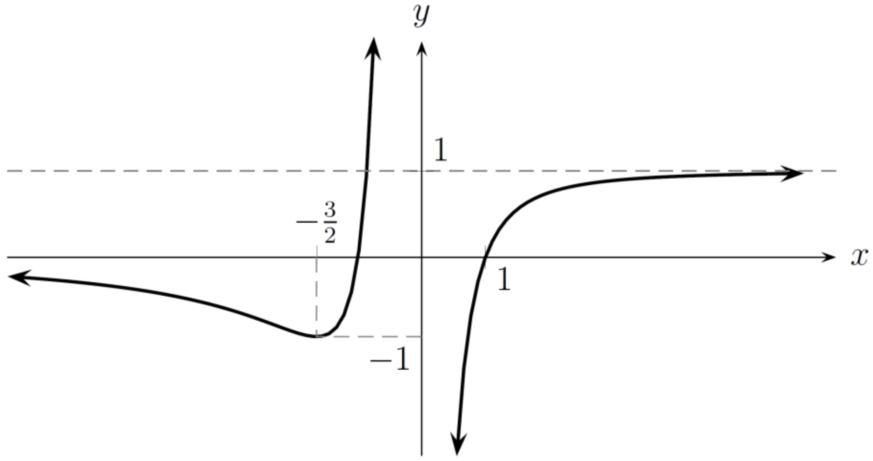
After T seconds the particle is X metres from the origin and is moving with velocity $\frac{1}{2}U\ ms^{-1}$.

(i) Show that $X = \log_e\left(\frac{2+U}{1+U}\right)$. **4**

(ii) Show that $U(T + X) = 1$. **4**

Question 14 (15 marks) **Marks**

(a) The graph of $y = f(x)$ is shown below. The lines $y = 1$ and the x- axis are the asymptotes.



Draw a neat one-third page sketch of the following, showing relevant features:

(i) $y = f(-x)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

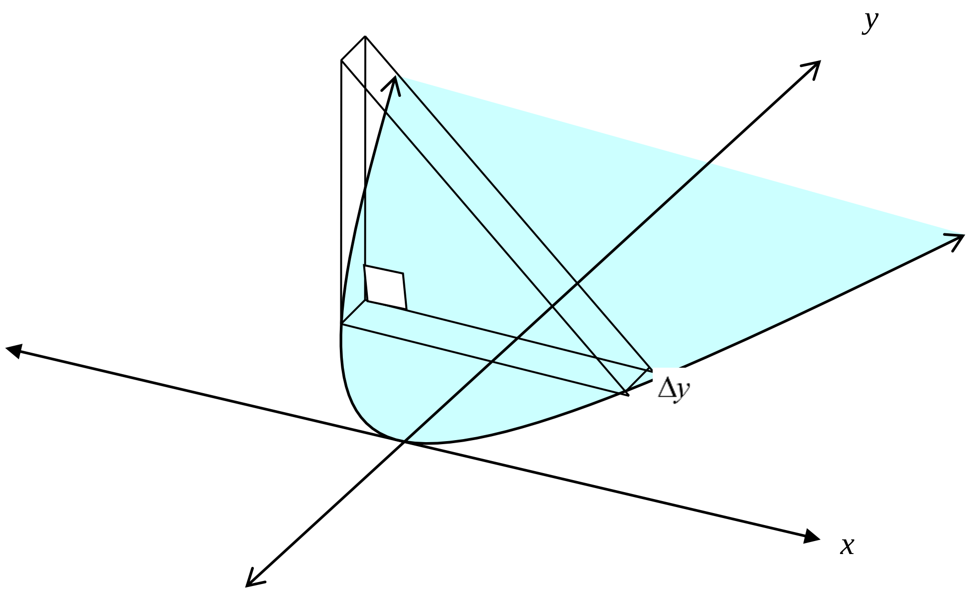
(iv) $y = (f(x))^2$ **2**

Question 14 continues overleaf…

Question 14 (continued…)

(b) A solid is formed by rotating about the y-axis the region bounded by the curve $y = \sin x$ and the x-axis between $0 \le x \le \pi$. Find the volume of this solid using the method of cylindrical shells. **4**

(c) A solid shape is formed as shown below. Its base is in the xy plane and is in the shape of a parabola $y = x^2$. The vertical cross section is in the shape of a right angled isosceles triangle. **3**



By using the method of slicing, calculate the volume of the solid between the values $y = 0$ and $y = 4$.

Examination continues overleaf…

Question 15 (15 marks) **Marks**

(a) Find $\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx$. **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate **4**

$$\int \frac{dx}{5 + 3\sin x + 4\cos x}$$

(c) (i) Find real numbers *A*, *B*, *C* and *D* such that: **2**

$$\frac{4}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

(ii) Hence evaluate in simplest form $\int \frac{4}{1-x^4}\,dx$ **2**

(d) (i) Let $I_n = \int_0^1 (1-x^r)^n\,dx$, where $r > 0$, for $n = 0, 1, 2, 3, \ldots$ **3**

Show that $I_n = \frac{nr}{nr+1} I_{n-1}$.

(ii) Hence or otherwise, find the value of $I_n = \int_0^1 (1-x^{\frac{3}{2}})^3\,dx$. **2**

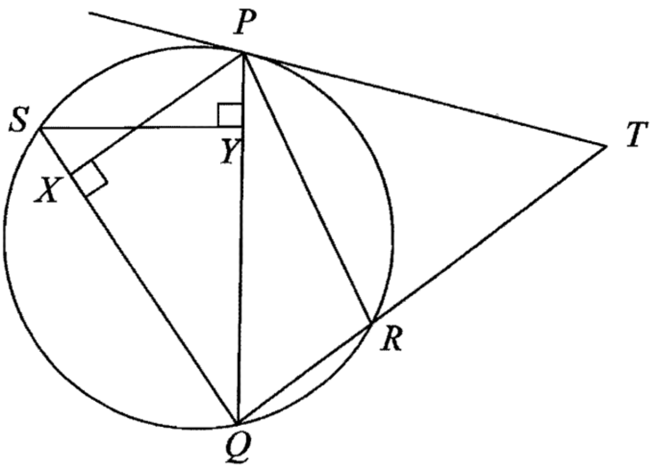
Question 16 (15 marks) **Marks**

(a) (i) Show that $\sin x + \sin 3x = 2\sin 2x\cos 2x$ **1**

(ii) Hence or otherwise solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

(b) Find the equation of the tangent to the curve $x^2y + 2x - 2xy = 0$ at (1, 2). **2**

(c) In the diagram below, *TP* is a tangent to the circle at *P*, and *TQ* is a secant cutting the circle at *R*



SQ is a chord of the circle such that *PX* and *SY* are perpendicular to *SQ* and *PQ* respectively.

(i) Prove that $\angle TRP = \angle TPQ$. **3**

(ii) Explain why *SPYX* is a cyclic quadrilateral. **1**

(iii) Prove that $\angle PYX = \angle PRQ$. **3**

(d) ΔABC has sides of length *a*, *b* and *c*. If $a^2 + b^2 + c^2 = ab + bc + ca$ show that ΔABC is an equilateral triangle. **3**

End of paper

Solutions Ext 2 2016

Q 11

a i) $(1-2i)^2$

$= 1-4i+4i^2$

$= 1-4i-4$ ✓

$= -3-4i$

ii) $z^2-5z+(7+i)=0$

$z = \dfrac{5 \pm \sqrt{25-4(7+i)}}{2}$

$= \dfrac{5 \pm \sqrt{25-28-4i}}{2}$

$= \dfrac{5 \pm \sqrt{-3-4i}}{2}$

$= \dfrac{5 \pm \sqrt{(1-2i)^2}}{2}$ ✓

$= \dfrac{5 \pm (1-2i)}{2}$

$= \dfrac{6-2i}{2}, \dfrac{4+2i}{2}$

$= 3-i, \; 2+i$ ✓

b) $z = 2+2i$

i) $|z| = \sqrt{4+4}$

$= 2\sqrt{2}$ ✓

$\arg z = \dfrac{\pi}{4}$ ✓

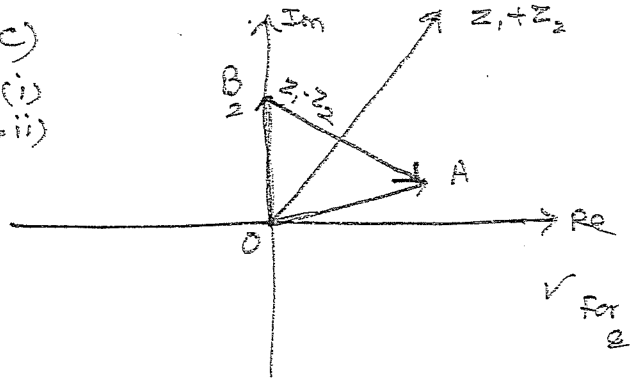
(ii) $z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z^3 = 16\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ✓

$= 16\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= -16+16i$ ✓

c) (i) & ii)



✓ for z_1 & z_2

✓ z_1+z_2 & z_1-z_2

iii) $\arg(z_1+z_2) = \dfrac{7\pi}{24}$

$\arg(z_1-z_2) = -\dfrac{5\pi}{24}$

d) i) $z^3+1=0$

$z^3 = -1$

Let $z = \text{cis}\,\theta$

$\text{cis}\,3\theta = -1$

$\text{cis}\,3\theta = \text{cis}(2n+1)\pi$

$3\theta = (2n+1)\pi$

$\theta = \dfrac{(2n+1)\pi}{3}$

as $-\pi < \theta \leq \pi$

$-\pi < \dfrac{(2n+1)}{3}\pi \leq \pi$

$-3\pi < (2n+1)\pi \leq 3\pi$

$-3 < (2n+1) \leq 3$

$-4 < 2n \leq 2$

$-2 < n \leq 1$

$\therefore n = -1, 0, 1$

$z_1 = \text{cis}\dfrac{(-2+1)\pi}{3} = \text{cis}\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

$z_2 = \text{cis}\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

$z_3 = \text{cis}\left(\dfrac{3}{3}\right)\pi = \text{cis}\,\pi = -1$

) $\alpha^3+1=0$

$(\alpha+1)(\alpha^2-\alpha+1)=0$

as α is not the real root

$\alpha+1\neq 0$

$\therefore \alpha^2-\alpha+1=0$

$\alpha^2=\alpha-1$

or Show it to be true with one of values of α.

If $\alpha=\frac{1+\sqrt{3}i}{2}$

$\alpha^2=\frac{-2+2\sqrt{3}i}{4}$

$=\frac{-1+\sqrt{3}i}{2}$

$=\frac{-2+1+\sqrt{3}i}{2}$

$\alpha^2=-1+\alpha$

i) $(1-\alpha)^6=(-(\alpha-1))^6$

$=(\alpha-1)^6$

$=\alpha^{12}$

$=(\alpha^3)^4$

$=(-1)^4=1$ ✓

Question 12

) $3x^3-5x^2-4x+3=0$

Hence required eq.

$3(x+1)^3-5(x+1)^2-4(x+1)+3=0$

$(x^3+3x^2+3x+1)-5(x^2+2x+1)-4x-4+3=0$

$x^3+9x^2+9x+3-5x^2-10x-5-4x-4+3=0$ ✓

$x^3+4x^2-5x-3=0$

) Double root at 2

$\Rightarrow P(2)=0$

$P'(2)=0$

$P(x)=x^4+ax^2+bx+28$

$P'(x)=4x^3+2ax+b$

$P(2)=2^4+a(2)^2+b(2)+28=0$

$=16+4a+2b+28=0$

$4a+2b+44=0$ ①

$P'(2)=4(2)^3+2a(2)+b=0$

$=32+4a+b=0$ ② ✓

$-$ ①

$b+12=0$

$b=-12$

Sub in ②

$4a-12+32=0$

$4a+20=0$

$a=-5$ ✓

C) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-2x}{a^2}\times\frac{b^2}{2y}$

$=-\frac{b^2}{a^2}\frac{x}{y}$

at P,

$\frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{a\cos\theta}{b\sin\theta}$

$=-\frac{b\cos\theta}{a\sin\theta}$ ✓

Equation of the tangent at P

$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$a\sin\theta y-ab\sin^2\theta=-b\cos\theta x+ab\cos^2\theta$

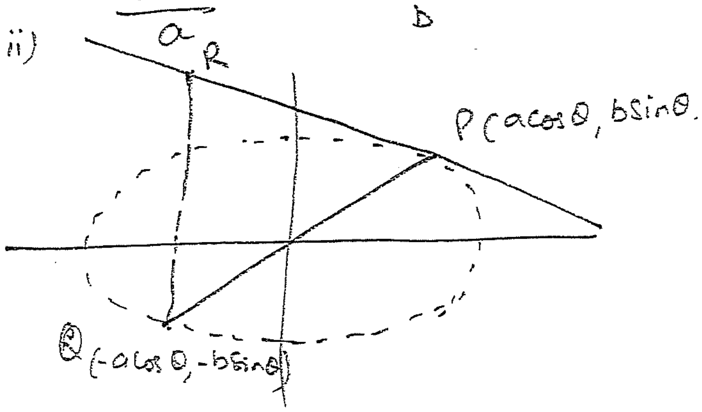
$b\cos\theta x+a\sin\theta y=ab(\cos^2\theta+\sin^2\theta)$ ✓

$b\cos\theta x+a\sin\theta y=ab$

Divide by ab

$\frac{\cos\theta}{a}x+\frac{\sin\theta}{b}y=1$

ii)



$Q: (-a\cos\theta, -b\sin\theta)$

For R: $x: (-a\cos\theta)$

Sub x-coordinate of R in equation of PQ

$$\frac{\cos\theta}{a}(-a\cos\theta) + \frac{\sin\theta}{b}y = 1$$

$$-\cos^2\theta + \frac{\sin\theta}{b}y = 1$$

$$y = \frac{b(1+\cos^2\theta)}{\sin\theta}$$ ✓

Length RQ

$$= \frac{b(1+\cos^2\theta)}{\sin\theta} + b\sin\theta$$

$$= \frac{b + b\cos^2\theta + b\sin^2\theta}{\sin\theta}$$

$$= \frac{2b}{\sin\theta}$$

Perpendicular distance of RQ from P

$$= 2a\cos\theta$$ ✓

Area of ΔPQR

$$= \frac{1}{2} \times \frac{2b}{\sin\theta} \times 2a\cos\theta$$

$$= 2ab\cot\theta$$

$$= \frac{2ab}{\tan\theta} = \frac{2ab}{|\tan\theta|}$$ as area > 0 ✓

Ratio of ar ΔPQR to ellipse

$$= \frac{2ab}{|\tan\theta|} \div \pi ab$$

$$= 2 : \pi|\tan\theta|$$

a) i) $$\frac{x}{a^2} - \frac{y}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$= \frac{b^2x}{a^2y}$$

at P,

$$\frac{dy}{dx} = \frac{b^2 . a\sec\theta}{a^2 . b\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$ ✓

Eq. of the tangent at P

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$a\tan\theta y - ab\tan^2\theta = b\sec\theta x - ab\sec^2\theta$$

$$b\sec\theta x - a\tan\theta y = ab(\sec^2\theta - \tan^2\theta)$$

$$b\sec\theta x - a\tan\theta y = ab$$

Divide by ab ✓

$$\frac{\sec\theta}{a}x - \frac{\tan\theta}{b}y = 1$$

ii) Eq. of directrix, $x = \frac{a}{e}$

at Q, eq. of PQ becomes

$$\frac{\sec\theta}{a} . \frac{a}{e} - \frac{\tan\theta}{b}y = 1$$

$$-\frac{\tan\theta}{b}y = 1 - \frac{\sec\theta}{e}$$

$$y = \left(\frac{\sec\theta}{e} - 1\right)\frac{b}{\tan\theta}$$

$$= \frac{b\sec\theta}{e\tan\theta} - \frac{b}{\tan\theta}$$ ✓

$\therefore$ for Q: $x = \frac{a}{e}$

$$y = \frac{b\sec\theta}{e\tan\theta} - \frac{b}{\tan\theta}$$

iii) $P: (a\sec\theta, b\tan\theta)$

$Q: \left(\frac{a}{e}, \frac{b\sec\theta}{e\tan\theta} - \frac{b}{\tan\theta}\right)$

$S: (ae, 0)$

gradient of PS

$$m_{PS} = \frac{b\tan\theta}{a\sec\theta - ae}$$

$$= \frac{b\tan\theta}{a(\sec\theta - e)}$$

gradient of QS

$$m_{QS} = \frac{\frac{b\sec\theta}{e\tan\theta} - \frac{b}{\tan\theta}}{\frac{a}{e} - ae}$$

$$= \frac{\frac{b\sec\theta - be}{e\tan\theta}}{\frac{a - ae^2}{e}}$$

$$= \frac{b(\sec\theta - e)}{e\tan\theta} \times \frac{e}{a(1-e^2)}$$

$$= \frac{b(\sec\theta - e)}{a(1-e^2)\tan\theta}$$

$m_{PS} \times m_{QS}$

$$= \frac{b\tan\theta}{a(\sec\theta - e)} \times \frac{b(\sec\theta - e)}{a(1-e^2)\tan\theta}$$

$$= \frac{b^2}{a^2(1-e^2)}$$

$$= \frac{b^2}{-a^2(e^2-1)}$$

$$= -\frac{b^2}{b^2}$$

$$= -1$$

Question 13

a) i) $ma = F - kv^2$

(Net force = Forward force − Resistance) ✓

$$a = \frac{1}{m}(F - kv^2)$$

$$\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$$

ii) $a = \frac{1}{m}(F - kv^2)$

$$v\frac{dv}{dx} = \frac{1}{m}(F - kv^2)$$

$$\frac{dv}{dx} = \frac{F - kv^2}{mv}$$

$$\frac{dx}{dv} = \frac{mv}{F - kv^2}$$ ✓

$$x = m\int \frac{v}{F - kv^2}\,dv$$

$$= -\frac{m}{2k}\int_{v_1}^{v_2} \frac{-2kv}{F - kv^2}\,dv$$

$$x = -\frac{m}{2k}\Big[\ln(F - kv^2)\Big]_{v_1}^{v_2}$$ ✓

$$= -\frac{m}{2k}\left[\ln(F - kv_2^2) - \ln(F - kv_1^2)\right]$$

$$= \frac{m}{2k}\left[\ln(F - kv_1^2) - \ln(F - kv_2^2)\right]$$ ✓

$$= \frac{m}{2k}\left[\ln\frac{F - kv_1^2}{F - kv_2^2}\right]$$

b) $v \downarrow \quad \downarrow mg \quad \uparrow R$

$ma = mg - R$

$ma = mg - \frac{1}{10}mv^2$

$a = g - \frac{1}{10}v^2$

$a = 10 - \frac{1}{10}v^2$

$a = \frac{100 - v^2}{10}$

$\frac{dv}{dt} = \frac{100-v^2}{10}$

$t = \int \frac{10}{100-v^2}\,dv$

$t = 10\int \frac{1}{100-v^2}\,dv$

For $\frac{1}{100-v^2} = \frac{1}{(10-v)(10+v)}$

$= \frac{A}{10-v} + \frac{B}{10+v}$

$\Rightarrow A(10+v) + B(10-v) = 1$

when $v = 10$

$20A = 1$

$A = \frac{1}{20}$

if $v = -10$

$20B = 1$

$B = \frac{1}{20}$ ✓

$\therefore t = 10\int \frac{1}{20(10-v)} + \frac{1}{20(10+v)}\,dv$

$= \frac{1}{2}\int \left(\frac{1}{10-v} + \frac{1}{10+v}\right)dv$

$= \frac{1}{2}\left[-\ln(10-v) + \ln(10+v)\right] + c$

$= \frac{1}{2}\left[\ln\frac{10+v}{10-v}\right] + c$

when $t=0$, $v=0 \Rightarrow c = 0$

$\therefore t = \frac{1}{2}\left[\ln\frac{10+v}{10-v}\right]$ ✓

$\therefore e^{2t} = \frac{10+v}{10-v}$

$(10-v)e^{2t} = 10+v$

$10e^{2t} - ve^{2t} = 10+v$

$10(e^{2t}-1) = v + ve^{2t}$

$v(1+e^{2t}) = 10(e^{2t}-1)$

$v = \frac{10(e^{2t}-1)}{e^{2t}+1}$ ✓

c) $ma = -m(v^2+v^3)$

$a = -(v^2+v^3)$

$v.\frac{dv}{dx} = -(v^2+v^3)$

$\frac{dv}{dx} = -(v+v^2)$

$x = \int -\frac{1}{v+v^2}\,dv$ ✓

$= -\int \frac{1}{v(1+v)}\,dv$

For $\frac{1}{v(1+v)} = \frac{A}{v} + \frac{B}{1+v}$

$A(1+v) + Bv = 1$

when $v=0$; when $v=1$

$A = 1$; $2A + B = 1$

$B = -1$

$\therefore x = -\int \frac{1}{v} - \frac{1}{1+v}\,dv$

$= \int \frac{1}{1+v} - \frac{1}{v}\,dv$

$= \left[\ln(1+v) - \ln v\right] + c$ ✓

$x = \ln\left[\frac{1+v}{v}\right] + c$

when $x=0$, $v = U$

$0 = \ln\left[\frac{1+U}{U}\right] + c$

$\therefore x = \ln\left[\frac{1+v}{v}\right] - \ln\left[\frac{1+U}{U}\right]$

$= \ln\left[\frac{(1+v)(U)}{v(1+U)}\right]$ ✓

(7)

$\therefore$ When $x = X$, $v = \frac{1}{2}U$

$\therefore$ From (1)

$$X = \ln\left[\frac{1+\frac{1}{2}U}{\frac{1}{2}U} \cdot \frac{U}{(1+U)}\right]$$ ✓

$$= \ln\left[\frac{2+U}{2} \cdot \frac{2}{U} \cdot \frac{U}{(1+U)}\right]$$

$$X = \ln\left[\frac{2+U}{1+U}\right]$$

ii) $a = -(v^2+v^3)$

$$\frac{dv}{dt} = -(v^2+v^3)$$

$$t = \int \frac{-1}{v^2+v^3}\, dv$$

$$= -\int \frac{1}{v^2(1+v)}\, dv$$

For $\frac{1}{v^2(1+v)} = \frac{Av+B}{v^2} + \frac{C}{1+v}$

$(Av+B)(1+v) + Cv^2 = 1$

when $v = 0$

$B = 1$

When $v = -1$

$C = 1$

When $v = 1$

$(A+B)2 + C = 1$

$2A + 2 + 1 = 1$

$A = -1$

$$t = -\int \frac{-v+1}{v^2} + \frac{1}{1+v}\, dv$$ ✓

$$= -\int \frac{1-v}{v^2} + \frac{1}{1+v}\, dv$$

$$= -\int \frac{1}{v^2} - \frac{1}{v} + \frac{1}{1+v}\, dv$$

$$= \int \frac{1}{v} - \frac{1}{v^2} - \frac{1}{1+v}\, dv$$

$$t = \ln v + \frac{1}{v} - \ln(1+v) + C$$

$$t = \frac{1}{v} + \ln\frac{v}{1+v} + C$$

When $t = 0$, $v = U$

$$0 = \frac{1}{U} + \ln\frac{U}{1+U} + C$$

$$\therefore t = \frac{1}{v} + \ln\frac{v}{1+v} - \frac{1}{U} - \ln\frac{U}{1+U}$$

$$= \frac{1}{v} - \frac{1}{U} + \ln\frac{v(1+U)}{(1+v)U}$$ ✓

When $t = T$, $v = \frac{1}{2}U$

$$T = \frac{1}{\frac{U}{2}} - \frac{1}{U} + \ln\frac{\frac{U}{2}(1+U)}{\left(1+\frac{U}{2}\right)U}$$

$$= \frac{2}{U} - \frac{1}{U} + \ln\frac{U(1+U)\cdot 2}{2(2+U)U}$$ ✓

$$T = \frac{1}{U} + \ln\frac{1+U}{2+U}$$

$$= \frac{1}{U} + \ln\left(\frac{2+U}{1+U}\right)^{-1}$$

$$T = \frac{1}{U} - \ln\left(\frac{2+U}{1+U}\right)$$ ✓

From part (i)

$$T = \frac{1}{U} - X$$

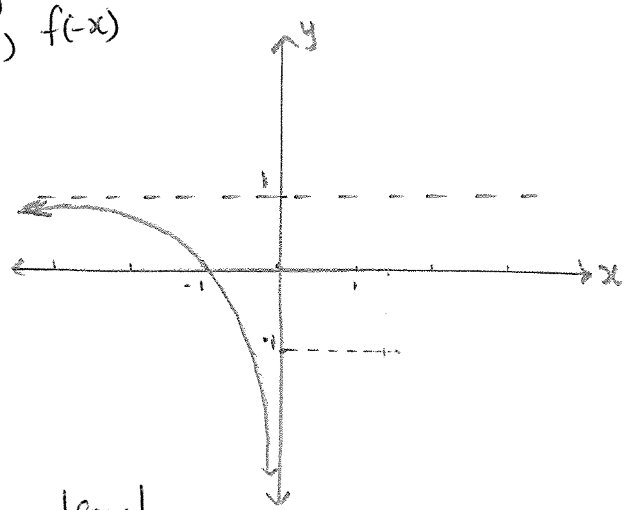
$$T + X = \frac{1}{U}$$

$$\therefore U(T+X) = 1$$

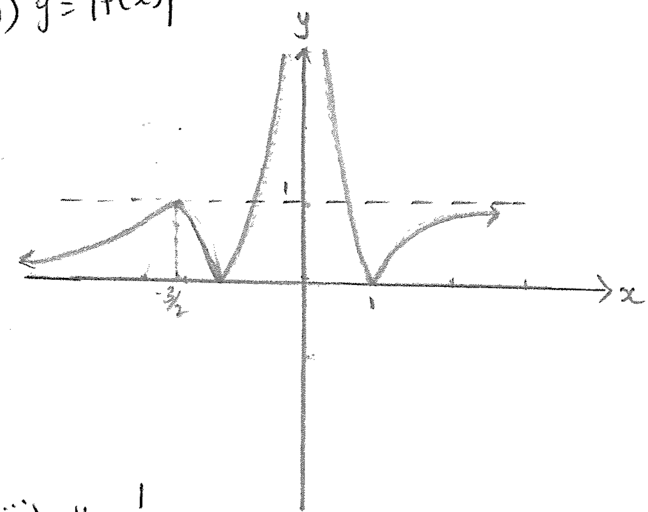
Q14

a)

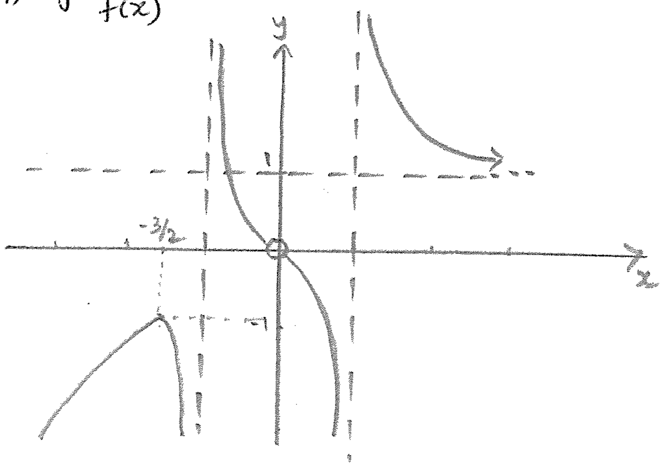
i) $f(-x)$



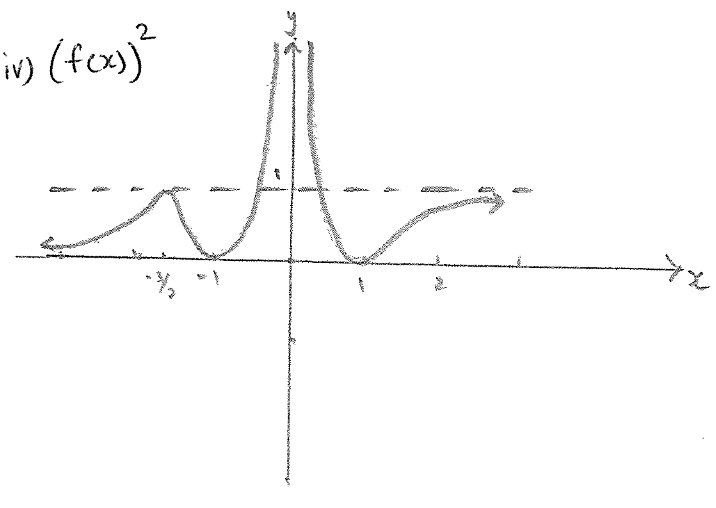
ii) $y = |f(x)|$



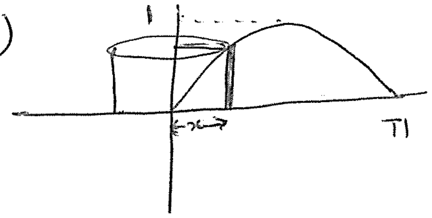
iii) $y = \frac{1}{f(x)}$



iv) $(f(x))^2$



b)



For the shell

radius $= r = x$

height $= h = y = \sin x$

thickness $= \delta x$

$\delta V = 2\pi x . h . \delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} \delta V$

$= 2\pi \int_0^{\pi} x \sin x \, dx$

For $\int_0^{\pi} x \sin x \, dx$

let $u = x$, $v = -\cos x$

$u' = 1$, $v' = \sin x$

$\therefore \int_0^{\pi} x \sin x \, dx = \left[-x \cos x\right]_0^{\pi} + \int_0^{\pi} \cos x \, dx$

$= \left[-\pi \cos \pi\right] + \left[\sin x\right]_0^{\pi}$

$= \pi$

$\therefore V = 2\pi . \pi$

$= 2\pi^2$ units^3

c) For triangular cross-section

base $= 2x =$ height

$\therefore$ area of the cross-section

$A = \frac{1}{2} \times b \times h = \frac{1}{2} \times 2x \times 2x$

$= 2x^2$

$\delta V =$ Area of cross-section $\times$ thickness

$\delta V = 2x^2 . \delta y$

$V = \int_0^4 2x^2 \, dy$

$= \int_0^4 2y \, dy$

$= \left[\frac{2y^2}{2}\right]_0^4 = 16$ units^3.

Question 15

(a) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

Let $e^x = u$

$e^x = \frac{du}{dx}$

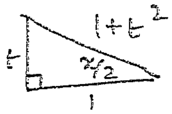
$\therefore \displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$ ✓

$= \sin^{-1} u + C$

$= \sin^{-1}(e^x) + C$ ✓

b) $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$



$\frac{dt}{dx} = \frac{1+t^2}{2}$

$\therefore dx = \frac{2}{1+t^2}\,dt$ ✓

$\Rightarrow \displaystyle\int \frac{dx}{5+3\sin x+4\cos x}$

$= \displaystyle\int \frac{1}{5+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2}{(1+t^2)}\,dt$ ✓

$= \displaystyle\int \frac{\cancel{1+t^2}}{5(1+t^2)+6t+4(1-t^2)}\cdot\frac{2}{\cancel{1+t^2}}\,dt$

$= \displaystyle\int \frac{2}{5+5t^2+6t+4-4t^2}\,dt$

$= \displaystyle\int \frac{2}{t^2+6t+9}\,dt$ ✓

$= \displaystyle\int \frac{2}{(t+3)^2}\,dt$

$= 2\displaystyle\int (t+3)^{-2}\,dt$

$= \frac{-2}{t+3} + C$

$= \frac{-2}{\tan\frac{x}{2}+3} + C$ ✓

c) (i) $\frac{4}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$

$\Rightarrow A(1+x)(1+x^2)+B(1-x)(1+x^2)+(Cx+D)(1-x^2)=4$

If $x=0$

$A+B+D=4$ — (1)

If $x=1$

$4A=4$

$A=1$

If $x=-1$

$4B=4$

$B=1$

From (1) $D=2$

If $x=2$

$A(3)(5)+B(-1)(5)+(2C+D)(-3)=4$

$15A-5B-6C-6=4$

$15-5-6C-6=4$

$C=0$

$\therefore \frac{4}{1-x^4} = \frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}$

✓✓ values of A, B, C, D (1 mark off for any incorrect value)

(ii) $\displaystyle\int \frac{4}{1-x^4}\,dx = \int \frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}\,dx$

$= -\ln(1-x)+\ln(1+x)+2\tan^{-1}x+C$ ✓

$= \ln\frac{1+x}{1-x} + 2\tan^{-1}x + C$ ✓

d) i)

$$I_n = \int_0^1 (1-x^r)^n \, dx$$

Let $u = (1-x^r)^n$

$u' = n(1-x^r)^{n-1} \cdot (-r x^{r-1})$

$= -nr x^{r-1} \cdot (1-x^r)^{n-1}$

and $v' = 1$

$\therefore v = x$

as $\int uv' = uv - \int vu'$

$$I_n = \left[(1-x^r)^n \cdot x\right]_0^1 + \int_0^1 nr x^{r-1} (1-x^r)^{n-1} \cdot x \, dx$$ ✓

$$= nr \int_0^1 x^r (1-x^r)^{n-1} dx$$

$$= nr \int_0^1 (1-1+x^r)(1-x^r)^{n-1} dx$$

$$= nr \int_0^1 (1-(1-x^r))(1-x^r)^{n-1} dx$$ ✓

$$= nr \int_0^1 (1-x^r)^{n-1} - (1-x^r)^n \, dx$$

$$= nr \int_0^1 (1-x^r)^{n-1} dx - nr \int_0^1 (1-x^r)^n dx$$

$$= nr I_{n-1} - nr I_n$$ ✓

$$I_n(1+nr) = nr I_{n-1}$$

$$I_n = \frac{nr}{1+nr} I_{n-1}$$

ii)

$$I_n = \int_0^1 \left(1-x^{\frac{3}{2}}\right)^3 dx$$

$r = \frac{3}{2}$, $n = 3$

$$I_n = I_3 = \frac{3\left(\frac{3}{2}\right)}{1+\frac{9}{2}} \cdot I_2$$

$$= \frac{9}{2} \cdot \frac{2}{11} I_2$$

$$= \frac{9}{11} I_2$$

$$= \frac{9}{11}\left[\frac{2 \cdot \frac{3}{2}}{1+2 \cdot \frac{3}{2}} \cdot I_1\right]$$

$$= \frac{9}{11}\left[\frac{3}{4} I_1\right]$$ ✓

$$= \frac{9}{11}\left(\frac{3}{4}\right)\left(\frac{\frac{3}{2}}{1+\frac{3}{2}} I_0\right)$$

$$= \frac{9}{11} \times \frac{3}{4} \times \frac{3}{5} I_0$$

for $I_0 = \int_0^1 \left(1-x^{\frac{3}{2}}\right)^0 dx$

$$= \int_0^1 1 \, dx$$

$$= [x]_0^1$$

$$= 1$$

$$\therefore I_n = I_3 = \frac{9}{11} \times \frac{3}{4} \times \frac{3}{5}$$ ✓

$$= \frac{81}{220}$$

Question 16

a) i) $\sin x + \sin 3x$

$= 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)$

$= 2\sin\frac{4x}{2}\cos\left(\frac{-2x}{2}\right)$

$= 2\sin 2x\cos(-x)$

$= 2\sin 2x\cos x$ ✓

ii) $\sin x + \sin 2x + \sin 3x = 0$

$\sin 2x + 2\sin 2x\cos x = 0$

$\sin 2x(1 + 2\cos x) = 0$

$\therefore \sin 2x = 0$

$2x = 0, \pi, 2\pi$

$x = 0, \frac{\pi}{2}, \pi$ ✓

$1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$ ✓

b) $x^2 y + 2x - 2xy = 0$

$y(x^2 - 2x) = -2x$

$y = \frac{-2x}{x^2 - 2x}$

$= \frac{-2}{x-2}$

$\frac{dy}{dx} = \frac{(x-2)(0) - (-2)}{(x-2)^2}$

$= \frac{2}{(x-2)^2}$ ✓

at (1,2)

$\frac{dy}{dx} = \frac{2}{1} = 2$

eq. of tangent

$y - 2 = 2(x-1)$

$2x - 4 = 0$ ✓

c) i) $\angle PSQ + \angle PRQ = 180$ (opp. angles of a cyclic quadrilateral) ✓

But

$\angle PRQ + \angle PRT = 180°$ (adjacent supplementary angles) ✓

$\therefore \angle PSQ = \angle PRT$

However

$\angle PSQ = \angle TPQ$ (angles in alt. segments) ✓

$\therefore \angle TPQ = \angle TRP$

ii) $\angle SXP = \angle SYP = 90°$ (converse of angles in the same segment) ✓

iii) $\angle PYX + \angle PSX = 180°$ (opp. angles of a cyclic quadrilateral)

$\angle PYX = 180 - \angle PSX$ — (1) ✓

~~= 180° −~~

But $\angle PSX + \angle PRQ = 180°$ (opp. angles of a cyclic quad. PSQR)

$\Rightarrow \angle PSX = 180° - \angle PRQ$ ✓

$\therefore$ from (1)

$\angle PYX = 180 - (180° - \angle PRQ)$ ✓

$\angle PYX = \angle PRQ$

$(a-b)^2 + (b-c)^2 + (a-c)^2$

$= a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + a^2 - 2ac + c^2$

$= 2(a^2 + b^2 + c^2) - 2ab - 2bc - 2ac$

$= 2(a^2 + b^2 + c^2 - ab - bc - ac)$

$= 0$ because $a^2 + b^2 + c^2 = ab + bc + ac$ (given)

Possible only if

$a-b=0$ $b-c=0$ $c-a=0$

$\Rightarrow a=b$, $b=c$, $c=a$

$\therefore \Delta ABC$ is an equilateral triangle.

MCQ

Q 1	A	6	B
2	C	7	B
3	B	8	B
4	B	9	C
5	D	10	B