NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2018 Trial HSC Examination

Monday, 13 August 2018

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

○ 12MAT1- Mr Sekaran ○ 12MAT2- Mrs Bhamra

Marker's use only

QUESTION	1-10	11	12	13	14	15	16	Total	%
MARKS	/10	/15	/15	/15	/15	/15	/15	/100	

Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided.

Questions **Marks**

1. Let $\arg(z) = \frac{\pi}{5}$ for a certain complex number z. What is $\arg(z^7)$? **1**

(A) $-\frac{7\pi}{5}$

(B) $-\frac{3\pi}{5}$

(C) $\frac{2\pi}{5}$

(D) $\frac{3\pi}{5}$

2. Which of following integrals uses the correct substitution for 1

$$\int_0^{\sqrt{3}} \frac{\ln(\tan^{-1}x)}{1+x^2}\,dx,$$

(A) $\int_0^{\sqrt{3}} \ln u \, du$

(B) $\int_0^{\frac{\pi}{6}} \frac{\ln u}{1+\tan^2 u} du$

(C) $\int_0^{\frac{\pi}{3}} \ln u \, du$

(D) $\int_0^{\frac{\pi}{3}} \frac{\ln u}{1+\tan^2 u} du$

3. What are the coordinates of the foci of the hyperbola $25y^2 - 16x^2 = 400$? 1

(A) $(0, \pm 4\sqrt{41}\,)$

(B) $(\pm 4\sqrt{41},\ 0)$

(C) $(0, \pm\sqrt{41}\,)$

(D) $(\pm\sqrt{41}, 0)$

4. Given that $w^5 = 1$ and w is a complex number, what is the value of 1

$$1 + w + w^2 + w^3 + w^4 + w^5 \text{ ?}$$

(A) 1

(B) 0

(C) w

(D) $-w$

5. What is the eccentricity of the ellipse given by the equation: 1

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

(A) $\frac{2}{3}$

(B) $\frac{\sqrt{14}}{3}$

(C) $\frac{2}{\sqrt{5}}$

(D) $\sqrt{\frac{14}{5}}$

6. Given that α, β and γ are the roots of the polynomial $2x^3 + 4x - 5 = 0$. 1

What is the value of $(\beta + \gamma - 3\alpha)(\alpha + \gamma - 3\beta)(\alpha + \beta - 3\gamma)$?

(A) 32

(B) 160

(C) -32

(D) -160

7. Which of the following is an expression of $\int \frac{dx}{\sqrt{7-6x-x^2}}$? 1

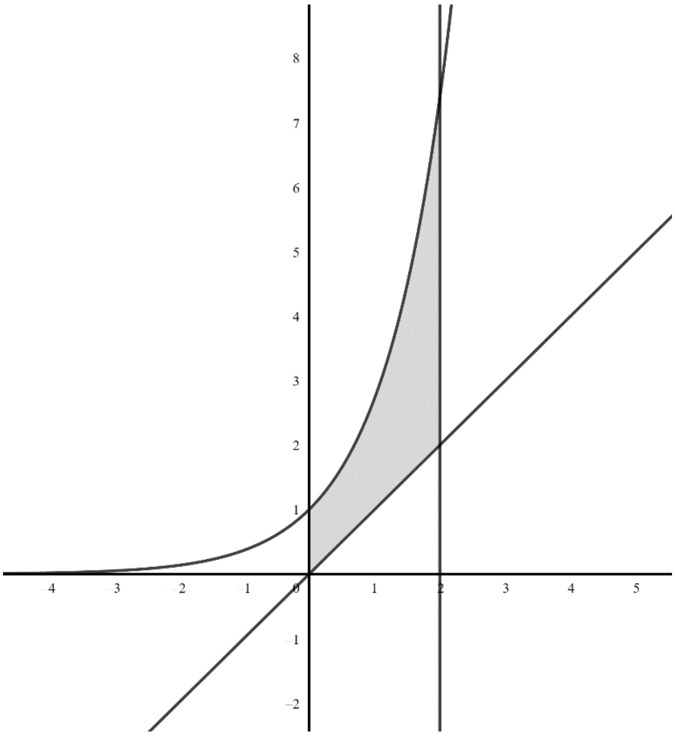
(A) $\sin^{-1}\left(\frac{x-3}{2}\right) + c$

(B) $\sin^{-1}\left(\frac{x-3}{4}\right) + c$

(C) $\sin^{-1}\left(\frac{x+3}{2}\right) + c$

(D) $\sin^{-1}\left(\frac{x+3}{4}\right) + c$

8. The region bounded by the curve $= e^x$, the line $y = x$, the y-axis and the line $x = 2$ is rotated about the y-axis to form a solid. 1



Using method of cylindrical shells, which of the following is an expression for the volume V of the solid formed?

(A) $2\pi \int_0^2 x(e^x - x)\ dx$

(B) $2\pi \int_0^2 (x - e^x)\ dx$

(C) $2\pi \int_0^2 (e^x - x)\ dx$

(D) $2\pi \int_0^2 x(x - e^x)\ dx$

9. A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to motion has magnitude $\frac{1}{20}mv^2$ where $v\ ms^{-1}$ is the speed of the particle and $g = 9.8\ ms^{-2}$ is the acceleration due to gravity. What is the terminal velocity of the particle? 1

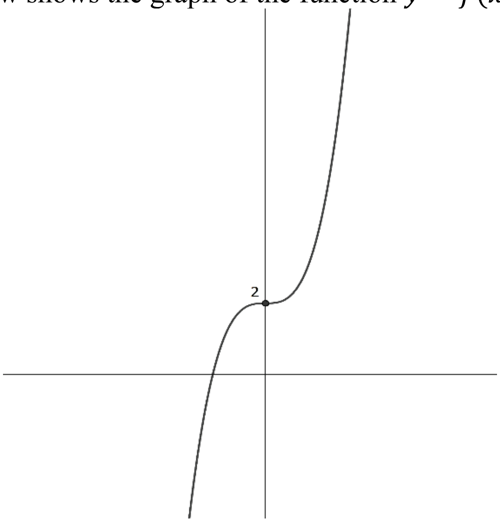
(A) $9.8\ ms^{-1}$

(B) $14\ ms^{-1}$

(C) $19.6\ ms^{-1}$

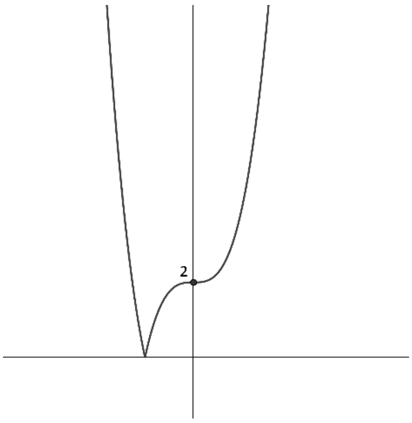
(D) $20\ ms^{-1}$

10. The diagram below shows the graph of the function $y = f(x)$: 1

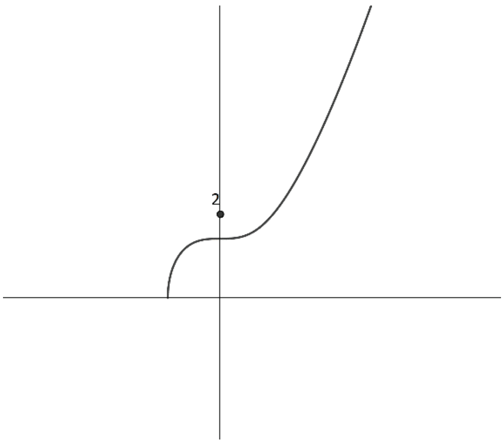


Which diagram represents the graph of $y = \sqrt{|f(x)|}$?

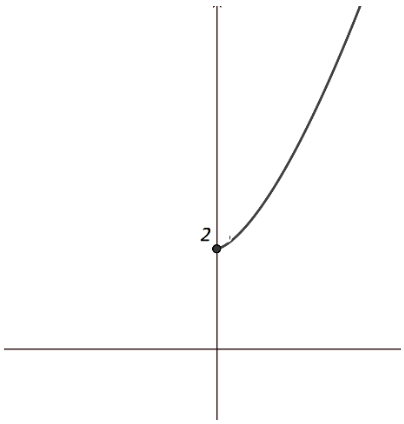
(A)



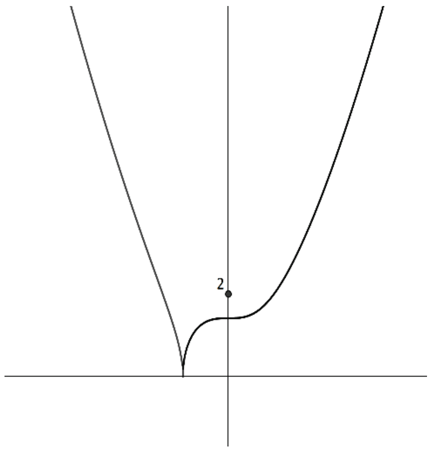
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) For $z = 1 - i$, $w = 3 - 2i$, find:

(i) $|z + w|$ 1

(ii) $z^2 - w^2$ 1

(b) Find the exact value of:

(i) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$ 2

(ii) $\int_0^{\sqrt{2}} \sqrt{4 - x^2}\,dx$ 3

(c) Find the equation of the tangent to the curve $3x^2 - 2xy + y^3 = 1$ at the point P (1, 1) to the curve. 3

(d) Sketch the region in the Argand diagram whose points satisfy: 3

$$|z - 5i| \leq 5 \text{ and } |z + 5| > |z - 5i|$$

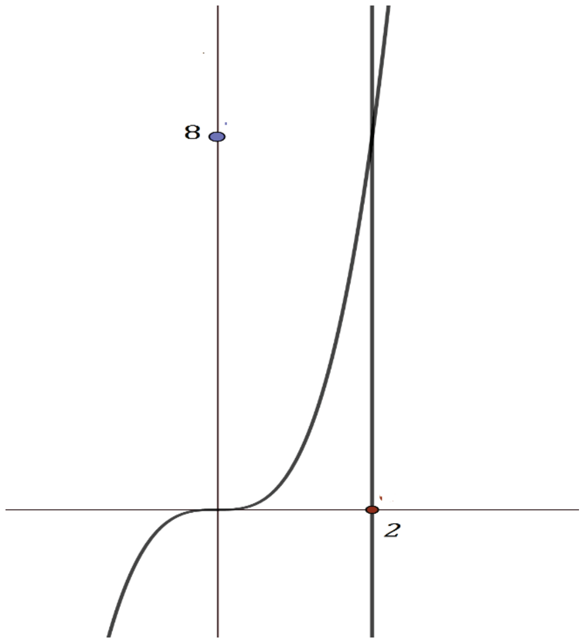
(e) Find: 2

$$\int \frac{\ln x}{x^2}\,dx$$

Examination continues overleaf….

Question 12 (15 marks)

(a) The region bounded by the curve $y = x^3$, the x-axis, $x = 0$ and $x = 2$ is rotated about the line $x = 2$. Find the volume of this solid using the method of slicing. 3



(b) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$

(i) Find $z_1 \div z_2$ in the form $a + ib$ where a and b are real. 1

(ii) Write z_1 and z_2 in modulus-argument form. 2

(iii) Write $\cos\frac{5\pi}{12}$ as a surd by equating equivalent expressions for $z_1 \div z_2$. 1

(c) (i) Find the values of A, B, C and D such that: 2

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

(ii) Hence integrate: 2

$$\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx$$

(f) (i) If α is a root of the polynomial $Q(x)$ with a multiplicity m, show that α is also a root of $Q'(x)$, with multiplicity $(m - 1)$. 1

(ii) If the following polynomial $Q(x)$ has a triple root, factorise $Q(x)$ into its linear factors 3
$Q(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24.$

Examination continues overleaf….

Question 13 (15 marks)

(a) Given that a, b and c are real positive numbers

(i) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$. 2

(ii) Show that 2

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$$

(iii) Given that $a^2 + b^2 + c^2 = 9$, prove that 2

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \geq \frac{3}{4}$$

(b) If z and w are complex numbers such that $|z| = |w|$, show that 3

$$\frac{1}{2}(z+w).\frac{1}{2}\overline{(z+w)} + \frac{1}{2}(z-w).\frac{1}{2}\overline{(z-w)} = z\bar{z}.$$

(c) A particle of mass m kg is projected vertically upwards with a speed of $U\ ms^{-1}$. At time t seconds the particle has vertical height x metres above the point of projection, speed $v\ ms^{-1}$ and acceleration $a\ ms^{-2}$. The particle moves under gravity in a medium where the resistance to motion has magnitude $\frac{m}{g}v^2$ Newtons where $g\ ms^{-1}$ is the acceleration due to gravity.

(i) Show that $a = -\frac{1}{g}(g^2 + v^2)$. 1

(ii) Show that 3

$$v = g\left(\frac{U - g\tan t}{g + U\tan t}\right)$$

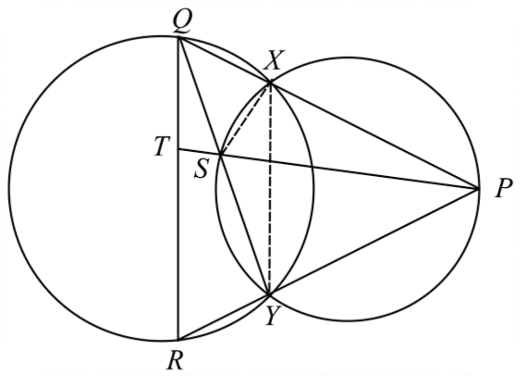
(iii) Find the time taken for the particle to reach its maximum height 1

(iv) Express x in terms of t. 1

Examination continues overleaf….

Question 14 (15 marks)

(a) The circles *XPYS* and *XYRQ* intersect at the points *X* and *Y*. *PXQ*, *PYR*, *QSY*, *PST* and *QTR* are straight lines.



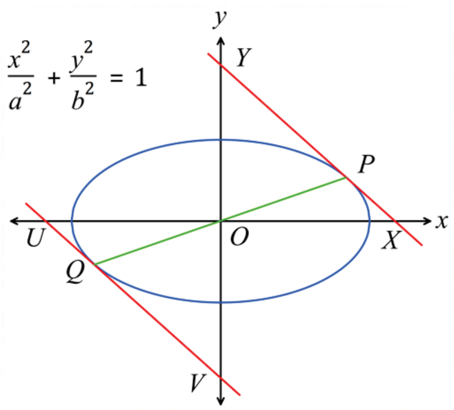
(i) Explain why $\angle STQ = \angle YRQ + \angle YPS$. 1

(ii) Show that $\angle YRQ + \angle YPS + \angle SXQ = 180°$. 2

(iii) Prove that *STQX* is a cyclic quadrilateral. 1

(iv) Let $\angle QPY = \alpha$ and $\angle PQY = \beta$. Show that $\angle STQ = \alpha + \beta$ 3

(a) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\varphi, b\sin\varphi)$ are the end points of a diameter of the ellipse shown below.



Tangents to the ellipse at *P*, *Q* cut the *x*-axis at *X*, *U* respectively, and the *y*-axis at *Y*, *V* respectively.

(i) Derive the expression for the gradient of the tangent to the ellipse and hence show that the tangent at *P* has the following equation.

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$ 2

(ii) Show that $\varphi = \theta \pm \pi$ 2

(iii) What are the coordinates of *X*, *Y*, *U*, *V* in terms of *a*, *b* and θ? 2

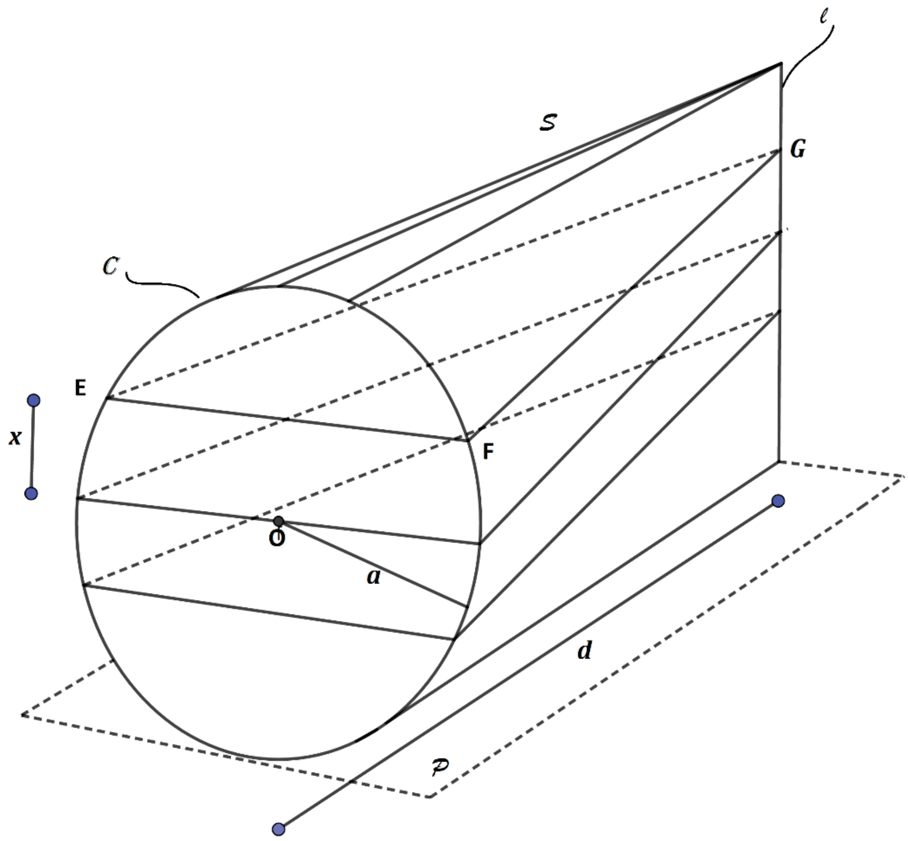
(iv) Show that the area of XYUV is 2

$$\frac{4ab}{|\sin 2\theta|}$$

Question 15 (15 marks)

(a) The solid $\boldsymbol{S}$ is generated by moving a straight edge so that it is always parallel to a fixed plane $\boldsymbol{P}$. It is constrained to pass through a circle $\boldsymbol{C}$ and line segment $\boldsymbol{l}$.

The circle $\boldsymbol{C}$, which forms a base for $\boldsymbol{S}$, has radius a and the line segment $\boldsymbol{l}$ is at a distance d from $\boldsymbol{C}$. Both $\boldsymbol{C}$ and $\boldsymbol{l}$ are perpendicular to $\boldsymbol{P}$. The perpendicular to $\boldsymbol{C}$ at its centre O bisects $\boldsymbol{l}$.



(i) Calculate the area of the triangular cross-section EFG which is parallel to $\boldsymbol{P}$ and distance x from the centre O of $\boldsymbol{C}$. 2

(ii) Calculate the volume of $\boldsymbol{S}$. 2

(b) (i) Prove the identity: 2

$$\cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$$

(iii) Show that $x = 2\sqrt{2}\cos A$ satisfies the cubic equation $x^3 - 6x + 2 = 0$ 2

given that $\cos 3A = -\frac{1}{2\sqrt{2}}$

(c) For $n = 1,2,3 \ldots$, S_n and T_n are two different sequences of positive integers.

Given that $S_n = T_1 + T_2 + T_3 + T_4 \ldots + T_n$

Also $S_1 = 6,\ S_2 = 20\ and\ S_n = 6S_{n-1} - 8S_{n-2}$ for $n = 3, 4, 5 \ldots$

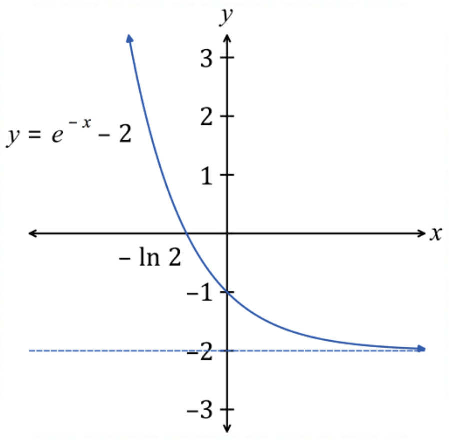
(i) Prove by mathematical induction that $S_n = 4^n + 2^n$, $n = 1,2,3\ldots$ 4

(ii) Hence or otherwise, find $T_n,\ n = 1, 2, 3 \ldots$ in simplest form 3

Examination continues overleaf….

Question 16 (15 marks)

(a) The graph of $f(x) = e^{-x} - 2$ is shown below:



Draw separate one-third page sketches of the following functions. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = |f(x)|$ 1

(ii) $y = \{f(x)\}^2$ 1

(iii) $y = \frac{1}{f(x)}$ 2

(iv) $y = \ln f(x)$ 1

(b) If a polynomial $P(x)$ is divided by $x^2 - u^2$, where $u \neq 0$, the remainder is $px + q$.

(i) Show that 2

$$p = \frac{1}{2u}[P(u) - P(-u)] \text{ and}$$

$$q = \frac{1}{2}[P(u) + P(-u)]$$

(ii) Find the remainder when $P(x) = x^n - u^n$, n is a positive integer, is divided by $x^2 - u^2$. 3

(Hint: Consider all possible cases for the value of n.)

(c) For

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx \qquad \text{n} = 1,2,3,\ldots$$

(i) Show that 3

$$I_{n+1} = \frac{2n-1}{2n} I_n + \frac{1}{n \times 2^{n+1}} \qquad n = 1,2,3,\ldots$$

(ii) Hence evaluate 2

$$\int_0^1 \frac{1}{(1+x^2)^3}\,dx$$

Ext 2: TRIAL EXAM SOLUTIONS 2018

Q1

$z = \text{cis}\frac{\pi}{5}$

$z^7 = \text{cis}\frac{7\pi}{5}$

$\arg z = -\frac{3\pi}{5}$ **[B]**

Q2 let $\tan^{-1}x = u$

$\frac{1}{1+x^2} = \frac{du}{dx}$

If $x=0$

$u = \tan^{-1}(0) = 0$

If $x = \sqrt{3}$

$u = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\therefore \int_0^{\sqrt{3}} \frac{\ln(\tan^{-1}x)}{1+x^2}\,dx$

$= \int_0^{\frac{\pi}{3}} \ln u \, du$ **[C]**

Q3. $25y^2 - 16x^2 = 400$

$\frac{y^2}{16} - \frac{x^2}{25} = 1$

$a^2 = 25, \quad b^2 = 16$

$a^2 = b^2(e^2-1)$

$\frac{25}{16} = e^2 - 1$

$e^2 = \frac{41}{16}$

$e = \pm \frac{\sqrt{41}}{4}$

foci $= (0 \pm be) = (0, \pm\sqrt{41})$ **[C]**

Q4 $\omega^5 - 1 = 0$

$(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4) = 0$

$\omega - 1 \neq 0$

$\therefore 1+\omega+\omega^2+\omega^3+\omega^4 = 0$

$\Rightarrow 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5$

$= 1+\omega(1+\omega+\omega^2+\omega^3+\omega^4) = 0$

Q5 $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$b^2 = a^2(1-e^2)$

$\frac{5}{9} = 1 - e^2$

$e^2 = 1 - \frac{5}{9} = \frac{4}{9}$

$e = \frac{2}{3}$ **[A]**

Q6. $2x^3 + 4x - 5 = 0$.

$\alpha+\beta+\gamma = 0, \quad \alpha\beta\gamma = \frac{5}{2}$

$(\beta+\gamma-3\alpha)(\alpha+\gamma-3\beta)(\alpha+\beta-3\gamma)$

$= (\alpha+\beta+\gamma-4\alpha)(\alpha+\beta+\gamma-4\beta)(\alpha+\beta+\gamma-4\gamma)$

$= (-4\alpha)(-4\beta)(-4\gamma)$

$= -64(\alpha\beta\gamma)$

$= -64 \times \frac{5}{2} = -160$ **[D]**

Q7 $\int \frac{dx}{\sqrt{7-6x-x^2}}$

$= \int \frac{dx}{\sqrt{16-(x^2+6x+9)}}$

$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$

$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$ **[D]**

Q8 Radius of the shell $= x$

height $= e^x - x$

$\delta V = 2\pi x(e^x - x)\,\delta x$

$\therefore$ Total $V = 2\pi \int_0^2 x(e^x - x)\,dx$ **[A]**

Q9 Net force $=$ mg $-$ Resistance

$ma = mg - \frac{1}{20} m v^2$

$a = 9.8 - \frac{1}{20} v^2$

For Terminal velocity, $a = 0$

$v^2 = 9.8 \times 20$

$= 196.0$

$v = 14 \text{ ms}^{-1}$ **[B]**

Q10. **[D]**

Q11

a) $z = 1 - i, \quad w = 3 - 2i$

i) $z + w = 4 - 3i$

$|z+w| = \sqrt{16+9}$

$= 5$ ✓

ii) $z^2 - w^2$

$= (1-i)^2 - (3-2i)^2$

$= (1-2i+i^2) - (9-12i+4i^2)$

$= -2i - (5-12i)$

$= -5 + 10i$ ✓

b) i) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$

let $u = x^2+2x+5$

$\frac{du}{dx} = 2x+2$

when $x = 2$

$u = 13$

when $x = 3$

$u = 20$

$\therefore \frac{1}{2}\int_2^3 \frac{2(x+1)}{\sqrt{x^2+2x+5}}\,dx$ ✓

$= \int_{13}^{20} u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{13}^{20}$

$= \sqrt{20} - \sqrt{13}$ ✓

ii) $I = \int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$

let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

If $x = 0$

$2\sin\theta = 0$

$\theta = 0$

If $x = \sqrt{2}$

$2\sin\theta = \sqrt{2}$

$\therefore \theta = \frac{\pi}{4}$

$\therefore I = \int_0^{\frac{\pi}{4}} \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\,d\theta$ ✓

$= \int_0^{\frac{\pi}{4}} 4\cos^2\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{4}} (\cos 2\theta + 1)\,d\theta$ ✓

$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{4}}$

$= 2\left[\frac{1}{2} + \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[2 + \pi\right]$ ✓

Q(c) $3x^2 - 2xy + y^3 = 1$

$6x - \left(2x\frac{dy}{dx} + 2y\right) + 3y^2\frac{dy}{dx} = 0$

$6x - 2y + (3y^2 - 2x)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2y-6x}{3y^2-2x}$ ✓

at $P(1,1)$

$\frac{dy}{dx} = \frac{2-6}{3-2} = -4$

$\therefore$ equation of the tangent at

$y - 1 = -4(x-1)$

$4x + y - 5 = 0$ ✓

$\therefore |z-5i| \le 5$

circle with centre $5i$, radius 5

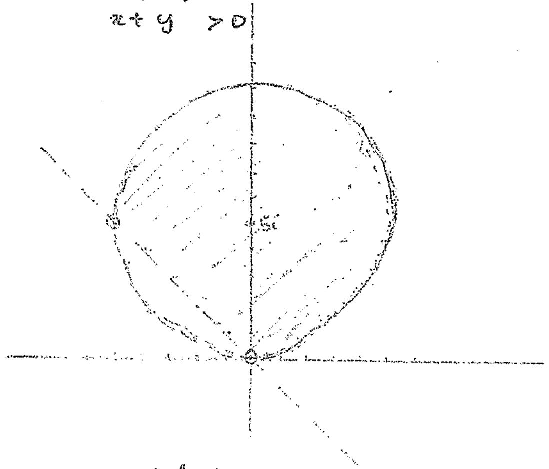
& $|z+5| > |z-5i|$

$|(x+5)+iy| > |x+(y-5)i|$

$\sqrt{(x+5)^2+y^2} > \sqrt{x^2+(y-5)^2}$

$10x+10y > 0$

$x+y > 0$



e) $I = \int \frac{\ln x}{x^2}\,dx$

Let $u = \ln x$, $v = -\frac{1}{x}$

$u' = \frac{1}{x}$, $v' = \frac{1}{x^2}$

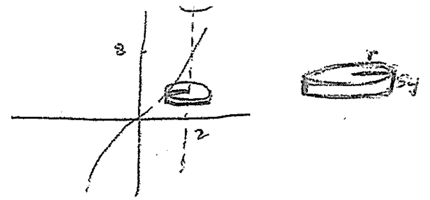
$\therefore$

$I = -\frac{1}{x}\cdot \ln x + \int \frac{1}{x^2}\,dx$ ✓

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

$= -\frac{1}{x}(1+\ln x) + C$ ✓

Q12
a)



Radius of the slice $= 2-x$

thickness $= \delta y$

Area of the cross-section of the slice $= \pi r^2$

$= \pi(2-x)^2$

$= \pi(2-y^{\frac{1}{3}})^2$

Volume of the slice

$\delta V = \pi(2-y^{\frac{1}{3}})^2 \delta y$

Total volume

$V = \lim_{\delta y \to 0} \sum_{y=0}^{8} \pi(2-y^{\frac{1}{3}})^2 \delta y$ ✓

$= \pi \int_0^8 (2-y^{\frac{1}{3}})^2\,dy$

$= \pi \int_0^8 4 - 4y^{\frac{1}{3}} + y^{\frac{2}{3}}\,dy$

$= \pi \left[4y - 4y^{\frac{4}{3}}\cdot\frac{3}{4} + y^{\frac{5}{3}}\cdot\frac{3}{5}\right]_0^8$ ✓

$= \pi\left[32 - 3(8)^{\frac{4}{3}} + \frac{3}{5}(8)^{\frac{5}{3}}\right]$

$= \pi\left[32 - 48 + \frac{3}{5}\times 32\right]$

$V = \frac{16\pi}{5}$ units^3 ✓

b) $z_1 = 1+i$

$z_2 = \sqrt{3}-i$

i) $\frac{z_1}{z_2} = \frac{(1+i)}{(\sqrt{3}-i)}\frac{(\sqrt{3}+i)}{(\sqrt{3}+i)}$

$= \frac{\sqrt{3}+i+\sqrt{3}i-1}{4}$

$= \frac{\sqrt{3}-1}{4} + \frac{(\sqrt{3}+1)}{4}i$ ✓

ii) $z_1 = 1+i$

$|z_1| = \sqrt{2}$

$\arg z_1 = \frac{\pi}{4}$

$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ ✓

$z_2 = \sqrt{3}-i$

$|z_2| = 2$

$\arg z_2 = -\frac{\pi}{6}$

$z_2 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ ✓

iii) $\frac{z_1}{z_2} = \frac{\sqrt{2}\,\text{cis}\frac{\pi}{4}}{2\,\text{cis}(-\frac{\pi}{6})}$

$= \frac{1}{\sqrt{2}}\,\text{cis}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{2}}\,\text{cis}\frac{5\pi}{12}$

Comparing part (i) & (iii)

$\frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{4}$

$\cos\frac{5\pi}{12} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓

c)

i) $\frac{5x^3-3x^2+2x-1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$\Rightarrow Ax(x^2+1) + B(x^2+1) + (Cx+D)(x^2)$

$= 5x^3 - 3x^2 + 2x - 1$ — (i

$= A(x^3+x) + B(x^2+1) + Cx^3 + Dx^2$

$= (A+C)x^3 + (B+D)x^2 + Ax + B$ (ii

Comparing coefficients in (i) & (ii)

$A+C = 5$

$B+D = -3$

$A = 2$ ✓

$B = -1$ ✓

$\Rightarrow C = 3$

$D = -2$

(ii) Hence $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$

$= \int \frac{2}{x} - \frac{1}{x^2} + \frac{3x-2}{x^2+1}\,dx$

$= \int \frac{2}{x} - \frac{1}{x^2} + \frac{3x}{x^2+1} - \frac{2}{x^2+1}\,dx$

$= \int \frac{2}{x} - \frac{1}{x^2} + \frac{3}{2}\cdot\frac{2x}{x^2+1} - \frac{2}{x^2+1}\,dx$ ✓

$= 2\ln x + \frac{1}{x} + \frac{3}{2}\ln(x^2+1) - 2\tan^{-1}x$ ✓

d

(i) $Q(x) = (x-\alpha)^m \cdot P(x)$

$Q'(x) = m(x-\alpha)^{m-1}\cdot P(x) + (x-\alpha)^m \cdot P'(x)$

$= (x-\alpha)^{m-1}\left[m\cdot P(x) + (x-\alpha)P'(x)\right]$ ✓

$Q'(x) = (x-\alpha)^{m-1} R(x)$

where $R(x) = m\,P(x) + (x-\alpha)P'(x)$

d (ii) α is a root of multiplicity 3

$Q(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$Q'(x) = 8x^3 + 27x^2 + 12x - 20$

$Q''(x) = 24x^2 + 54x + 12$

α will be root of $Q'(x)$ with $m=2$

& a root of $Q''(x)$ with $m=1$

$\therefore Q''(\alpha) = 24\alpha^2 + 54\alpha + 12 = 0$ ✓

$= 4\alpha^2 + 9\alpha + 2 = 0$

$(4\alpha+1)(\alpha+2) = 0$

$\Rightarrow \alpha = -\frac{1}{4}, \quad \alpha = -2$

Test:

$Q(-2) = 2(-2)^4 + 9(-2)^3 + 6(-2)^2 - 20(-2) - 24$

$= 32 - 72 + 24 + 40 - 24$

$= 0$ ✓

$\Rightarrow \alpha = -2$ is a root of $m = 3$

$Q(x) = (x+2)^3 P(x)$

$P(x) = Q(x) \div (x+2)^3$

$$\begin{array}{r} 2x - 3 \\ x^3+6x^2+12x+8 \overline{)\, 2x^4 + 9x^3 + 6x^2 - 20x - 24} \\ 2x^4 + 12x^3 + 24x^2 + 16x \\ \hline -3x^3 - 18x^2 - 36x - 24 \\ -3x^3 - 18x^2 - 36x - 24 \\ \hline x \end{array}$$

$\therefore Q(x) = (x+2)^3(2x-3)$ ✓

Q13

a) (i) $(a-b)^2 \geq 0$

$a^2 + b^2 - 2ab \geq 0$

$a^2 + b^2 \geq 2ab$ ①

Similarly

$b^2 + c^2 \geq 2bc$ ②

& $a^2 + c^2 \geq 2ac$ ③ ✓

Add ①, ② & ③

$2(a^2+b^2+c^2) \geq 2(ab+bc+ac)$ ✓

$a^2+b^2+c^2 \geq ab+bc+ac$

(ii) From part (i)

$(a-b)^2 \geq 0$

$\therefore (x-y)^2 \geq 0$

$x^2 + y^2 \geq 2xy$

Let $x = \sqrt{\frac{a}{b}}$, $y = \sqrt{\frac{b}{a}}$

$\Rightarrow x^2 + y^2 \geq 2xy$

$\frac{a}{b} + \frac{b}{a} \geq 2$

Similarly

$\frac{b}{c} + \frac{c}{b} \geq 2$

& $\frac{a}{c} + \frac{c}{a} \geq 2$ ✓

Also
$$\begin{cases} \frac{a}{b} + \frac{b}{a} + \frac{a}{a} \geq 3 \\ \frac{b}{c} + \frac{c}{b} + \frac{b}{b} \geq 3 \\ \frac{a}{c} + \frac{c}{a} + \frac{c}{c} \geq 3 \end{cases}$$

Add

$\frac{a}{b} + \frac{b}{a} + \frac{a}{a} + \frac{b}{c} + \frac{c}{b} + \frac{b}{b} + \frac{a}{c} + \frac{c}{a} + \frac{c}{c} \geq$

$\frac{1}{a}(a+b+c) + \frac{1}{b}(a+b+c) + \frac{1}{c}(a+b+c) \geq$ ✓

$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$

iii) From part ii)

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}$

If $x = 1+ab$

$y = 1+bc$

$z = 1+ac$

$\therefore \frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac}$

$\geq \frac{9}{1+ab+1+bc+1+ac}$ ✓

$= \frac{9}{3+ab+bc+ac}$

$\geq \frac{9}{3+a^2+b^2+c^2}$ ✓ [From part (i) $a^2+b^2+c^2 \geq ab+bc+ac$]

$= \frac{9}{12}$

$\therefore \frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \geq \frac{3}{4}$

b) $|z| = |w|$

$|z|^2 = |w|^2$

LHS

$\therefore \frac{1}{4}\left[(z+w)\overline{(z+w)} + (z-w)\overline{(z-w)}\right]$

$= \frac{1}{4}\left[(z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w})\right]$ ✓

$= \frac{1}{4}\left[z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} + z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}\right]$

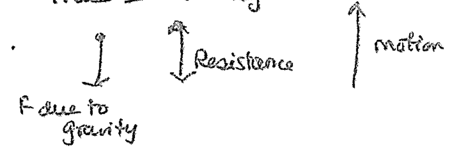
$= \frac{1}{4}\left[2z\bar{z} + 2w\bar{w}\right]$

$= \frac{1}{2}\left[z\bar{z} + w\bar{w}\right]$ ✓

$= \frac{1}{2} \cdot 2z\bar{z} = z\bar{z}$ RHS

[as $z\bar{z} = |z|^2$, $w\bar{w} = |w|^2$ $\therefore z\bar{z} = w\bar{w}$ $\because |z| = |w|$] ✓

αc mass $= m$ kg



(i) $\therefore$ Net force $= -(mg + \text{Resistance})$

$ma = -mg - \frac{m}{g}v^2$ ✓

$a = -g - \frac{v^2}{g}$

$a = -\frac{1}{g}(g^2 + v^2)$

(ii) $\therefore \frac{dv}{dt} = -\frac{1}{g}(g^2+v^2)$

$\frac{dt}{dv} = \frac{-g}{g^2+v^2}$

$t = -g\int \frac{1}{g^2+v^2}\, dv$

$t = -\tan^{-1}\left(\frac{v}{g}\right) + c$ ✓✓

when $t = 0$, $v = U$

$\therefore -\tan^{-1}\frac{U}{g} + c = 0$

$c = \tan^{-1}\frac{U}{g}$

$\therefore t = -\tan^{-1}\frac{v}{g} + \tan^{-1}\frac{U}{g}$

Let $\tan^{-1}\frac{U}{g} = \alpha$ $\quad \therefore \frac{U}{g} = \tan\alpha$

& $\tan^{-1}\frac{v}{g} = \beta$ $\quad \frac{v}{g} = \tan\beta$

$\therefore t = \alpha - \beta$

& $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\tan t = \frac{\frac{U}{g} - \frac{v}{g}}{1 + \frac{U}{g}\cdot\frac{v}{g}}$

$= \frac{U-v}{g} \times \frac{g^2}{g^2+Uv}$

$\tan t = \frac{g(U-v)}{g^2+Uv}$

(continued ..

$\tan t(g + Uv) = gU - gv$

$g^2 \tan t + Uv \tan t + gv = gU$ ✓

$(U\tan t + g)v = gU - g^2 \tan t$

$v = \dfrac{g(U - g\tan t)}{g + U\tan t}$

iii) At max height, $v = 0$

$\dfrac{g(U - g\tan t)}{g + U\tan t} = 0$

$U - g\tan t = 0$

$\tan t = \dfrac{U}{g}$

$t = \tan^{-1}\left(\dfrac{U}{g}\right)$ ✓

iv) $\dfrac{dx}{dt} = \dfrac{g(U - g\tan t)}{g + U\tan t}$

$= \dfrac{g\left(U - \frac{g\sin t}{\cos t}\right)}{g + U\frac{\sin t}{\cos t}}$

$= \dfrac{g(U\cos t - g\sin t)}{g\cos t + U\sin t}$

$x = g\displaystyle\int \dfrac{U\cos t - g\sin t}{g\cos t + U\sin t}\,dt$

$x = g\ln(g\cos t + U\sin t) + c$ ✓

Q14

a) i) In $\triangle RPT$

$\angle PTQ = \angle TRP + \angle RPT$ [ext. angle is equal to sum of interior opposite angles]

$\therefore \angle STQ = \angle YRQ + \angle YPS$

ii) In quad. QRYX

$\angle TRY + \angle QXY = 180°$ (opposite angles of a cyclic quadrilateral)

$\angle YRQ + \angle QXS + \angle SXY = 180°$

but $\angle SXY = \angle YPS$ (angles in the same segment)

$\therefore \angle YRQ + \angle YPS + \angle SXQ = 180°$ — (2)

iii) From part (i)

$\angle YRQ + \angle YPS = \angle STQ$

$\therefore$ from (2)

$\angle STQ + \angle SXQ = 180°$

as $\angle STQ$ & $\angle SXQ$ are opp. angle of a quadrilateral STQ

$\Rightarrow$ STQX is a cyclic quad.

iv) $\angle XYS = \angle XPS$ & $\angle SXY = \angle SPY$ angles in the same segm

$\alpha = \angle QPY = \angle XPS + \angle SPY$
$= \angle XYS + \angle SXY = \angle XSQ$ [ext. angle of a triangl = sum of int. opp. ang

$\angle STQ = \pi - \angle SXQ$ (opp. angles of a cyclic quadrilat PXQ → ast. li

$= \angle PXS$

$= \angle XQS + \angle XSQ$ (ext angle of a triangle = su of int. opposite an

$= \angle PQY + \angle QPY$

$= \alpha + \beta$

b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

i) $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2x}{a^2} \times \dfrac{b^2}{2y}$

$= -\dfrac{b^2x}{a^2y}$

at $P(a\cos\theta, b\sin\theta)$

$\dfrac{dy}{dx} = -\dfrac{b^2 \cdot a\cos\theta}{a^2 \cdot b\sin\theta}$

$= \dfrac{-b\cos\theta}{a\sin\theta}$

$\therefore$ Equation of tangent at P is:

$y - b\sin\theta = \dfrac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\Rightarrow ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$\Rightarrow bx\cos\theta + ay\sin\theta = ab$

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

ii) $\angle UOQ = \angle XOP = \theta$ (vertically opposite angles)

$\phi = 180° + \theta$ ie $180° + \angle UOQ$

$= \pi + \theta$

or $\phi = -(\pi - \angle UOQ)$

$= -(\pi - \theta)$

$= \theta - \pi$

$\therefore \phi = \theta \pm \pi$

iii) Eq. of tangent at P

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

at X, $y = 0$

$x = \dfrac{a}{\cos\theta}$ $\therefore X: \left(\dfrac{a}{\cos\theta}, 0\right)$

$\dfrac{y\sin\theta}{b} = 1$

$\therefore y = \dfrac{b}{\sin\theta}$

$Y: \left(0, \dfrac{b}{\sin\theta}\right)$

for tangent Q, equation

$\dfrac{x\cos\phi}{a} + \dfrac{y\sin\phi}{b} = 1$

$\dfrac{x\cos(\pi+\theta)}{a} + \dfrac{y\sin(\pi+\theta)}{b} = 1$

$-\dfrac{x\cos\theta}{a} + -\dfrac{y\sin\theta}{b} = 1$

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = -1$

$\therefore$ at V, $x = 0$

$y = \dfrac{-b}{\sin\theta}$ $\therefore V: \left(0, \dfrac{-b}{\sin\theta}\right)$

at U = $y = 0$

$\therefore x = \dfrac{-a}{\cos\theta}$

$U: \left(\dfrac{-a}{\cos\theta}, 0\right)$ ✓ all 4 correct

(iv) Diagonals of XYUV: $YV \perp UX$ & diagonals bisect each other

$\therefore$ XYUV is a rhombus

$\therefore$ area $= \frac{1}{2} \times (UX)(YV)$

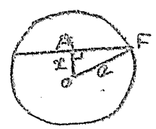
$= \dfrac{1}{2} \cdot \dfrac{2a}{|\cos\theta|} \cdot \dfrac{2b}{|\sin\theta|}$

$= \dfrac{4ab}{|2\cos\theta\sin\theta|}$ ✓

$= \dfrac{4ab}{|\sin 2\theta|}$

Q15 :

a) i)



$AF = \sqrt{a^2 - x^2}$

$\therefore$ base of the triangle EFG

$= 2\sqrt{a^2-x^2}$ ✓

$\therefore$ area of the triangle

$= \frac{1}{2} \cdot 2\sqrt{a^2-x^2} \cdot d$

$= d\sqrt{a^2-x^2}$ ✓

ii) $V = \int_{-a}^{a} d\sqrt{a^2-x^2}\,dx$

$= d\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$ ✓

but $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$

area of the semi circle $x^2+y^2=a^2$

$\therefore V = d\ \frac{1}{2}\pi a^2$ ✓

$= \frac{\pi}{2} d a^2$ units^3

b) $\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (\cos^2 A - \sin^2 A)\cos A - 2\sin A \cos A . \sin A$

$= \cos^3 A - \sin^2 A \cos A - 2\sin^2 A \cos A$

$= \cos^3 A - 3\sin^2 A \cos A$ ✓

$= \cos^3 A - 3(1-\cos^2 A)\cos A$

$= \cos^3 A - 3\cos A + 3\cos^3 A$ ✓

$= 4\cos^3 A - 3\cos A$

ii) If $x = 2\sqrt{2}\cos A$ satisfies the equation

$x^3 - 6x + 2 = 0$

$\Rightarrow (2\sqrt{2}\cos A)^3 - 6(2\sqrt{2}\cos A) + 2 = 0$

$16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A + 2 = 0$.

Divide by $16\sqrt{2}$

$\cos^3 A - \frac{12\sqrt{2}}{16\sqrt{2}}\cos A + \frac{2}{16\sqrt{2}} = 0$

$\cos^3 A - \frac{3}{4}\cos A = \frac{-2}{16\sqrt{2}}$ ✓

$\frac{1}{4}\cos 3A = \frac{-2}{16\sqrt{2}}$

$\cos 3A = \frac{-8}{16\sqrt{2}}$ ✓

$\cos 3A = \frac{-1}{2\sqrt{2}}$

$\therefore x = 2\sqrt{2}\cos A$ satisfies the given equation.

c) $S_n = 4^n + 2^n$

Prove it true for $n=1$, $n=2$

$S_1 = 4+2 = 6$ True

$S_2 = 4^2+2^2 = 20$ True

Proven true for $n=1$ & $n=2$ ✓

Assume it to be true for $n=k$;

$S_k = 4^k + 2^k$

To prove it true for $n=k+1$

$S_{k+1} = 4^{k+1} + 2^{k+1}$

As given

$S_n = 6S_{n-1} - 8S_{n-2}$

$\therefore S_{k+1} = 6S_k - 8S_{k-1}$ ✓

$= 6(4^k+2^k) - 8(4^{k-1}+2^{k-1})$

$= 6(4^k+2^k) - 8(\frac{4^k}{4} + \frac{2^k}{2})$

$= 6(4^k+2^k) - 2.4^k - 4.2^k$

$= (6-2)4^k + (6-4)2^k$ ✓

$= 4.4^k + 2.2^k$

$= 4^{k+1} + 2^{k+1}$

Hence proven true for $n=k+1$

$\therefore$ The statement is true for all $n \geq 1$, by principle of mathematical induction. ✓

ii) $S_n = T_1 + T_2 + \dots + T_n$

$\Rightarrow S_1 = T_1$

$S_2 = T_1 + T_2 = S_1 + T_2$

$S_3 = T_1 + T_2 + T_3 = S_2 + T_3$

$\Rightarrow S_n = S_{n-1} + T_n$

$\therefore T_n = S_n - S_{n-1}$

$= 4^n + 2^n - 4^{n-1} - 2^{n-1}$ ✓

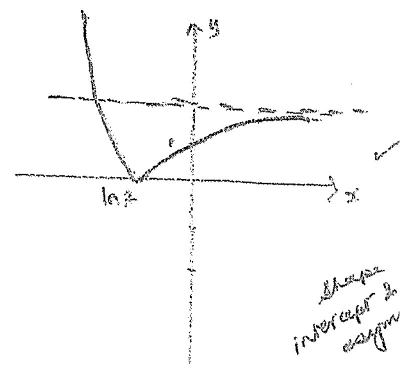
$= 4^n - \frac{4^n}{4} + 2^n - \frac{2^n}{2}$

$= (1-\frac{1}{4})4^n + (1-\frac{1}{2})2^n$

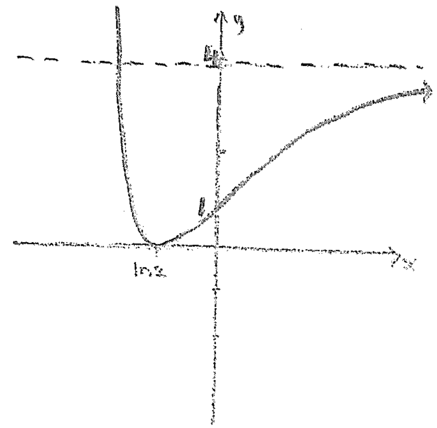
$= \frac{3}{4}.4^n + \frac{1}{2}.2^n$ ✓

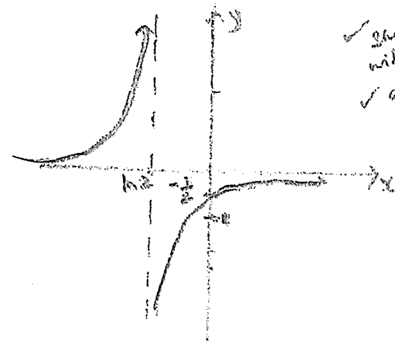
$T_n = 3\ 4^{n-1} + 2^{n-1}$ ✓

16(a)

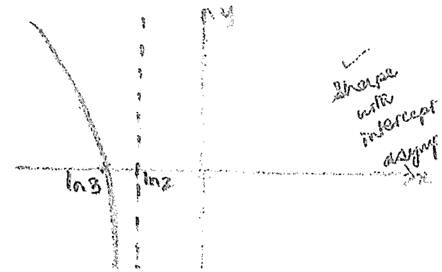


✓ shape & intercept & asym





✓ shape with int
✓ asym



✓ shape with intercept asym

d) $P(x)$ divided by x^2-u^2

i) $P(x) = (x^2-u^2)Q(x) + (px+q)$

$P(u) = pu + q$ ①

$P(-u) = -pu + q$ ②

① + ②

$2q = P(u)+P(-u)$

$q = \frac{1}{2}[P(u)+P(-u)]$

& ① − ②

$P(u) - P(-u) = 2pu$

$p = \frac{1}{2u}[P(u)-P(-u)]$

(ii) $P(x) = x^n - u^n$

If n is even

$P(u) = u^n - u^n = 0$

& $P(-u) = (-u)^n - u^n = 0$

$\Rightarrow p = \frac{1}{2u}[0-0] = 0$

$q = \frac{1}{2}[0] = 0$

∴ n is even remainder $= 0$

If n is odd

$P(u) = u^n - u^n = 0$

$P(-u) = (-u)^n - u^n = -2u^n$

∴ $p = \frac{1}{2u}[2u^n] = u^{n-1}$

$q = \frac{1}{2}[-2u^n] = -u^n$

remainder $= u^{n-1}.x - u^n$

$= u^{n-1}[x-u]$

(c) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}dx$

Let $u = \frac{1}{(1+x^2)^n}$ & $v = x$

$u' = \frac{-2nx}{(1+x^2)^{n+1}}$ $v' = 1$

∴ $I_n = \left[\frac{x}{(1+x^2)^n}\right]_0^1 + 2n\int \frac{x^2}{(1+x^2)^{n+1}}dx$

$= \frac{1}{2^n} + 2n\int (1+x^2-1)(1+x^2)^{-n-1}dx$

$= \frac{1}{2^n} + 2n\int (1+x^2)^{-n} - (1+x^2)^{-n-1}dx$

$= \frac{1}{2^n} + 2n I_n - 2n I_{n+1}$

∴ $2n I_{n+1} = \frac{1}{2^n} + 2n I_n - I_n$

$= \frac{1}{2^n} + (2n-1)I_n$

$I_{n+1} = \frac{1}{2n.2^n} + \frac{2n-1}{2n}I_n$

$= \frac{1}{n.2^{n+1}} + \frac{2n-1}{2n}I_n$

ii) $\int_0^1 \frac{1}{(1+x^2)^3}dx = I_3$ $n=2$

$I_3 = \frac{4-1}{4}I_2 + \frac{1}{2.2^3}$ —(f)

$= \frac{3}{4}I_2 + \frac{1}{16}$

$I_2 = \frac{1}{2}I_1 + \frac{1}{2^2}$

$= \frac{1}{2}I_1 + \frac{1}{4}$

$I_1 = \int_0^1 \frac{1}{1+x^2}dx = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$

∴ $I_2 = \frac{\pi}{8} + \frac{1}{4}$

$I_3 = \frac{3}{4}\left[\frac{\pi}{8}+\frac{1}{4}\right] + \frac{1}{16}$

$= \frac{3}{4}\left[\frac{\pi}{8}+1\right] + \frac{1}{16} = \frac{1}{32}[3\pi+8]$