NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2017 HSC Course Assessment Task 4 (Trial HSC)
Thursday August 3, 2017

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order after this paper, tie into one bundle with the string provided and hand to examination supervisors.
- A NESA Reference Sheet is provided.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MAT.1 – Mrs Bhamra ◯ 12MAT.2 – Mr Lam

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 10 minutes for this section

Mark your answers on the answer grid provided (labelled as page 13).

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers
- $\mathbb{C}$ – set of all complex numbers

Questions **Marks**

1. What is the number of asymptotes on the graphs of $y = \dfrac{x^3}{x^2-1}$? **1**

(A) 1 (B) 2 (C) 3 (D) 4

2. The equation $x^3 - y^3 + 3xy + 1 = 0$ is an implicit function in x and y. **1**

Which of the following is the expression for the gradient function?

(A) $\dfrac{y^2-x}{x^2+y}$ (B) $\dfrac{y^2+x}{x^2-x}$ (C) $\dfrac{x^2+y}{y^2-x}$ (D) $\dfrac{x^2-y}{y^2+x}$

3. The polynomial $P(z) = z^4 - 4z^3 + Az + 20$, where $A \in \mathbb{R}$, has $(3+i)$ as one of its zeros. **1**

Which of the following expression is $P(z)$ expressed as a product of two real quadratic factors?

(A) $\left(z^2-2z+2\right)\left(z^2-6z+10\right)$ (C) $\left(z^2-2z+2\right)\left(z^2+6z+10\right)$

(B) $\left(z^2+2z+2\right)\left(z^2-6z+10\right)$ (D) $\left(z^2+2z+2\right)\left(z^2+6z+10\right)$

4. A particle of mass m is moving horizontally in a straight line. It experiences a resistive force of magnitude $2m\left(v+v^2\right)$ N when its speed is v metres per second. **1**

At time t seconds, the particle has a displacement of x metres from a fixed point O.

Which of the following is the correct expression for x in terms of v?

(A) $x = -\frac{1}{2}\int \frac{1}{1+v}\,dv$

(B) $x = -\frac{1}{2}\int \frac{1}{v(1+v)}\,dv$

(C) $x = \frac{1}{2}\int \frac{1}{1+v}\,dv$

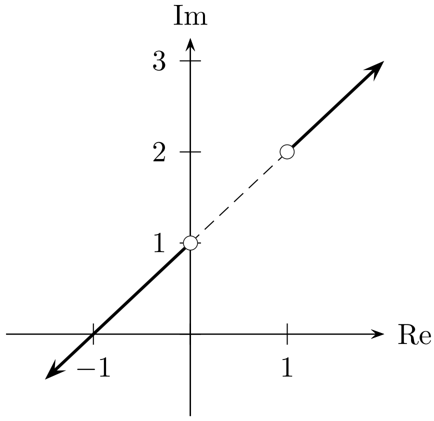
(D) $x = \frac{1}{2}\int \frac{1}{v(1+v)}\,dv$

5. Which of the following could be $f(x)$ if **1**

$$\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2\cos x\,dx$$

(A) $3x^2$ (B) x^3 (C) $-x^3$ (D) $-3x^2$

6. Which of the following defines the locus of the complex number z, as sketched below? **1**



(A) $\arg\left(\dfrac{z-i}{z-1-2i}\right) = \pi$

(B) $\arg(z+i) = \arg(z-1-2i)$

(C) $\arg\left(\dfrac{z+i}{z-1-2i}\right) = \pi$

(D) $\arg(z-i) = \arg(z-1-2i)$

7. Multiplying a non-zero complex number by $\dfrac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}$ results in a rotation about the origin on an Argand diagram. **1**

What is the rotation?

(A) Clockwise by $\dfrac{\pi}{12}$

(B) Clockwise by $\dfrac{7\pi}{12}$

(C) Anticlockwise by $\dfrac{\pi}{12}$

(D) Anticlockwise by $\dfrac{7\pi}{12}$

8. Without evaluating the integrals, which of the following is greater than zero? **1**

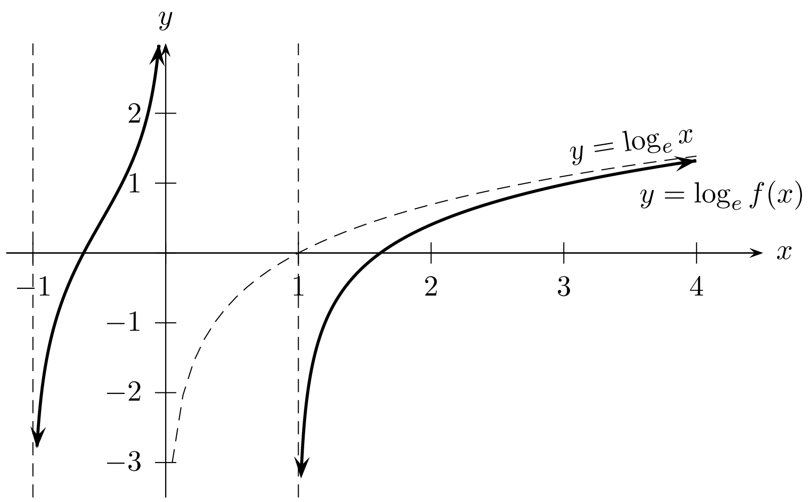
(A) $\displaystyle\int_{-1}^{1} \tan^{-1}(\sin x)\ dx$

(B) $\displaystyle\int_{-1}^{1} \frac{2x}{\sin^2 x}\, dx$

(C) $\displaystyle\int_{-1}^{1} \left((e^x)^3 + x^7\right)\ dx$

(D) $\displaystyle\int_{-1}^{1} \frac{x^5}{\cos^3 x}\, dx$

9. The following is a graph of $y = \log_e f(x)$ over its natural domain. The graph $y = \log_e f(x)$ has vertical asymptotes at $x = 0$, $x = \pm 1$, and an upper bound of $y = \log_e x$ as $x \to \infty$. **1**



Which of the following is the correct expression for $f(x)$?

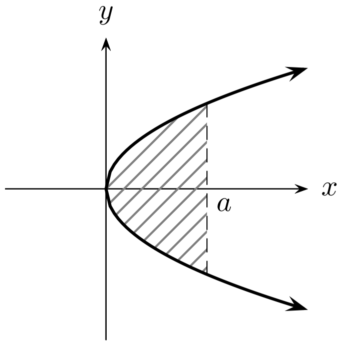
(A) $y = x - \dfrac{1}{x}$

(B) $y = x + \dfrac{1}{x}$

(C) $y = -\dfrac{1}{x}$

(D) $y = \dfrac{1}{x}$

10. A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex $(0, 0)$ and the line $x = a$ about the y axis. **1**



Which of the following integrals gives the volume of this area by *slicing*?

(A) $2\pi\sqrt{a}\int_0^a z^{\frac{3}{2}}\,dz$

(B) $4\pi\sqrt{a}\int_0^a z^{\frac{3}{2}}\,dz$

(C) $\pi\int_0^{2a}\left(a^2 - \dfrac{z^4}{16a^2}\right)\,dz$

(D) $2\pi\int_0^{2a}\left(a^2 - \dfrac{z^4}{16a^2}\right)\,dz$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 50 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) Find $\displaystyle\int e^{-2x}\cos x\,dx$. **4**

(b) Evaluate $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$. **3**

(c) i. Using the substitution $x = a - u$, show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

ii. Hence evaluate $\displaystyle\int_0^1 x(1-x)^{2017}\,dx$, giving your answer as the simplest fraction. **2**

(d) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3-\cos x-2\sin x}\,dx$$

Examination continues overleaf. . .

Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) i. Express $z = 1 + \sqrt{3}i$ in modulus-argument form. **1**

ii. Hence or otherwise, show that $z^7 - 64z = 0$. **2**

(b) Find $\sqrt{6i - 8}$, and hence solve the equation **4**

$$2z^2 - (3 + i)z + 2 = 0$$

(c) Find the complex number $z = a + ib$, where $a, b \in \mathbb{R}$, such that $2\overline{z} - iz = 1 + 4i$. **2**

(d) Sketch the region in the Argand diagram of the point z such that it satisfies all of **3**

$$\begin{cases} |\text{Arg}(z)| < \frac{\pi}{3} \\ z + \overline{z} < 4 \\ |z| > 2 \end{cases}$$

(e) $ABCD$ is a quadrilateral in the complex plane such that the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ represent complex numbers a, b, c and d respectively.

P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. M and N are midpoints of PR and QS respectively.

i. Show that the vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$ both represent the complex number **2**

$$\frac{1}{4}(a + b + c + d)$$

ii. Hence explain the type of quadrilateral that $PQRS$ is. **1**

Examination continues overleaf. . .

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) α, β and γ are non-zero and the roots of the cubic equation

$$x^3 + px + q = 0$$

i. Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of p and q. **1**

ii. Hence or otherwise, form a cubic equation with roots **3**

$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma} \text{ and } \frac{\gamma}{\alpha\beta}$$

(b) i. Find A, B and $C \in \mathbb{R}$ such that **3**

$$\frac{4x^2 - 5x - 7}{(x-1)(x^2 + x + 2)} \equiv \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 2}$$

ii. Hence evaluate **2**

$$\int_{-1}^{0} \frac{4x^2 - 5x - 7}{(x-1)(x^2 + x + 2)}\, dx$$

(c) i. Prove for all A: **2**

$$\cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$$

ii. Show that $x = 2\sqrt{2}\cos A$ satisfies the cubic equation **2**

$$x^3 - 6x = -2$$

provided $\cos 3A = -\dfrac{1}{2\sqrt{2}}$.

iii. Hence or otherwise, find all three roots to the equation $x^3 - 6x + 2 = 0$, correct to four decimal places. **2**

Examination continues overleaf. . .

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) i. Determine the real values of p for which the equation

$$\frac{x^2}{3+p} + \frac{y^2}{8+p} = 1$$

defines

(α) an ellipse **1**

(β) a hyperbola **2**

ii. For the value $p = -4$ in the above equation, find the **2**

- eccentricity
- coordinates of the foci, and
- the equations of the directrices

of the conic.

iii. Draw a neat sketch of the conic in part (ii), indicating the foci, vertices and directrices. **2**

(b) i. Prove that the equation of the tangent at the point $\left(t, \frac{1}{t}\right)$ to the hyperbola $xy = 1$ is $x + t^2y = 2t$. **2**

ii. The tangent at a point P on the hyperbola $xy = 1$ meets the y axis at A, and the normal at P meets the x axis at B. **3**

Show that the equation of the locus of the midpoint of AB as P loves along the hyperbola is

$$x = \frac{1 - y^4}{2y}$$

(c) $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ are the endpoints of a focal chord of the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **3**

Show that $e = \dfrac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$.

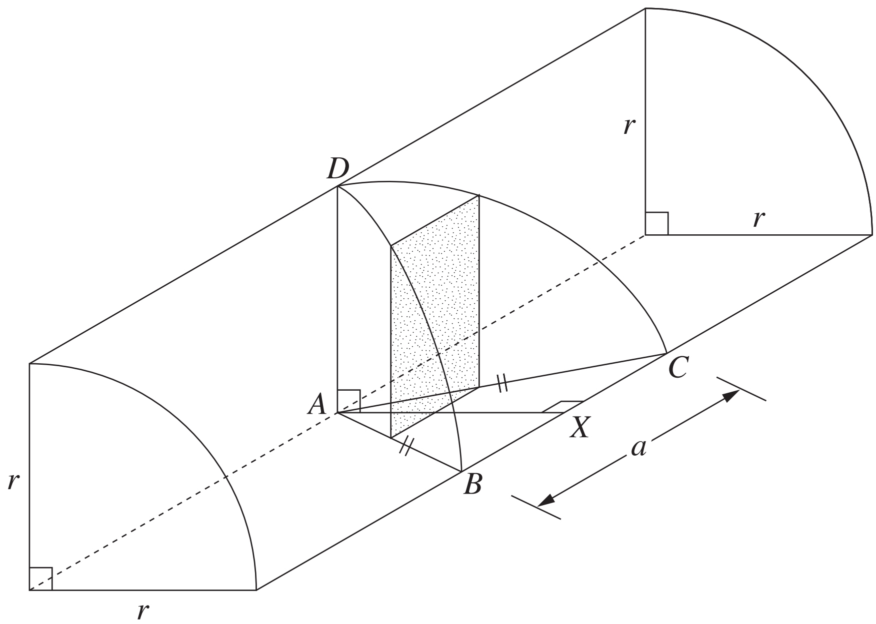
Examination continues overleaf. . .

Question 15 (15 Marks) Commence a NEW booklet. **Marks**

(a) By using the method of cylindrical shells, find the volume of the solid generated when the region bounded by the curve $y = \log_e x$, the x axis and the lines $x = 1$ and $x = e$ is rotated about the y axis. **3**

(b) The solid $ABCD$ is cut from a quarter cylinder of radius r as shown. Its base is an isosceles $\triangle ABC$ with $AB = AC$. The length of BC is a and the midpoint of BC is X. **3**

The cross-sections perpendicular to AX are rectangles. A typical cross-section is shown shaded in the diagram.



Find the volume of the solid $ABCD$.

(c) A particle of unit mass moves in a straight line and experiences a resistive force of $v + v^3$, where v is its velocity.

Initially, the particle is at the origin and travelling to the right with speed Q metres per second.

i. Show that the displacement of the particle is given by **2**

$$x = \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right)$$

ii. Show that at time t which has elapsed, the velocity-time relationship is given by **3**

$$t = \frac{1}{2}\log_e\left(\frac{Q^2\left(1 + v^2\right)}{v^2\left(1 + Q^2\right)}\right)$$

iii. Find v^2 as a function of time. **2**

iv. Find the limiting values of v and x as $t \to \infty$. **2**

Examination continues overleaf. . .

Question 16 (15 Marks) Commence a NEW booklet. **Marks**

(a) Let $f(x) = \dfrac{1-x}{x}$. On separate one-third page diagrams, sketch the following, clearly showing any asymptotes.

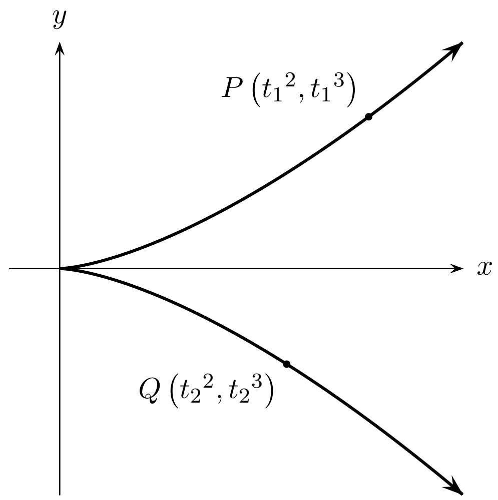
i. $y = f(x)$ **1**

ii. $y = f(|x|)$ **2**

iii. $y = e^{f(x)}$ **2**

iv. $y^2 = f(x)$. In addition, briefly describe the behaviour of the curve at $x = 1$. **3**

(b) Consider the curve defined parametrically by $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$.



Let $P\left(t_1{}^2, t_1{}^3\right)$ and $Q\left(t_2{}^2, t_2{}^3\right)$ be two distinct points on the curve.

i. Express the equation of the curve in terms of x and y only. **1**

ii. Show that the equation of the chord PQ is given by **2**

$$(t_1 + t_2)\, y - \left(t_1{}^2 + t_1 t_2 + t_2{}^2\right) x + t_1{}^2 t_2{}^2 = 0$$

iii. Hence or otherwise, show that the equation of the tangent to the curve at a point corresponding to t, where $t \neq 0$, is given by **1**

$$2y - 3tx + t^3 = 0$$

iv. Let $R(x_0, y_0)$ be a point in the plane such that $x_0{}^3 > y_0{}^2 > 0$. **4**

Prove that there are exactly three tangents from R to the curve.

(*Hint:* differentiate the expression given in part (iii))

End of paper.

2017 Mathematics Extension 2 HSC Course Assessment Task 4 STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

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2. Which topics did I need more help with, and what parts specifically?

- Q5, 8, 10, 11 - Integration

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- Q6, 7, 12 - Complex Numbers

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- Q3, 13, 16(b) - Polynomials

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- Q14 - Conics

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- Q15(a)(b) - Volumes

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- Q15(c) - Mechanics (Resisted Motion)

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- Q1, 2, 9, 16(a) - Graphs

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3. What other parts from the feedback session can I take away to refine my solutions for future reference?

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Sample Band 6/E4 Responses

Section I

1. (C) **2.** (C) **3.** (B) **4.** (A) **5.** (B)
6. (D) **7.** (B) **8.** (C) **9.** (A) **10.** (D)

Section II

Question 11 (Bhamra)

(a) (4 marks)

- ✓ [1] for the correct expression by applying integration by parts the first time.
- ✓ [1] for correctly applying integration by parts, the second time.
- ✓ [1] for observing the original in the second application of integration by parts
- ✓ [1] for final answer.

Let $I = \int e^{-2x} \cos x\, dx$.

$$\begin{array}{|ll} u = e^{-2x} & dv = \cos x \\ du = -2e^{-2x} & v = \sin x \end{array}$$

$$\begin{aligned} I &= uv - \int v\, du \\ &= e^{-2x} \sin x - \int -2e^{-2x} \sin x\, dx \\ &= e^{-2x} \sin x + 2\int e^{-2x} \sin x\, dx \end{aligned}$$

For the rightmost integral,

$$\begin{array}{|ll} u = e^{-2x} & dv = \sin x \\ du = -2e^{-2x} & v = -\cos x \end{array}$$

$$\begin{aligned} \therefore I &= e^{-2x} \sin x + 2\left(-e^{-2x} \cos x - 2\int e^{-2x} \cos x\, dx\right) \\ &= e^{-2x} \sin x - 2e^{-2x} \cos x - 4I + C_1 \\ 5I &= e^{-2x} \sin x - 2e^{-2x} \cos x + C_1 \\ I &= \frac{1}{5}e^{-2x}(\sin x - 2\cos x) + C \end{aligned}$$

Alternatively, let $u = \cos x$ and $dv = e^{-2x}$. Correct application should result in the same primitive.

(b) (3 marks)

- ✓ [1] for correctly manipulation of integrand.
- ✓ [1] for correct resolution into index form.
- ✓ [1] for final answer.

$$\begin{aligned} \int \frac{x}{\sqrt{1-x}}\, dx &= -\int \left(\frac{1-x-1}{\sqrt{1-x}}\right) dx \\ &= -\int \left(\sqrt{1-x} - \frac{1}{\sqrt{1-x}}\right) dx \\ &= -\int \left((1-x)^{\frac{1}{2}} + (1-x)^{-\frac{1}{2}}\right) dx \\ &= \frac{2}{3}(1-x)^{\frac{3}{2}} + 2(1-x)^{\frac{1}{2}} + C \end{aligned}$$

Alternatively, make the substitution $u^2 = 1-x$, or $u = 1-x$.

(c) i. (2 marks)

- ✓ [1] for substitution transformation.
- ✓ [1] for correct proof.

Letting $x = a - u$,

$$\begin{array}{|ll} x = a-u & x = 0, u = a \\ dx = -du & x = a, u = 0 \end{array}$$

$$\begin{aligned} \therefore \int_0^a f(x)\, dx &= \int_{u=a}^{u=0} f(a-u)\,(-du) \\ &= \int_0^a f(a-u)\, du \\ &= \int_0^a f(a-x)\, dx \end{aligned}$$

by interchanging u for x.

ii. (2 marks)

- ✓ [1] for integrand transformation.
- ✓ [1] for final answer.

$$\begin{aligned} \int_0^1 x(1-x)^{2017}\, dx &= \int_0^1 x^{2017}(1-x)\, dx \\ &= \int_0^1 \left(x^{2017} - x^{2018}\right) dx \\ &= \left[\frac{x^{2018}}{2018} - \frac{x^{2019}}{2019}\right]_0^1 \\ &= \frac{1}{2018} - \frac{1}{2019} \\ &= \frac{1}{(2018)(2019)} \end{aligned}$$

(d) (4 marks)

- ✓ [1] for correct t transformation, including corresponding differential.
- ✓ [1] for simplification of the new denominator in terms of t.
- ✓ [1] for transformation into an integrand leading to the inverse trig integral.
- ✓ [1] for final answer.

Letting $t = \tan\frac{x}{2}$, then $x = 2\tan^{-1} t$:

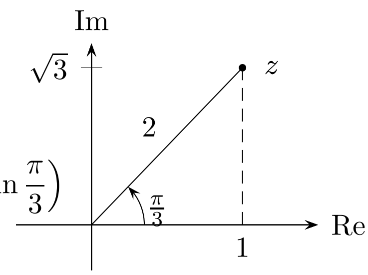
$$\left| \begin{array}{ll} x = 2\tan^{-1} t & x = 0, t = 0 \\ dx = \dfrac{2}{1+t^2}\,dt & x = \dfrac{\pi}{2}, t = 1 \end{array} \right.$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{3 - \cos x - 2\sin x}\,dx \\
&= \int_{t=0}^{t=1} \frac{1}{3 - \frac{1-t^2}{1+t^2} - 2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,dt \\
&= \int_0^1 \frac{2\,dt}{3 + 3t^2 - 1 + t^2 - 4t} \\
&= \int_0^1 \frac{2\,dt}{4t^2 - 4t + 2} \\
&= \int_0^1 \frac{2\,dt}{4t^2 - 4t + 1 + 1} \\
&= \int_0^1 \frac{2\,dt}{(2t-1)^2 + 1} \\
&= \left[\tan^{-1}(2t-1)\right]_0^1 \\
&= \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{2}
\end{aligned}$$

Question 12 (Bhamra)

(a) i. (1 mark)

$$\begin{aligned}
z &= 1 + \sqrt{3}i \\
&= 2e^{i\frac{\pi}{3}} \\
&= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)
\end{aligned}$$



ii. (2 marks)

- z^7 by De Moivre's Theorem:

$$\begin{aligned}
z^7 &= 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right) \\
&= 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)
\end{aligned}$$

- $64z$:

$$\begin{aligned}
64z &= 64 \times 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\
&= 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)
\end{aligned}$$

Hence $z^7 - 64z = 0$.

(b) (4 marks)

- ✓ [1] for formation of simultaneous equations in terms of a and b.
- ✓ [1] for solutions to $\sqrt{-8+6i}$
- ✓ [1] for applying quadratic formula to complex quadratic.
- ✓ [1] for both final solutions.

- Evaluate $\sqrt{6i - 8} = \sqrt{-8+6i}$. Letting $z^2 = -8 + 6i$ where $z = a + ib$ and then comparing real and imaginary parts:

$$\begin{aligned}
(a+ib)^2 &\equiv -8 + 6i \\
a^2 - b^2 + i(2ab) &\equiv -8 + 6i
\end{aligned}$$

$$\begin{cases} a^2 - b^2 = -8 & (1) \\ ab = 3 & (2) \end{cases}$$

Substituting (2) into (1):

$$\begin{aligned}
a^2 - \left(\frac{3}{a}\right)^2 &= -8 \\
a^4 - 9 &= -8a^2 \\
a^4 + 8a^2 - 9 &= 0 \\
\left(a^2 + 9\right)\left(a^2 - 1\right) &= 0 \\
\therefore a = \pm 1 \qquad b &= \pm 3 \\
\therefore \sqrt{-8+6i} &= \pm(1+3i)
\end{aligned}$$

- Solving $2z^2 - (3+i)z + 2 = 0$ via the quadratic formula:

$$\begin{aligned} z &= \frac{(3+i) \pm \sqrt{(3+i)^2 - 16}}{4} \\ &= \frac{(3+i) \pm \sqrt{9+6i-1-16}}{4} \\ &= \frac{(3+i) \pm \sqrt{-8+6i}}{4} \\ &= \frac{(3+i) \pm (1+3i)}{4} \\ z_1 &= \frac{3+i+1+3i}{4} = 1+i \\ z_2 &= \frac{3+i-1-3i}{4} = \frac{1}{2} - \frac{1}{2}i \end{aligned}$$

(c) (2 marks) Let $z = a + ib$,

$$\begin{aligned} 2\overline{z} - iz &= 1+4i \\ 2(a-ib) - i(a+ib) &= 1+4i \\ 2a - 2ib - ai + b &= 1+41 \\ 2a + b - i(a+2b) &= 1+4i \end{aligned}$$

Equating real and imaginary parts,

$$\begin{cases} 2a + b = 1 & (1) \\ -a - 2b = 4 & (2) \end{cases}$$

Multiplying (1) by 2, and adding to (2):

$$\begin{cases} 4a + 2b = 2 & (1) \times 2 \\ -a - 2b = 4 & (2) \end{cases}$$

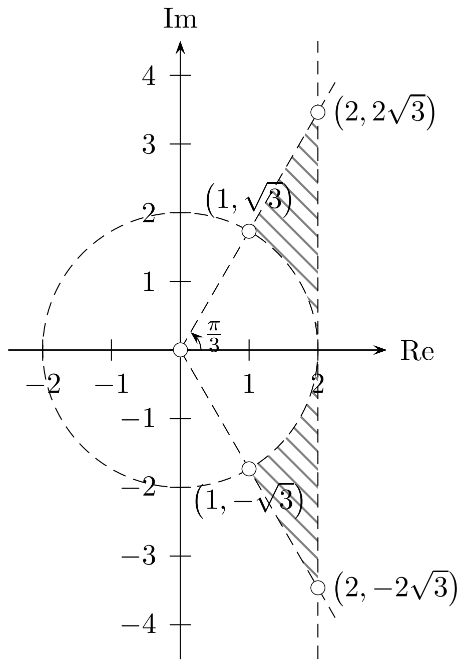
$$\begin{aligned} 3a &= 6 \\ a &= 2 \end{aligned}$$

Substitute into (2):

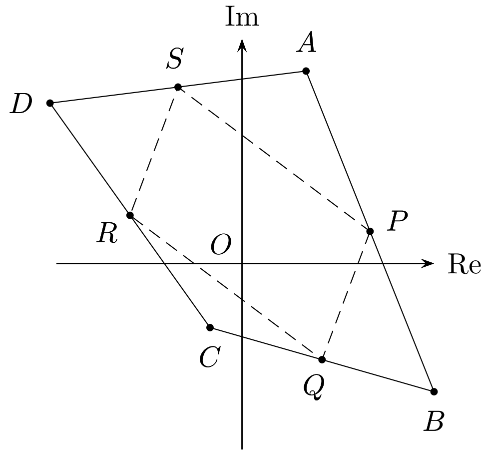
$$\begin{aligned} -2 - 2b &= 4 \\ -2b &= 6 \\ b &= -3 \\ \therefore z &= 2 - 3i \end{aligned}$$

(d) (3 marks)

- $|\text{Arg}(z)| < \frac{\pi}{3}$ draw straight lines angled at $\pm\frac{\pi}{3}$ to positive x axis
- $z + \overline{z} < 4$: results in region $2x < 4$
- $|z| > 2$: region outside the circle of radius 2, centre origin



(e) i. (2 marks)



- $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$.

  Hence $\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$:

$$\begin{aligned} \overrightarrow{OP} &\Rightarrow a + \frac{1}{2}(b-a) \\ &= \frac{1}{2}(a+b) \end{aligned}$$

- Similarly, $\overrightarrow{OR}$ is represented by

$$\frac{1}{2}(c+d)$$

- As M is the midpoint of PR,

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OR}\right)\\ &\Rightarrow \frac{1}{2}\left(\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right)\\ &= \frac{1}{4}(a+b+c+d)\end{aligned}$$

- N is the midpoint of $\overrightarrow{QS}$, and $\overrightarrow{ON}$ represents

$$\begin{aligned}\overrightarrow{ON} &= \frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OS}\right)\\ &\Rightarrow \frac{1}{2}\left(\frac{1}{2}(b+c)+\frac{1}{2}(a+d)\right)\\ &= \frac{1}{4}(a+b+c+d)\end{aligned}$$

ii. (1 mark) As M and N are coincident, the diagonals of $PQRS$ bisect each other. $PQRS$ is a parallelogram.

Question 13 (Sharma)

(a) i. (1 mark)

$$x^3+0x^2+px+q=0$$

With roots α, β and γ:

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

Finding the elementary symmetric functions,

$$\begin{aligned}\alpha+\beta+\gamma &= 0\\ \alpha\beta+\beta\gamma+\alpha\gamma &= \frac{p}{1}=p\\ \alpha\beta\gamma &= -\frac{q}{1}=-q\end{aligned}$$
$$\therefore \alpha^2+\beta^2+\gamma^2 = (0)^2-2(p)=-2p$$

ii. (3 marks)
✓ [1] for identifying new roots of the new equation.
✓ [1] for correct polynomial with roots α^2, β^2 and γ^2.
✓ [1] for correct polynomial with roots $\frac{\alpha^2}{-q}$ etc.

$$\frac{\alpha}{\beta\gamma}=\frac{\alpha^2}{\alpha\beta\gamma}\qquad \frac{\beta}{\alpha\gamma}=\frac{\beta^2}{\alpha\beta\gamma}\qquad \frac{\gamma}{\alpha\beta}=\frac{\gamma^2}{\alpha\beta\gamma}$$

Hence, new equation will have roots

$$\frac{\alpha^2}{-q},\frac{\beta^2}{-q},\frac{\gamma^2}{-q}$$

In $x^3+px+q=0$, let $x\mapsto\sqrt{x}$, which produces roots α^2, β^2 and γ^2:

$$\begin{aligned}\left(\sqrt{x}\right)^3+p\left(\sqrt{x}\right)+q&=0\\ x\sqrt{x}+p\sqrt{x}+q&=0\\ \sqrt{x}(x+p)&=-q\\ x(x+p)^2&=q^2\\ x\left(x^2+2px+p^2\right)&=q^2\\ x^3+2px^2+p^2x-q^2&=0\end{aligned}$$

Now let $x\mapsto -qx$ to produce the required roots:

$$\begin{aligned}(-qx)^3+2p\left(-qx\right)^2+p^2(-qx)-q^2&=0\\ -q^3x^3+2pq^2x^2-p^2qx-q^2&=0\\ \therefore q^2x^3-2pqx^2+p^2x+q&=0\end{aligned}$$

(b) i. (3 marks)
✓ [1] for each correct constant.

$$\frac{4x^2-5x-7}{(x-1)\left(x^2+x+2\right)}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+2}$$
$$4x^2-5x-7\equiv A(x^2+x+2)+(Bx+C)(x-1)$$

- Let $x=1$,

$$\begin{aligned}4-5-7&\equiv A(1+1+2)+0\\ 4A&=-8\\ A&=-2\end{aligned}$$

- Let $x=0$:

$$\begin{aligned}-7&\equiv -2(0+0+2)+(B(0)+C)(0-1)\\ -7&\equiv -4-C\\ C&=3\end{aligned}$$

- Let $x=2$,

$$\begin{aligned}&4(2^2)-5(2)-7\\ &\equiv -2(2^2+2+2)+(2B+3)(2-1)\\ 16-10-7&\equiv -16+2B+3\\ 2B&=12\\ B&=6\end{aligned}$$

$$\therefore \frac{4x^2-5x-7}{(x-1)(x^2+x+2)} \equiv \frac{-2}{x-1}+\frac{6x+3}{x^2+x+2}$$

ii. (2 marks)

✓ [1] for correct integral.

✓ [1] for correct final answer.

$$\begin{aligned}&\int_{-1}^{0} \frac{4x^2-5x-7}{(x-1)(x^2+x+2)}\,dx\\ &= \int_{-1}^{0}\left(-\frac{2}{x-1}+\frac{3(2x+1)}{x^2+x+2}\right)dx\\ &= -2\left[\ln(x-1)\right]_{-1}^{0}+3\left[\ln\left(x^2+x+2\right)\right]_{-1}^{0}\end{aligned}$$

By symmetry,

$$\int_{-1}^{0} -\frac{2}{x-1}\,dx = \int_{2}^{3}\frac{2}{x-1}\,dx$$

(Alternatively, use absolute values inside the logarithm)

$$\begin{aligned}&\int_{-1}^{0}\left(-\frac{2}{x-1}+\frac{3(2x+1)}{x^2+x+2}\right)dx\\ &= 2\left[\ln(x-1)\right]_{2}^{3}+3\left[\ln\left(x^2+x+2\right)\right]_{-1}^{0}\\ &= 2\ln 2+3\left(\ln 2-\ln(1-1+2)\right)\\ &= 2\ln 2\end{aligned}$$

(c) i. (2 marks)

$$\begin{aligned}&\cos 3A\\ &= \cos(A+2A)\\ &= \cos A\cos 2A-\sin A\sin 2A\\ &= \cos A\left(2\cos^2 A-1\right)-\sin A\left(2\sin A\cos A\right)\\ &= 2\cos^3 A-\cos A-2\left(1-\cos^2 A\right)\cos A\\ &= 2\cos^3 A-\cos A-2\cos A+2\cos^3 A\\ &= 4\cos^3 A-3\cos A\\ &\therefore \frac{1}{4}\cos 3A = \cos^3 A-\frac{3}{4}\cos A\end{aligned}$$

Alternatively, use De Moivre's theorem to expand and equate real parts to obtain required result.

ii. (2 marks)

✓ [1] for x^3-6x in terms of $\cos A$.

✓ [1] for correctly showing required result.

$$\begin{aligned}x &= 2\sqrt{2}\cos A\\ x^3 &= 16\sqrt{2}\cos^3 A\\ \therefore x^3-6x &= 16\sqrt{2}\cos^3 A-6\left(2\sqrt{2}\cos A\right)\\ &= 16\sqrt{2}\cos^3 A-12\sqrt{2}\cos A\\ &= 4\sqrt{2}\left(4\cos^3 A-3\cos A\right)\\ &= 4\sqrt{2}\cos 3A\end{aligned}$$

If $\cos 3A = -\frac{1}{2\sqrt{2}}$,

$$\begin{aligned}x^3-6x &= \overset{2}{\cancel{4}}\cancel{\sqrt{2}}\left(-\frac{1}{\cancel{2}\cancel{\sqrt{2}}}\right)\\ &= -2\end{aligned}$$

Hence $x^3-6x=-2$ has a solution of $x=2\sqrt{2}\cos A$ provided $\cos 3A = -\frac{1}{2\sqrt{2}}$.

iii. (2 marks)

✓ [1] for first quadrant related angle.

✓ [1] for all three roots.

$$\begin{aligned}\cos 3A &= -\frac{1}{2\sqrt{2}}\\ 3A &= 1.93\cdots\end{aligned}$$

$3A = 1.93\cdots$ lies in the 2nd quadrant. The first quadrant (related) angle is

$$\pi-1.93\cdots = 1.209\cdots$$

Hence,

$$\begin{aligned}3A &= \pi-1.209\cdots \quad \text{(2nd quadrant)}\\ &\text{or } \pi+1.209\cdots \quad \text{(3rd quadrant)}\\ &\text{or } 3\pi-1.209\cdots \quad \text{(6th quadrant)}\\ A &= \frac{\pi-1.209\cdots}{3}, \frac{\pi+1.209}{3}, \frac{3\pi-1.209\cdots}{3}\\ \therefore x &= 2\sqrt{2}\cos\left(\frac{\pi-1.209\cdots}{3}\right)\approx 2.2618\cdots\\ &\text{or } 2\sqrt{2}\cos\left(\frac{\pi+1.209\cdots}{3}\right)\approx 0.3399\cdots\\ &\text{or } 2\sqrt{2}\cos\left(\frac{3\pi-1.209\cdots}{3}\right)\approx -2.6017\cdots\end{aligned}$$

Question 14 (Sharma)

(a) i.(α) (1 mark)

$$\frac{x^2}{3+p}+\frac{y^2}{8+p}=1$$

If the conic is an ellipse, then both of $3+p>0$ and $8+p>0$:

$$3+p>0 \qquad 8+p>0$$
$$p>-3 \qquad p>-8$$

Hence $p>-3$ for both inequalities to be satisfied.

(α) (2 marks)

✓ [1] for the first condition $p>-3$ or $p<-8$, and explanation of which represents the most restrictive condition.

✓ [1] for rejecting the invalid solution.

If the conic is a hyperbola, then either of the following pairs of inequalities must hold true simultaneously:

– $3+p>0$ and $8+p<0$

$$p>-3 \text{ and } p<-8$$

which is not possible to hold simultaneously.

– $3+p<0$ and $8+p>0$

$$p<-3 \text{ and } p>-8$$

Hence $-8<p<-3$.

ii. (2 marks)

✓ [−1] for each new error.

$$p=-4: \qquad \frac{x^2}{3-4}+\frac{y^2}{8-4}=1$$
$$\frac{x^2}{-1}+\frac{y^2}{4}=1$$
$$\therefore \frac{y^2}{4}-x^2=1$$

Applying relationship for eccentricity:

$$b^2=a^2\left(e^2-1\right)$$
$$1=4\left(e^2-1\right)$$
$$e^2=\frac{5}{4}$$
$$e=\frac{\sqrt{5}}{2}$$

Finding the foci with $a=2$ and $e=\frac{\sqrt{5}}{2}$:

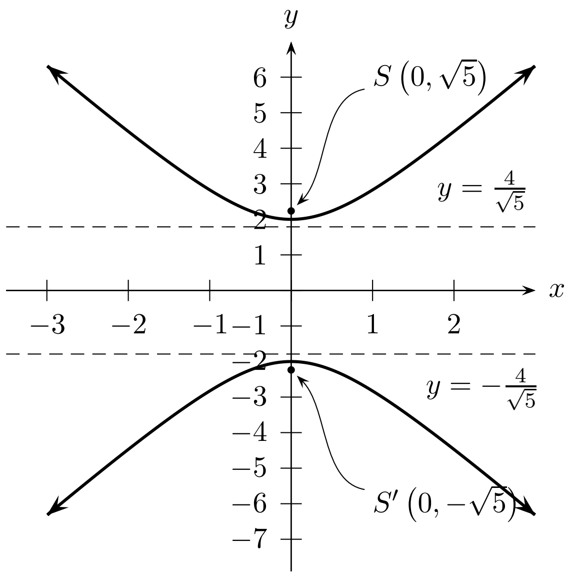
$$S\,(0,\pm ae)=\left(0,\pm\sqrt{5}\right)$$

Finding the directricies:

$$y=\pm\frac{a}{e}=\pm\frac{2}{\frac{\sqrt{5}}{2}}=\pm\frac{4}{\sqrt{5}}$$

iii. (2 marks)

✓ [−1] for each new error.



(b) i. (2 marks)

$$y=\frac{1}{x}$$
$$\frac{dy}{dx}=-x^{-2}\Big|_{x=t}=-\frac{1}{t^2}$$

Equation of the tangent at $\left(t,\frac{1}{t}\right)$ via the point-gradient formula:

$$\frac{y-\frac{1}{t}}{x-t}=-\frac{1}{t^2}$$
$$t^2y-t=-x+t$$
$$x+t^2y=2t$$

ii. (3 marks)

✓ [1] for correct equation of the normal at $\left(t,\frac{1}{t}\right)$.

✓ [1] for correct coordinates of A and B.

✓ [1] for required proof.

- The equation of the normal at $\left(t, \frac{1}{t}\right)$ via the point-gradient formula:

$$\begin{aligned} \frac{y - \frac{1}{t}}{x - t} &= t^2 \\ y - \frac{1}{t} &= t^2 x - t^3 \\ ty - 1 &= t^3 x - t^4 \\ t^3 x - ty &= t^4 - 1 \end{aligned}$$

- At $x = 0$, the tangent is at

$$\begin{aligned} 0 + t^2 y &= 2t \\ y &= \frac{2}{t} \\ \therefore A &\left(0, \frac{2}{t}\right) \end{aligned}$$

- At $y = 0$, the normal is at

$$\begin{aligned} t^3 x - 0 &= t^4 - 1 \\ x &= \frac{t^4 - 1}{t^3} \\ \therefore B &\left(\frac{t^4 - 1}{t^3}, 0\right) \end{aligned}$$

- The midpoint of AB:

$$\left(\frac{\frac{t^4-1}{t^3} + 0}{2}, \frac{\frac{2}{t} + 0}{2}\right) = \left(\frac{t^4 - 1}{2t^3}, \frac{1}{t}\right)$$

$$\therefore x_M = \frac{t^4 - 1}{2t^3} \qquad y_M = \frac{1}{t}$$

Substituting $t = \frac{1}{y}$ into the x equation:

$$\begin{aligned} x &= \frac{\left(\frac{1}{y}\right)^4 - 1}{2\left(\frac{1}{y}\right)^3} = \frac{\frac{1}{y^4} - 1}{\frac{2}{y^3}} \frac{\times y^4}{\times y^4} \\ &= \frac{1 - y^4}{2y} \end{aligned}$$

(c) (3 marks)

✓ [1] for obtaining m_{PQ}.

✓ [1] for using the point gradient formula for the equation of PQ.

✓ [1] for final solution.

$$\begin{cases} x_P = a\cos\alpha & x_Q = a\cos\beta \\ y_P = b\sin\alpha & y_Q = b\sin\beta \end{cases}$$

The gradient of the chord PQ:

$$m_{PQ} = \frac{b\sin\beta - b\sin\alpha}{a\cos\beta - a\cos\alpha} = \frac{b(\sin\beta - \sin\alpha)}{a(\cos\beta - \cos\alpha)}$$

Using the point-gradient formula, the equation of PQ is:

$$\frac{y - b\sin\alpha}{x - a\cos\alpha} = \frac{b(\sin\beta - \sin\alpha)}{a(\cos\beta - \cos\alpha)}$$

As the focal chord passes through $(ae, 0)$:

$$\begin{aligned} \frac{-b\sin\alpha}{ae - a\cos\alpha} &= \frac{b(\sin\beta - \sin\alpha)}{a(\cos\beta - \cos\alpha)} \\ -\sin\alpha(\cos\beta - \cos\alpha) &= (e - \cos\alpha)(\sin\beta - \sin\alpha) \\ e &= \frac{-\sin\alpha\cos\beta + \sin\alpha\cos\alpha}{\sin\beta - \sin\alpha} + \cos\alpha \\ &= \frac{-\sin\alpha\cos\beta + \cancel{\sin\alpha\cos\alpha} + \sin\beta\cos\alpha - \cancel{\sin\alpha\cos}}{\sin\beta - \sin\alpha} \\ &= \frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\sin\beta - \sin\alpha} \\ &= \frac{\sin(\beta - \alpha)}{\sin\beta - \sin\alpha} \end{aligned}$$

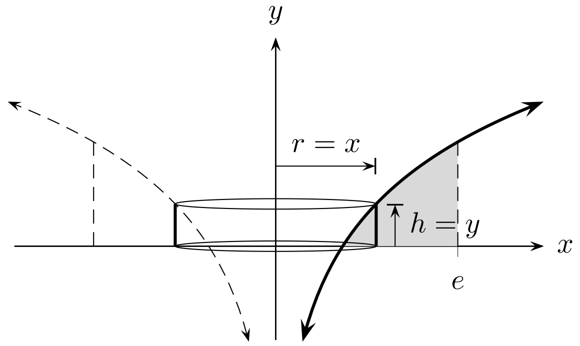
Question 15 (Lam)

(a) (3 marks)

✓ [1] for the correct working, leading to the curved surface area of the cylinder.

✓ [1] for the applying integration by parts.

✓ [1] for the final answer.



- Surface area of cylinder: $SA = 2\pi rh$
- Radius of each individual cylinder:

$$r = x$$

- Height of each individual cylinder:

$$h = y$$

- Hence,

$$\begin{aligned}\delta V &= \sum A\,\delta x\\ &= \sum 2\pi xy\,\delta x\\ &= 2\pi\sum x\ln x\,\delta x\end{aligned}$$

Taking $\delta x \to 0$:

$$\begin{aligned}V &= 2\pi\lim_{\delta x\to 0}\sum_{x=1}^{x=e} x\ln x\,\delta x\\ &= 2\pi\int_1^e x\ln x\,dx\end{aligned}$$

$$\left|\begin{array}{ll} u = \ln x & dv = x\\ du = \dfrac{1}{x} & v = \dfrac{1}{2}x^2\end{array}\right.$$

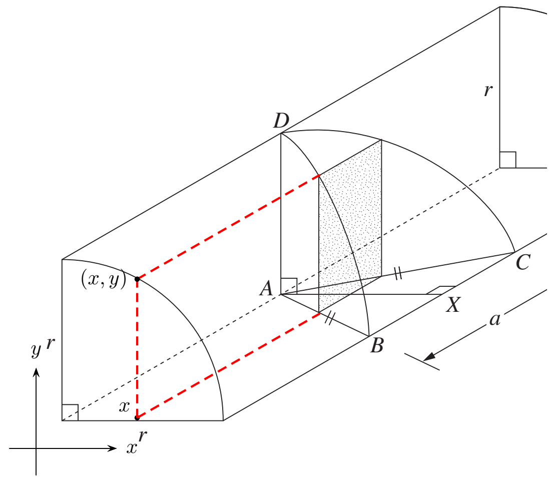
$$\begin{aligned}\therefore V &= 2\pi\left(\left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x^{\not 2}\cdot\frac{1}{\not x}\,dx\right)\\ &= 2\pi\left(\frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^e\right)\\ &= 2\pi\left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}\right)\\ &= \frac{\pi}{2}\left(e^2+1\right)\end{aligned}$$

(b) (3 marks)

✓ [1] for similarity ratio relationship in $\triangle ABC$.

✓ [1] for correct area expression.

✓ [1] for final answer.

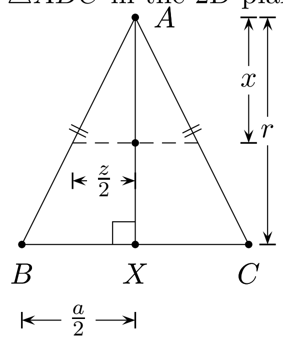


- From the diagram: at a point x along the horizontal, the corresponding vertical value is y. As it is a quarter circle,

$$y = \sqrt{r^2 - x^2}$$

which becomes the height of the rectangle.

- Drawing $\triangle ABC$ in the 2D plane:



From similar triangles,

$$\begin{aligned}\frac{r}{\frac{a}{2}} &= \frac{x}{\frac{z}{2}}\\ \therefore \frac{2r}{a} &= \frac{2x}{z}\\ z &= \frac{ax}{r}\end{aligned}$$

- From the area of the rectangles,

$$A = bh = zy = \frac{ax}{r}\sqrt{r^2 - x^2}$$

the volume is the sum of these areas:

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=r} A\,\delta x = \int_0^r \frac{ax}{r}\sqrt{r^2 - x^2}\,dx$$
$$= \frac{a}{r} \times \left(-\frac{1}{2}\right) \int_0^r -2x\left(r^2 - x^2\right)^{\frac{1}{2}}\,dx$$
$$= -\frac{a}{\cancel{2}r}\left[\frac{\cancel{2}}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_0^r$$
$$= -\frac{a}{2r}\left(0 - \left(r^2\right)^{\frac{3}{2}}\right) = \frac{ar^2}{3}$$

(c) i. (2 marks)

- ✓ [1] for separation of variables to obtain $\tan^{-1} x = -t + C_1$.
- ✓ [1] for final required result.

$$F = m\ddot{x} = -\left(v + v^3\right)$$

As $m = 1$,

$$\ddot{x} = v\frac{dv}{dx} = -\left(v + v^3\right)$$
$$\frac{1}{1+v^2}\,dv = -dx$$

Integrating both sides,

$$\tan^{-1} v = -x + C_1$$

When $t = 0$, $v = Q$, $x = 0$.

$$\therefore C_1 = \tan^{-1} Q$$
$$\therefore x = \tan^{-1} Q - \tan^{-1} v$$

Letting

$$\alpha = \tan^{-1} Q \qquad \beta = \tan^{-1} v$$
$$\tan\alpha = Q \qquad \tan\beta = v$$
$$\therefore \tan x = \tan(\alpha - \beta)$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$= \frac{Q - v}{1 + Qv}$$
$$\therefore x = \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right)$$

ii. (3 marks)

- ✓ [1] for correct constants in the partial fraction decomposition.
- ✓ [1] for obtaining the correct primitive.
- ✓ [1] for final required result.

$$\frac{dv}{dt} = -v - v^3$$
$$\frac{dv}{v\left(1+v^2\right)} = -dt$$

Applying the partial fraction decomposition,

$$\underbrace{\frac{1}{v\left(1+v^2\right)} \equiv \frac{A}{v} + \frac{Bv + C}{1+v^2}}_{\times v(1+v^2)}$$
$$1 \equiv A\left(1 + v^2\right) + v(Bv + C)$$

- When $v = 0$, $A = 1$ by inspection.
- When $v = 1$,

$$1 \equiv 1(1+1) + 1(B + C)$$
$$1 \equiv 2 + B + C$$
$$\therefore B + C = -1$$

- When $v = 2$,

$$1 \equiv 1(1+4) + 2(2B + C)$$
$$1 \equiv 5 + 4B + 2C$$
$$4B + 2C = -4$$
$$2B + C = -2$$

- Solving simultaneously,

$$\begin{cases} B + C = -1 & (1) \\ 2B + C = -2 & (2) \end{cases}$$

(2) − (1):

$$B = -2 - (-1) = -1$$
$$\therefore C = 0$$

$$\int \frac{dv}{v\left(1+v^2\right)} = \int \left(\frac{1}{v} - \frac{v}{1+v^2}\right)\,dv$$
$$= \int -dt$$
$$\ln v - \frac{1}{2}\ln\left(1 + v^2\right) = -t + C_2$$

When $t=0$, $v=Q$:

$$C_2 = \ln Q - \frac{1}{2}\ln\left(1+Q^2\right)$$

$$\ln v - \frac{1}{2}\ln\left(1+v^2\right) = -t + \ln Q - \frac{1}{2}\ln\left(1+Q^2\right)$$

$$\begin{aligned}t &= \ln Q - \ln v + \frac{1}{2}\left(\ln\left(1+v^2\right) - \ln\left(1+Q^2\right)\right)\\ &= \ln\frac{Q}{v} + \frac{1}{2}\ln\left(\frac{1+v^2}{1+Q^2}\right)\\ &= \frac{1}{2}\ln\frac{Q^2}{v^2} + \frac{1}{2}\ln\left(\frac{1+v^2}{1+Q^2}\right)\\ &= \frac{1}{2}\ln\left(\frac{Q^2\left(1+v^2\right)}{v^2\left(1+Q^2\right)}\right)\end{aligned}$$

iii. (2 marks)

✓ [1] for e^{2t} as the subject.

✓ [1] for final result required.

$$t = \frac{1}{2}\ln\left(\frac{Q^2\left(1+v^2\right)}{v^2\left(1+Q^2\right)}\right)$$

$$2t = \ln\left(\frac{Q^2\left(1+v^2\right)}{v^2\left(1+Q^2\right)}\right)$$

$$\underbrace{e^{2t} = \frac{Q^2\left(1+v^2\right)}{v^2\left(1+Q^2\right)}}_{\times v^2(1+Q^2)}$$

$$e^{2t}v^2 + e^{2t}Q^2v^2 = Q^2 + Q^2v^2$$

$$v^2\left(e^{2t} + e^{2t}Q^2 - Q^2\right) = Q^2$$

$$\therefore v^2 = \frac{Q^2}{e^{2t}\left(1+Q^2\right) - Q^2}\,\frac{\times e^{-2t}}{\times e^{-2t}} = \frac{Q^2e^{-2t}}{\left(1+Q^2\right) - Q^2e^{-2t}}$$

iv. (2 marks)

$$t\to\infty, e^{-2t}\to 0$$

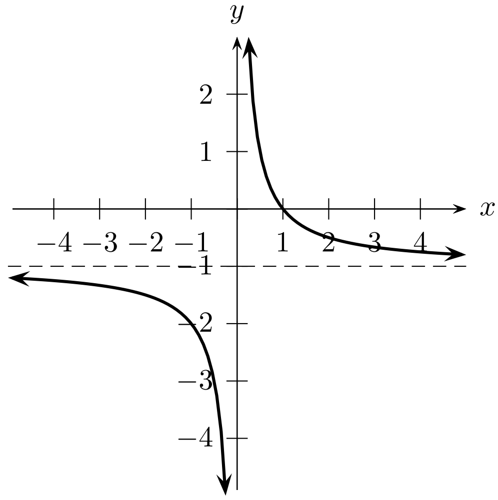
$$\therefore v^2 \to \frac{0}{0-Q^2} = 0$$

$$x \to \tan^{-1}\left(\frac{Q-0}{1+Q(0)}\right) = \tan^{-1}Q$$

Question 16 (Lam)

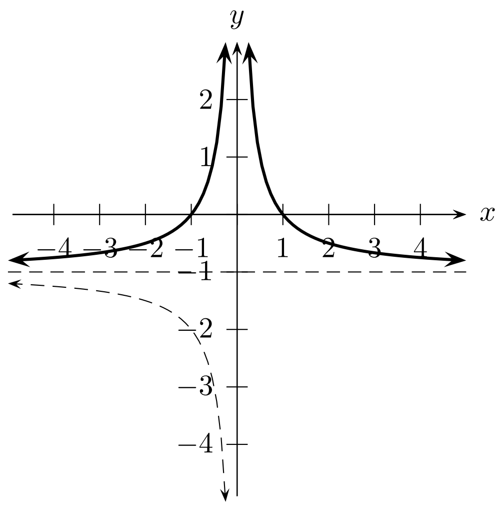
(a) i. (1 mark)

$$y = \frac{1-x}{x} = \frac{1}{x} - 1 = -1 + \frac{1}{x}$$



ii. (2 marks)

$$y = f\left(|x|\right)$$



iii. (2 marks)

✓ [−1] for each new error/feature not shown. Features are:

- Shape (both branches)
- Open circle at $x = 0$ for negative branch
- y asymptote of $y = \frac{1}{e}$

$$y = e^{f(x)}$$

Asymptotes:

- $x \to \infty$:

$$y = f(x) = -1 + \frac{1}{x} \to -1^+$$
$$\therefore y = e^{f(x)} \to e^{-1} = \left(\frac{1}{e}\right)^+$$

- $x \to -\infty$:

$$y = f(x) = -1 + \frac{1}{x} \to -1^-$$
$$\therefore y = e^{f(x)} \to e^{-1} = \left(\frac{1}{e}\right)^-$$

- $x \to 0^-$:

$$y = f(x) \to -\infty$$
$$\therefore y = e^{f(x)} \to 0^+$$

iv. (3 marks)

✓ [−1] for each new error/feature not shown. Features are:

- Shape (both branches)
- Smooth curve at $(1, 0)$
- x asymptote of $x = 0$

✓ [1] for description of the behaviour at $x = 1$, based on graph drawn.

$$y^2 = f(x)$$

- $y = \sqrt{f(x)}$ requires $f(x) \geq 0$:

$$\frac{1-x}{x} \geq 0$$
$$\frac{1}{x} - 1 \geq 0$$
$$\frac{1}{x} \geq 1$$
$$\therefore 0 < x \leq 1$$

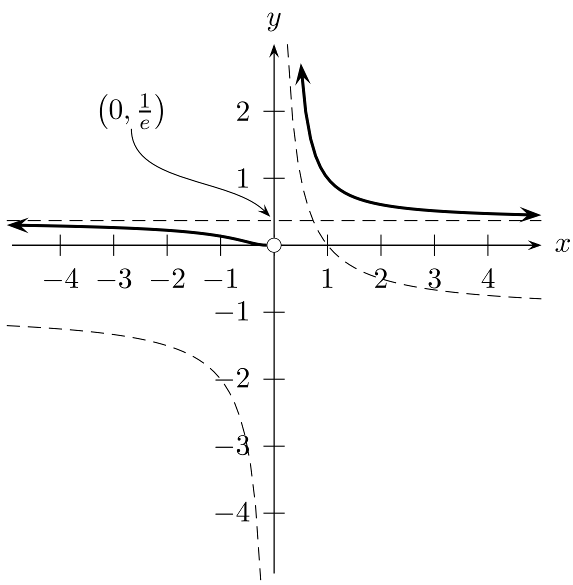
Hence the domain is $0 < x < 1$.

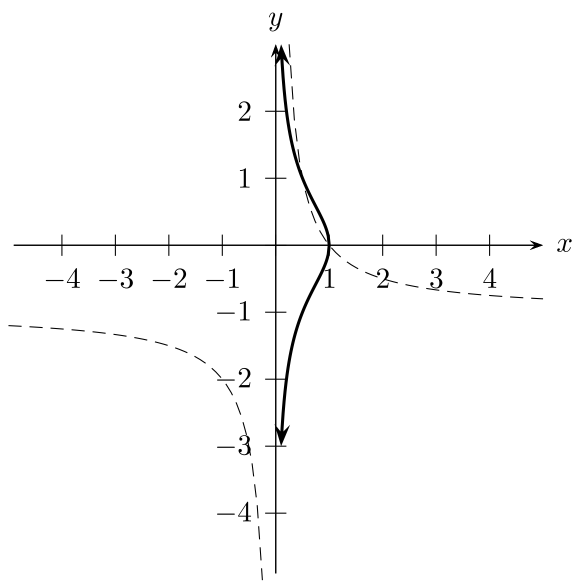
- Also, the other branch is

$$y = -\sqrt{f(x)}$$

Same domain, but below x axis.

- At $x = 1$, the curve reaches its maximum x value/at the upper limit of its domain and has a **vertical tangent**.





(b) i. (1 mark)

✓ [0] if $y = x\sqrt{x}$ (without $\pm$ symbol)

$$\begin{cases} x = t^2 & \Rightarrow x^3 = t^6 \\ y = t^3 & \Rightarrow y^2 = t^6 \end{cases}$$

Equating and removing the parameter,

$$y^2 = x^3$$

ii. (2 marks)

Finding the gradient of PQ:

$$\begin{aligned} m_{PQ} &= \frac{{t_1}^3 - {t_2}^3}{{t_1}^2 - {t_2}^2} \\ &= \frac{\cancel{(t_1 - t_2)}\left({t_1}^2 + t_1t_2 + {t_2}^2\right)}{\cancel{(t_1 - t_2)}(t_1 + t_2)} \\ &= \frac{{t_1}^2 + t_1t_2 + {t_2}^2}{t_1 + t_2} \end{aligned}$$

Using the point-gradient formula,

$$\begin{aligned} \frac{y - {t_1}^3}{x - {t_1}^2} &= \frac{{t_1}^2 + t_1t_2 + {t_2}^2}{t_1 + t_2} \\ \left(y - {t_1}^3\right)(t_1 + t_2) &= \left(x - {t_1}^2\right)\left({t_1}^2 + t_1t_2 + {t_2}^2\right) \\ y(t_1 + t_2) - \cancel{{t_1}^4} - \cancel{{t_1}^3t_2} &= x\left({t_1}^2 + t_1t_2 + {t_2}^2\right) \\ &\quad - \cancel{{t_1}^4} - \cancel{{t_1}^3t_2} - {t_1}^2{t_2}^2 \\ \therefore (t_1 + t_2)\,y - \left({t_1}^2 + t_1t_2 + {t_2}^2\right)x + {t_1}^2{t_2}^2 &= 0 \end{aligned}$$

iii. (1 mark) When $t_1 = t_2$, the chord becomes a tangent.

$$\begin{aligned} 2t_1y - \left({t_1}^2 + {t_1}^2 + {t_1}^2\right)x + {t_1}^4 &= 0 \\ 2t_1y - 3{t_1}^2x + {t_1}^4 &= 0 \\ \therefore 2y - 3tx + t^3 &= 0 \end{aligned}$$

iv. (4 marks)

✓ [1] for differentiating w.r.t. t and checking for stationary points.

✓ [1] for identifying cubic in terms of t will have three roots via fundamental theorem of algebra.

✓ [1] each for testing values of $f\left(\pm\sqrt{x_0}\right)$.

✓ [1] for final reasoning why there are three real roots and therefore three unique tangents.

Tangent to any point $R(x_0, y_0)$, where ${x_0}^3 > {y_0}^2 > 0$ is

$$t^3 - 3tx_0 + 2y_0 = 0$$

Letting $f(t) = t^3 - 3tx_0 + 2y_0$. There are up to three unique solutions to this equation in terms of t. Need all three to be real and no complex conjugate pairs. Differentiating,

$$f'(t) = 3t^2 - 3x_0 = 3\left(t^2 - x_0\right)$$

Hence stationary points are located at $t = \pm\sqrt{x_0}$. Check actual $f\left(\sqrt{x_0}\right)$ and $f\left(-\sqrt{x_0}\right)$:

$$\begin{aligned} f\left(\sqrt{x_0}\right) &= \left(\sqrt{x_0}\right)^3 - 3\left(\sqrt{x_0}\right)x_0 + 2y_0 \\ &= -2x_0\sqrt{x_0} + 2y_0 \\ &= -2\sqrt{{x_0}^3} + 2y_0 \\ &= 2\left(y_0 - \sqrt{{x_0}^3}\right) \end{aligned}$$

As ${x_0}^3 > {y_0}^2 > 0$, then $\sqrt{{x_0}^3} > y_0$

$$\therefore f\left(\sqrt{x_0}\right) < 0$$

Testing $f\left(-\sqrt{x_0}\right)$,

$$\begin{aligned} f\left(-\sqrt{x_0}\right) &= \left(-\sqrt{x_0}\right)^3 - 3\left(-\sqrt{x_0}\right)x_0 + 2y_0 \\ &= 2x_0\sqrt{x_0} + 2y_0 \\ &= 2\sqrt{{x_0}^3} + 2y_0 \\ &= 2\left(\sqrt{{x_0}^3} - y_0\right) \end{aligned}$$

As ${x_0}^3 > {y_0}^2 > 0$, then $\sqrt{{x_0}^3} > y_0$

$$\therefore f\left(-\sqrt{x_0}\right) > 0$$

In $f(t)$, one of the y values of the stationary points is negative and the other is the positive. Hence they are on opposing sides of the horizontal axis, and $t^3 - 3tx + y = 0$ always has three real solutions for all values of (x_0, y_0) where ${x_0}^3 > {y_0}^2 > 0$ and three unique tangents.