NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2019 Year 12 Course Assessment Task 4 (Trial Examination)

Monday August 12, 2019

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MAT.1 – Mr Lam ◯ 12MAT.2 – Mrs Bhamra

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 10 minutes for this section

Mark your answers on the answer grid provided (labelled as page 13).

Questions **Marks**

1. Which relationship between m and n will result in the graph of $y = \dfrac{1}{x^2 + mx - n}$ having no vertical asymptotes? **1**

(A) $m^2 < 4n$

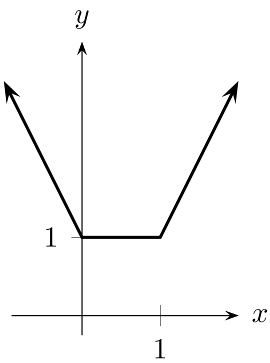
(B) $m^2 > 4n$

(C) $m^2 = -4n$

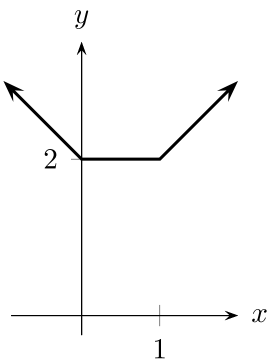
(D) $m^2 < -4n$

2. Which of the following graphs is the graph of $y = |x| + |x - 1|$? **1**

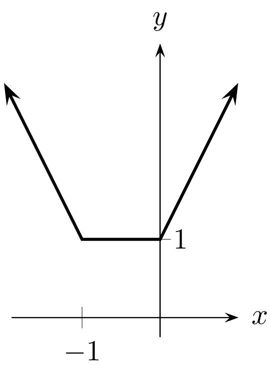
(A)



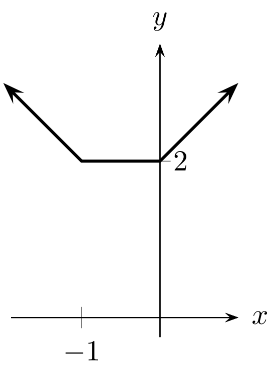
(C)



(B)



(D)



3. The equation $x^4 + px + q = 0$, where $p \neq 0$. $q \neq 0$, has roots α, β, γ and δ. **1**

What is the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$ (B) $p^2 - 2q$ (C) $p^4 - 2q$ (D) $p^4 - 2q$

4. A stone of mass m is dropped from rest and falls in a medium where the resistance is directly proportional to the square of the velocity v. Suppose mk is the constant of proportionality and the displacement downwards from the initial position is x at time t. The acceleration due to gravity is g. **1**

Which of the following is true?

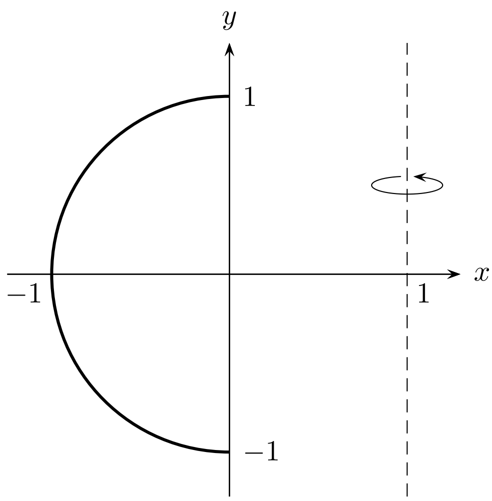
(A) The terminal velocity is $\dfrac{g}{k}$.

(B) As $t \to \infty$, $x \to L$ where L is a positive constant.

(C) The equation of motion is given by $v\dfrac{dv}{dx} = g - kv^2$.

(D) The time for the stone to reach velocity V is given by $\displaystyle\int_0^V \left(g - kv^2\right)\, dv$.

5. The diagram shows the graph of $x^2 + y^2 = 1$ for $-1 \le x \le 0$. The region bounded by the graph and the y axis is rotated about the line $x = 1$ to form a solid. **1**



Which integral represents the volume of the solid?

(A) $2\pi \displaystyle\int_{-1}^{0} (1 + x)\sqrt{1 - x^2}\, dx$

(B) $2\pi \displaystyle\int_{-1}^{0} (1 - x)\sqrt{1 - x^2}\, dx$

(C) $4\pi \displaystyle\int_{-1}^{0} (1 + x)\sqrt{1 - x^2}\, dx$

(D) $4\pi \displaystyle\int_{-1}^{0} (1 - x)\sqrt{1 - x^2}\, dx$

6. What is the maximum y value reached by the ellipse with the equation **1**

$$\frac{3(x+3)^2}{5} + \frac{(y-4)^2}{6} = 3$$

(A) $-4 + 3\sqrt{2}$ (B) $4 + \sqrt{5}$ (C) $3\sqrt{2}$ (D) $4 + 3\sqrt{2}$

7. Which of the following is an expression for the eccentricity of the ellipse **1**

$$\frac{x^2}{k} + \frac{y^2}{k-1} = 1$$

where $k > 1$?

(A) $\dfrac{\sqrt{2k-1}}{k}$

(B) $\dfrac{1}{\sqrt{k}}$

(C) $\sqrt{\dfrac{2k-1}{k}}$

(D) $\dfrac{\sqrt{2k^2-2k+1}}{k}$

8. Let $z = a + ib$ where $a \neq 0$ and $b \neq 0$. **1**

Which of the following statements is *false*?

(A) $z - \overline{z} = 2ib$

(B) $|z|^2 = |z|\,|\overline{z}|$

(C) $|z| + |\overline{z}| = |z + \overline{z}|$

(D) $\text{Arg}(z) + \text{Arg}\left(\overline{z}\right) = 0$

9. Which of the following is the minimum value of $\text{Arg}(z)$ if **1**

$$\left|z - \sqrt{2} - i\sqrt{2}\right| = 1$$

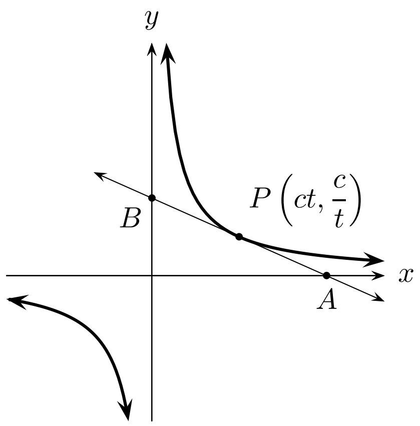
(A) $-\dfrac{5\pi}{12}$

(B) $-\dfrac{\pi}{12}$

(C) $\dfrac{\pi}{12}$

(D) $\dfrac{5\pi}{12}$

10. The equation of the tangent to the rectangular hyperbola $xy = c^2$ at $P\left(ct, \dfrac{c}{t}\right)$ is given by $x + t^2 y = 2ct$. The tangent cuts the x and y axes at A and B respectively. **1**



Which of the following statements is *false*?

(A) P is the centre of the circle that passes through A and B.

(B) The area of $\triangle AOB$ is $2c^2$ square units.

(C) The distance AB is $\sqrt{4c^2t^2 + \dfrac{4c^2}{t^2}}$

(D) $AP > BP$

Examination continues overleaf. . .

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 50 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) For $z = 3i$ and $w = 1 + i$, find the values of:

i. $|z - w|$ **1**

ii. $\dfrac{z}{w}$, with a real denominator. **1**

(b) i. Show that $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ is a root of the equation $z^3 = i$. **2**

ii. On an Argand diagram, neatly plot all three roots of i. **2**

(c) i. Find the points of intersection on the curves given by **3**

$$|z - i| = 1 \text{ and } \mathrm{Re}(z) = -\frac{1}{\sqrt{3}}\,\mathrm{Im}(z)$$

ii. Sketch above the two curves on the Argand diagram to show the points of intersection. **1**

(d) i. Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively. **2**

Find the complex number represented by B.

ii. The square is now rotated about O through $45°$ in an anticlockwise direction to $OA'B'C'$. **3**

Find the complex numbers presented by the points A', B' and C' in the form $z(a + ib)$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) Solve $\sin^{-1}(4x+1) = \cos^{-1} x$. **2**

(b) i. Find the values of A, B, C and D such that **2**

$$\frac{x^3 + 2x^2 + 4x + 2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

ii. Hence evaluate **2**

$$\int_0^2 \frac{x^3 + 2x^2 + 4x + 2}{(x^2+1)(x^2+4)}\, dx$$

(c) Find all the roots of the equation **3**

$$18x^3 + 3x^2 - 28x + 12 = 0$$

given two of its roots are equal.

(d) Given the roots of the equation $x^3 + ax^2 + bx + c = 0$ form a geometric sequence, show that $\left(\dfrac{b}{a}\right)^3 = c$. **3**

(e) A curve has equation $ye^{-2x} = 2x + y^2$.

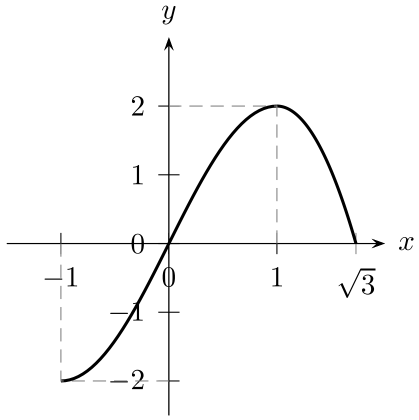
i. Find $\dfrac{dy}{dx}$ in terms of x and y. **2**

ii. Find the equation of the normal to the curve at $P(0, 1)$. **1**

Examination continues overleaf...

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) The graph of $y = f(x)$ is shown. On one-third page diagrams, sketch the following graphs, clearly indicating any asymptotes, intercepts with the axes and other important features.

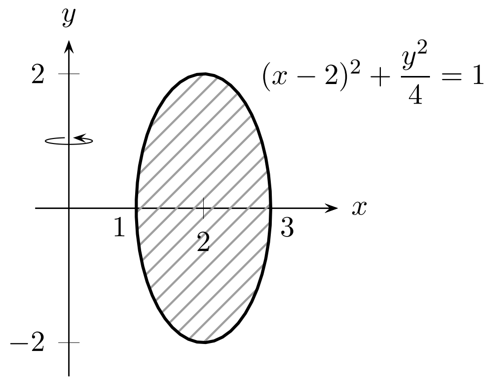


i. $y = |f(x)|$ **1**

ii. $y = \sqrt{f(x)}$ **2**

iii. $y = f^{-1}(x)$ **2**

(b) The region enclosed by the ellipse $(x-2)^2 + \dfrac{y^2}{4} = 1$ is rotated one complete revolution about the y axis.



i. Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by **2**

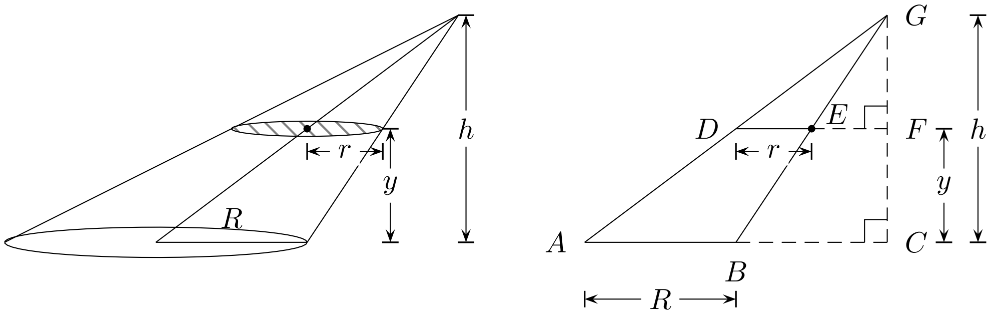
$$V = 8\pi \int_1^3 x\sqrt{1-(x-2)^2}\,dx$$

ii. Hence find the volume of the solid of revolution in simplest exact form. **4**

Question 13 continued overleaf...

Question 13 continued from previous page...

(c) The diagram shows an oblique cone of base radius R and perpendicular height h.



A horizontal cross section of the cone is taken at height y. This cross section is a circle of radius r as shaded in the diagram.

i. By considering the ratio of sides in two pairs of similar triangles, show that **1**

$$r = \left(\frac{h-y}{h}\right) R$$

ii. Show that the volume of the oblique cone is given by **3**

$$\frac{1}{3}\pi R^2 h$$

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) Find the following:

i. $\displaystyle\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx.$ **2**

ii. $\displaystyle\int \frac{dx}{\sqrt{6x - x^2}}.$ **2**

iii. $\displaystyle\int \ln\left(x^2 + 1\right)\, dx.$ **3**

(b) By using the substitution $x = 2 + \sin^2\theta$ for $0 \le \theta \le \frac{\pi}{2}$, evaluate **4**

$$\int_{\frac{9}{4}}^{\frac{5}{2}} \frac{dx}{\sqrt{(3-x)(x-2)}}$$

(c) i. Use the result $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ to show that **2**

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\, dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

ii. Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx.$ **2**

Question 15 (15 Marks) Commence a NEW booklet. **Marks**

(a) $P(a\cos\theta, b\sin\theta)$ is a point in the first quadrant on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $Q(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, where $0<b<a$.

i. Sketch the ellipse, hyperbola and their common auxiliary circle $x^2+y^2=a^2$ on the same diagram, showing the angle θ and related points P and Q. Show clearly how the positions of P and Q are determined by the value of θ, $0<\theta<\dfrac{\pi}{2}$. **2**

ii. Given the tangent to the ellipse at P is **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad \text{(Do NOT prove this)}$$

Deduce that this tangent cuts the x axis vertically below Q.

iii. Given the tangent to the hyperbola at Q has the equation **4**

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1 \qquad \text{(Do NOT prove this)}$$

show that this tangent, and the tangent to the ellipse at P intersect at $T\left(a, b\tan\frac{\theta}{2}\right)$. Show both tangents on the sketch.

iv. Without any further working, sketch a second diagram showing both curves, the common auxiliary circle, the points P and Q, and the corresponding tangents intersecting at T if $\frac{\pi}{2}<\theta<\pi$. **2**

(b) Given $I_n=\displaystyle\int_0^1 \frac{1}{(1+x^2)^n}\,dx$, $n\in\mathbb{Z}^+$

i. Show that **3**

$$I_{n+1}=\frac{2n-1}{2n}I_n+\frac{1}{n\times 2^{n+1}}$$

ii. Hence evaluate $\displaystyle\int_0^1 \frac{1}{(1+x^2)^3}\,dx$. **2**

Examination continues overleaf...

Question 16 (15 Marks) Commence a NEW booklet. **Marks**

(a) Two particles move in the same vertical line in a medium with resistance that varies directly with velocity. Particle A is projected vertically upwards from the ground with an initial velocity of u metres per second. At the same instant, Particle B is released from a height h metres above the ground.

i. For Particle A, show that the expression for its height x metres after a time of t seconds is given by **4**

$$x = \frac{g + ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$$

where g is acceleration due to gravity, and k is a constant.

ii. Assuming that the height at time t of Particle B is given by **3**

$$h - \frac{gt}{k} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2}$$

Prove that the particles are at the same height above the ground at time T, where

$$T = \frac{1}{k}\ln\left(\frac{u}{u - kh}\right)$$

(b) i. Use mathematical induction to show that if $f(x) = e^{2x}$, then the n-th derivative of $f(x)$ is **3**

$$f^{(n)}(x) = 2^n e^{2x}$$

for all positive integers n.

ii. A power series of a function can be expressed as **2**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots$$

where $2! = 2 \times 1$, $3! = 3 \times 2 \times 1$ etc.
Show that

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

iii. Given $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$, find **3**

$$\lim_{x\to 0} \frac{1 + 2x + 2x^2 - e^{2x}}{\tan x - x}$$

End of paper.

2019 Mathematics Extension 2 Assessment Task 4 STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

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2. Which topics did I need more help with, and what parts specifically?

- Q11 - Complex Numbers

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- Q12 - Polynomials, implicit differentiation

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- Q13 - Graphs & Curve Sketching, Volume (cylindrical shells & similar cross sections)

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- Q14, 15(b) - Integration

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- Q15(a) - Conic Sections

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- Q16 - Mechanics, Induction (3 Unit), Harder 3 Unit.

3. What other parts from the feedback session can I take away to refine my solutions for future reference?

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Sample Band E4 Responses

Section I

1. (D) **2.** (A) **3.** (A) **4.** (C) **5.** (D)
6. (D) **7.** (B) **8.** (C) **9.** (C) **10.** (D)

Section II

Question 11 (Bhamra)

(a) i. (1 mark)

$$\begin{aligned} z &= 3i \qquad w = 1+1 \\ |z-w| &= |(3i)-(1+i)| \\ &= |-1+2i| \\ &= \sqrt{1^2+2^2} \\ &= \sqrt{5} \end{aligned}$$

ii. (1 mark)

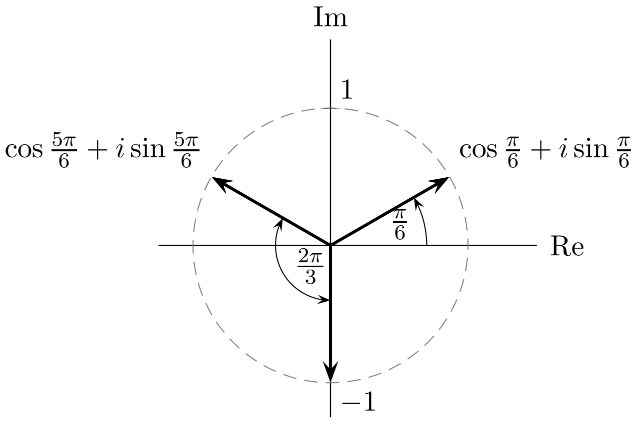
$$\begin{aligned} \frac{z}{w} &= \frac{3i}{1+i} \, {\scriptstyle \frac{\times(1-i)}{\times(1-i)}} \\ &= \frac{3+3i}{2} \end{aligned}$$

(b) i. (2 marks)

$$\begin{aligned} \frac{\sqrt{3}}{2}+\frac{1}{2}i &= \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} \\ \left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^3 &= \left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^3 \\ &= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} \\ &= i \end{aligned}$$

Hence $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ is a root of the equation $z^3 = i$, by using De Moivre's Theorem.

ii. (2 marks)



(c) i. (3 marks)

✓ [1] for both equations.

✓ [1] for each pair of correct x and y values.

- $|z-i| = 1$

$$\begin{aligned} |x+iy-i| &= 1 \\ |x+i(y-1)| &= 1 \\ \therefore x^2+(y-1)^2 &= 1 \end{aligned}$$

- $\text{Re}(z) = -\frac{1}{\sqrt{3}}\,\text{Im}(z)$:
letting $z = x+iy$,

$$\begin{aligned} x &= -\frac{1}{\sqrt{3}}y \\ \therefore y &= -\sqrt{3}x \end{aligned}$$

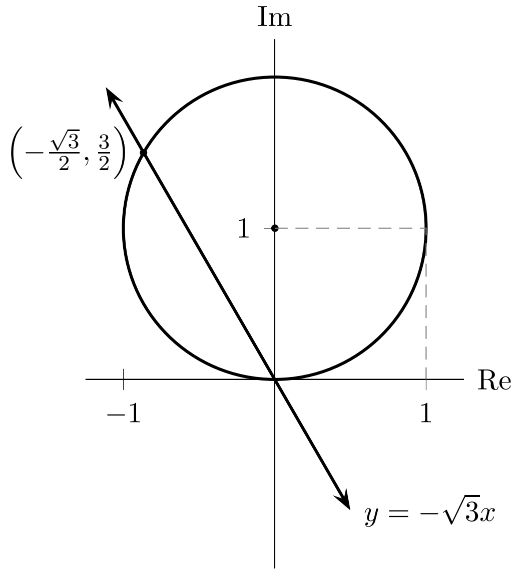
- Solving simultaneously,

$$\begin{aligned} x^2+\left(-\sqrt{3}x-1\right)^2 &= 1 \\ x^2+\left(\sqrt{3}x+1\right)^2 &= 1 \\ x^2+\left(3x^2+2x\sqrt{3}+\cancel{1}\right) &= \cancel{1} \\ 4x^2+2x\sqrt{3} &= 0 \\ 2x\left(2x+\sqrt{3}\right) &= 0 \\ \therefore x = 0 \text{ or } &-\frac{\sqrt{3}}{2} \end{aligned}$$

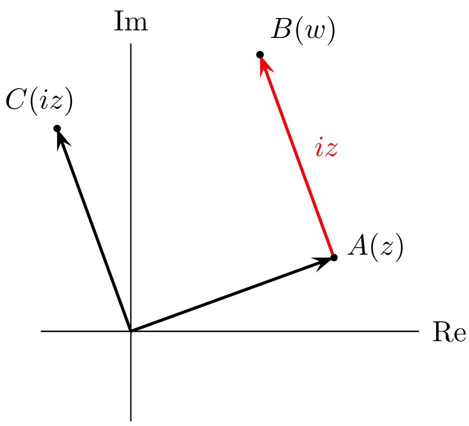
As $y = -\sqrt{3}x$, the corresponding y values are:

$$y = 0 \text{ or } \frac{3}{2}$$

ii. (1 mark)



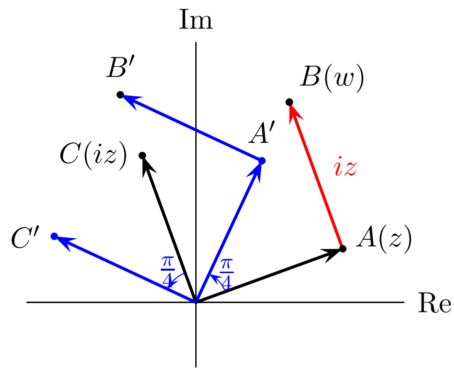
(d) i. (2 marks)



The vector $\overrightarrow{OB}$, represented by the complex number w is :

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$$
$$w = z + iz = (1+i)z$$

ii. (3 marks)



- Rotation by $\frac{\pi}{4}$ equates to multiplication by

$$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

- $A \to A'$:

$$A' : z\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

- $C \to C'$:

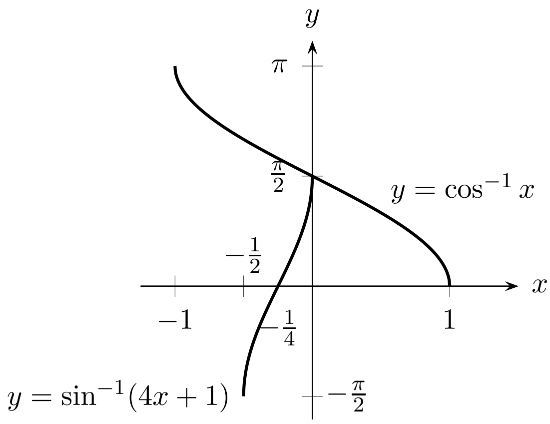
$$\begin{aligned} C' :\ & iz\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= z\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \end{aligned}$$

- $B \to B'$:

$$\begin{aligned} B' :\ & (1+i)z\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= z\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}\right) \\ &= \frac{2}{\sqrt{2}}iz \\ &= \sqrt{2}iz \end{aligned}$$

Question 12 (Bhamra)

(a) (2 marks)



$$\therefore \sin^{-1}(4x+1) = \cos^{-1}x \text{ when } x = 0 \text{ only}$$

(b) i. (2 marks)

$$\underbrace{\frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)}}_{\times(x^2+1)(x^2+4)} = \underbrace{\frac{Ax+B}{x^2+1}}_{\times(x^2+1)(x^2+4)} + \underbrace{\frac{Cx+D}{x^2+4}}_{\times(x^2+1)(x^2+4)}$$

$$\begin{aligned} & x^3+2x^2+4x+2 \\ \equiv\ & (Ax+B)\left(x^2+4\right) + (Cx+D)\left(x^2+1\right) \\ \equiv\ & Ax^3+4Ax+Bx^2+4B \\ & Cx^3+Cx+Dx^2+D \\ \equiv\ & (A+C)\,x^3+(B+D)x^2+(4A+C)x+(4B+D) \end{aligned}$$

- Equating coefficients of x^3 and x:

$$\begin{cases} A+C=1 & (1) \\ 4A+C=4 & (2) \end{cases}$$

(2) − (1):

$$\begin{aligned} 3A &= 3 \\ A &= 1 \\ \therefore C &= 0 \end{aligned}$$

- Equating coefficients of x^2 and x^0:

$$\begin{cases} B + D = 2 & (3) \\ 4B + D = 2 & (4) \end{cases}$$

(4) − (3):

$$\begin{aligned} 3B &= 0 \\ B &= 0 \\ \therefore D &= 2 \end{aligned}$$

Hence

$$\frac{x^3 + 2x^2 + 4x + 2}{(x^2+1)(x^2+4)} = \frac{x}{x^2+1} + \frac{2}{x^2+4}$$

ii. (2 marks)

$$\begin{aligned} &\int_0^2 \frac{x^3 + 2x^2 + 4x + 2}{(x^2+1)(x^2+4)}\,dx \\ &= \int_0^2 \left(\frac{x}{x^2+1} + \frac{2}{x^2+4}\right) dx \\ &= \left[\frac{1}{2}\ln\left(x^2+1\right) + \tan^{-1}\frac{x}{2}\right]_0^2 \\ &= \frac{1}{2}\ln 5 + \tan^{-1} 1 \\ &= \frac{1}{2}\ln 5 + \frac{\pi}{4} \end{aligned}$$

(c) (3 marks)

✓ [1] for finding the values of α.
✓ [1] for testing whether $\alpha = \frac{2}{3}$ is a root.
✓ [1] for the other root.

$$\begin{aligned} P(x) &= 18x^3 + 3x^2 - 28x + 12 \\ P'(x) &= 18\left(3x^2\right) + 6x - 28 \\ &= 27x^2 + 3x - 14 \end{aligned}$$

For a double root, $P(\alpha) = P'(\alpha) = 0$:

$$\begin{aligned} 27\alpha^2 + 3\alpha - 14 &= 0 \\ (3\alpha - 2)(9\alpha + 7) &= 0 \\ \therefore \alpha &= \frac{2}{3} \text{ or } -\frac{7}{9} \end{aligned}$$

Evaluate $P\left(\frac{2}{3}\right)$ to see whether this is the double root or not:

$$\begin{aligned} P\left(\frac{2}{3}\right) &= 18\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 - 28\left(\frac{2}{3}\right) + 12 \\ &= 0 \end{aligned}$$

Hence $\alpha = \frac{2}{3}$ is the double root. The full factorisation is

$$18x^3 + 3x^2 - 28x + 12 = (3x-2)(3x-2)(2x+3)$$

The final root is $x = -\frac{3}{2}$.

(d) (3 marks)

✓ [1] for writing Vieta's formulae.
✓ [1] for substantial progress to find α.
✓ [1] for showing required result.

$$x^3 + ax^2 + bx + c = 0$$

Let the roots be α, $\frac{\alpha}{r}$ and αr, where r is the common ratio.

- Product of roots:

$$\begin{aligned} \alpha \times \frac{\alpha}{r} \times \alpha r &= -c \\ \therefore \alpha^3 &= -c \qquad (\dagger) \end{aligned}$$

- Sum of roots:

$$\begin{aligned} \alpha + \alpha r + \frac{\alpha}{r} &= -a \\ \alpha\left(1 + r + \frac{1}{r}\right) &= -a \qquad (\ddagger) \end{aligned}$$

- Pairs of roots:

$$\begin{aligned} \alpha r\left(\frac{\alpha}{r}\right) + \alpha^2 r + \alpha^2\left(\frac{1}{r}\right) &= b \\ \alpha^2\left(1 + r + \frac{1}{r}\right) &= b \qquad (\S) \end{aligned}$$

Evaluating (§) ÷ (‡):

$$\alpha = -\frac{b}{a}$$

Taking the cube of both sides, and using (†):

$$\begin{aligned} \alpha^3 = -\frac{b^3}{a^3} &= -c \\ \therefore \left(\frac{b}{a}\right)^3 &= c \end{aligned}$$

(e) i. (2 marks)

$$\frac{d}{dx}\left(ye^{-2x}\right) = \frac{d}{dx}\left(2x + y^2\right)$$

Using the product rule,

$$\left|\begin{array}{ll} u = y & v = e^{-2x} \\ u' = \dfrac{dy}{dx} & v' = -2e^{-2x} \end{array}\right.$$

$$\therefore -2ye^{-2x} + e^{-2x}\frac{dy}{dx} = 2 + 2y\frac{dy}{dx}$$

$$\left(e^{-2x} - 2y\right)\frac{dy}{dx} = 2 + 2ye^{-2x}$$

$$\frac{dy}{dx} = \frac{2 + 2ye^{-2x}}{e^{-2x} - 2y}$$

ii. (1 mark)
At $x = 0$ and $y = 1$:

$$\frac{dy}{dx} = \frac{2 + 2e^0}{e^0 - 2} = -4$$

$$\therefore m_\perp = \frac{1}{4}$$

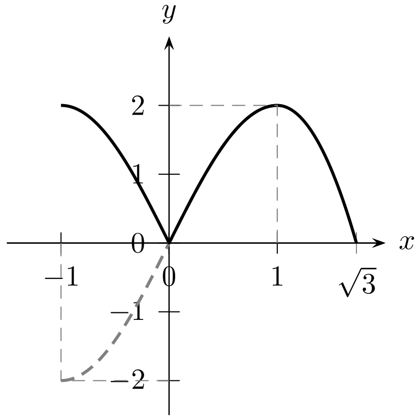
Applying the point-gradient formula,

$$y - 1 = \frac{1}{4}(x - 0)$$

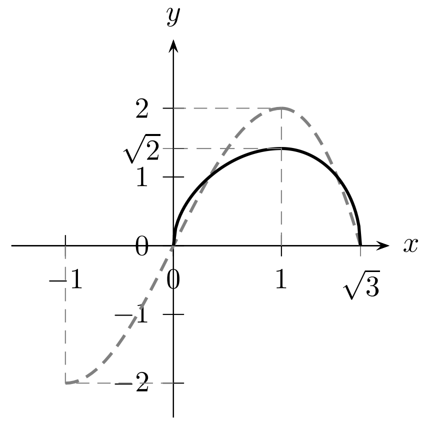
$$y = \frac{1}{4}x + 1$$

Question 13 (Bhamra)

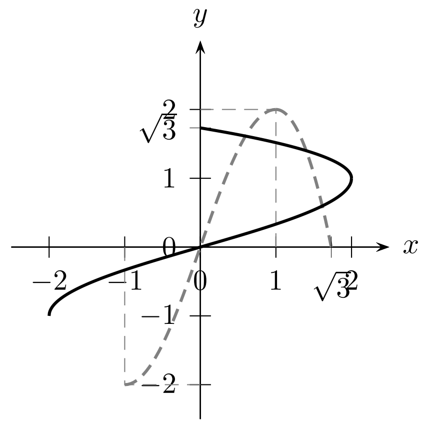
(a) i. (1 mark) $y = |f(x)|$



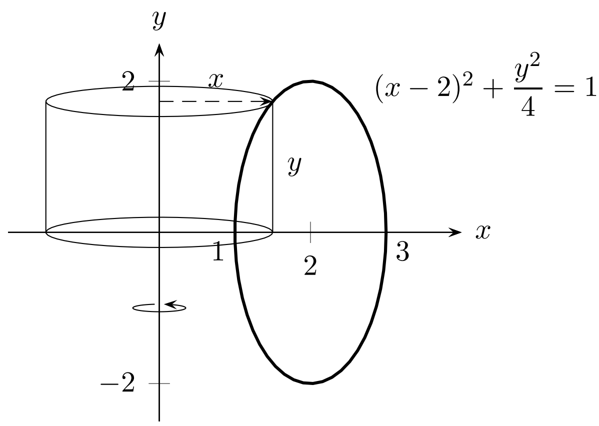
ii. (2 marks) $y = \sqrt{f(x)}$



iii. (2 marks) $y = f^{-1}(x)$



(b) i. (2 marks)



- Surface area of the cylinder element:

$$SA = 2\pi rh = 2\pi xy$$

- Change subject of the ellipse to y, and take only positive root to obtain the top half:

$$\begin{aligned} (x-2)^2 + \frac{y^2}{4} &= 1 \\ \frac{y^2}{4} &= 1-(x-2)^2 \\ y &= 2\sqrt{1-(x-2)^2} \\ \therefore SA &= 2\pi \times x \times 2\sqrt{1-(x-2)^2} \end{aligned}$$

- Sum of the areas produces the volume, where δx is the thickness of the area. Also, double the volume to include the bottom half of the ellipse:

$$\begin{aligned} V &= 2\lim_{\delta x\to 0}\sum_{x=1}^{x=3} SA\,\delta x \\ &= 2\int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx \\ &= 8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx \end{aligned}$$

ii. (4 marks)

- ✓ [1] for using the correct substitution to obtain the new limits.
- ✓ [1] for the new volume expression with revised integral in u.
- ✓ [1] for correct integration.
- ✓ [1] for final result.

$$V = 8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$$

Let $u = x-2$:

$$\begin{array}{|ll} x=1 & u = 1-2 = -1 \\ x=3 & u = 3-2 = 1 \end{array}$$

$$du = dx$$

$$\begin{aligned} V &= 8\pi\int_{u=-1}^{u=1}(u+2)\sqrt{1-u^2}\,du \\ &= 8\pi\int_{-1}^1 u\sqrt{1-u^2}\,du \;(\to 0) \\ &\quad + 8\pi\int_{-1}^1 2\sqrt{1-u^2}\,du \end{aligned}$$

The first integral has an odd intergrand over a balanced period, hence the outcome is zero. The other integral being the area of a semi circle of radius 1, from $x=-1$ to $x=1$. Hence

$$\begin{aligned} V &= 8\pi\left(0 + 2\times\frac{1}{2}\pi\times 1^2\right) \\ &= 8\pi^2 \end{aligned}$$

(c) i. (1 mark) Apply the similarity ratio for $\triangle GDE \,|||\, \triangle GAB$ and also $\triangle GEF \,|||\, \triangle GBC$:

$$\begin{aligned} \frac{r}{R} &= \frac{GE}{GB} \\ \frac{GE}{GB} &= \frac{h-y}{h} \end{aligned}$$

Equating,

$$\begin{aligned} \frac{r}{R} &= \frac{h-y}{h} \\ r &= \left(\frac{h-y}{h}\right)\times R \end{aligned}$$

ii. (3 marks)

- ✓ [1] for correct expression for δV.
- ✓ [1] for substantial progress.
- ✓ [1] for showing the final result required.

Let the thickness of each slice be δy.

$$\begin{aligned} \delta V &= \pi r^2 \delta y \\ &= \pi\left(\frac{h-y}{h}\right)^2 R^2\delta y \\ V &= \lim_{\delta y\to 0}\sum_{y=0}^{y=h}\pi\left(\frac{h-y}{h}\right)^2 R^2\delta y \\ &= \pi\int_0^h\left(\frac{h-y}{h}\right)^2 R^2\,dy \\ &= -\frac{\pi R^2}{3h^2}\Big[(h-y)^3\Big]_0^h \\ &= \frac{1}{3}\pi R^2 h \end{aligned}$$

Question 14 (Lam)

(a) i. (2 marks)

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx$$

Make the substitution $u = e^x + e^{-x}$:

$$\begin{aligned}\frac{du}{dx} &= e^x - e^{-x} \\ \therefore du &= \left(e^x - e^{-x}\right)\, dx \\ \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx &= \int \frac{du}{u^2} = \int u^{-2}\, du \\ &= -u^{-1} + C \\ &= -\frac{1}{e^x + e^{-x}} + C\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}&\int \frac{dx}{\sqrt{6x - x^2}} \\ &= \int \frac{dx}{\sqrt{-(x^2 - 6x + 9) + 9}} \\ &= \int \frac{dx}{\sqrt{9 - (x-3)^2}} \\ &= \sin^{-1}\left(\frac{x-3}{3}\right) + C\end{aligned}$$

Alternatively, let $x = 6\sin^2\theta$ to obtain $2\sin^{-1}\left(\frac{\sqrt{x}}{6}\right) + C$. This only differs by a vertical shift compared to the preferred solution.

iii. (3 marks)

- ✓ [1] Uses integration by parts with phantom term successfully.
- ✓ [1] For manipulation and substantial progress towards answer.
- ✓ [1] for correctly evaluating the $\int v\, du$ expression.

$$\int \ln\left(x^2+1\right)\, dx$$

Use integration by parts and insert 'phantom' term:

$$\left|\begin{array}{ll} u = \ln\left(x^2+1\right) & v = x \\ du = \dfrac{2x}{x^2+1} & dv = 1 \end{array}\right.$$

$$\begin{aligned}&\int 1 \times \ln\left(x^2+1\right)\, dx \\ &= uv - \int v\, du \\ &= x\ln\left(x^2+1\right) - \int \frac{2x^2}{x^2+1}\, dx \\ &= x\ln\left(x^2+1\right) - 2\int \frac{x^2+1-1}{x^2+1}\, dx \\ &= x\ln\left(x^2+1\right) - 2\int \left(1 - \frac{1}{x^2+1}\right)\, dx \\ &= x\ln\left(x^2+1\right) - 2\left(x - \tan^{-1} x\right) + C\end{aligned}$$

(b) (4 marks)

- ✓ [1] for using the substitution correctly to change the x expression into the θ expression in the integrand.
- ✓ [1] for using the substitution correctly to change the x integration limits into θ limits.
- ✓ [1] for correctly changing differential from dx into $d\theta$.
- ✓ [1] for final result required.

$$\int_{\frac{9}{4}}^{\frac{5}{2}} \frac{dx}{\sqrt{(3-x)(x-2)}}$$

Let $x = 2 + \sin^2\theta$,

$$\begin{aligned}3 - x &= 3 - \left(2 + \sin^2\theta\right) \\ &= 1 - \sin^2\theta \\ &= \cos^2\theta\end{aligned}$$

$$\begin{aligned}x - 2 &= \left(2 + \sin^2\theta\right) - 2 \\ &= \sin^2\theta\end{aligned}$$

Also,

$$\begin{aligned}\text{When } x = \frac{9}{4}, \quad \sin^2\theta &= \frac{1}{4}\\ \sin\theta &= \frac{1}{2}\\ &= \frac{\pi}{6}\\ \text{When } x = \frac{5}{2}, \quad \sin^2\theta &= \frac{1}{2}\\ \sin\theta &= \frac{1}{\sqrt{2}}\\ \theta &= \frac{\pi}{4}\end{aligned}$$

Differentiating,

$$\begin{aligned}\frac{dx}{d\theta} &= 2\sin\theta\cos\theta\\ \therefore dx &= (2\sin\theta\cos\theta)\ d\theta\end{aligned}$$

Make the substitution and perform integration:

$$\begin{aligned}\int_{\frac{9}{4}}^{\frac{5}{2}} \frac{dx}{\sqrt{(3-x)(x-2)}} &= \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{4}} \frac{2\cancel{\sin\theta\cos\theta}\ d\theta}{\cancel{\sqrt{\sin^2\theta\cos^2\theta}}}\\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\ d\theta\\ &= 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \frac{\pi}{6}\end{aligned}$$

(c) i. (2 marks)

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan x}{1+\tan x}+\frac{1-\tan x}{1+\tan x}\right)\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)\ dx\end{aligned}$$

ii. (2 marks)

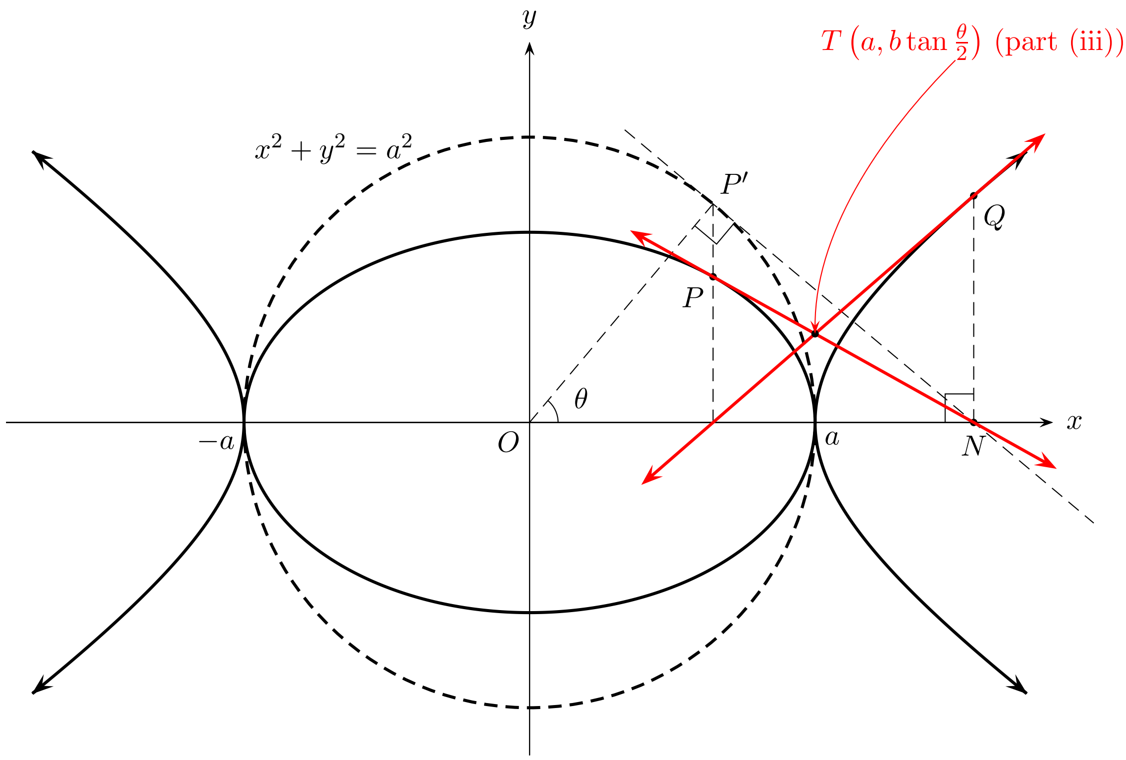
$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)\ dx\\ &= \int_0^{\frac{\pi}{4}} (\ln 2 - \ln(1+\tan x))\ dx\\ &= \int_0^{\frac{\pi}{4}} \ln 2\ dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\ dx\end{aligned}$$

Rearranging,

$$\begin{aligned}2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\ dx &= \int_0^{\frac{\pi}{4}} \ln 2\ dx\\ &= \Big[x\ln 2\Big]_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{4}\ln 2\\ \therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\ dx &= \frac{\pi}{8}\ln 2\end{aligned}$$

Question 15(Lam)

(a) i. (2 marks)



ii. (2 marks)

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

At $y=0$,

$$x=\frac{a}{\cos\theta}=a\sec\theta$$

As Q has x coordinates $(a\sec\theta, b\tan\theta)$, the tangent to the ellipse cuts directly below Q.

iii. (4 marks)

✓ [1] each for showing the x and y coordinates of T.

✓ [1] for each correctly drawn tangent (both must intersect at $x=a$)

$$\begin{cases}\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1 & (1)\\ \dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1 & (2)\end{cases}$$

Divide (1) by $\cos\theta$:

$$\frac{x}{a}+\frac{y\tan\theta}{b}=\sec\theta \qquad (\dagger)$$

$(\dagger)+(2)$:

$$\frac{x}{a}\cancel{(1+\sec\theta)}=\cancel{1+\sec\theta}$$
$$\therefore x=a$$

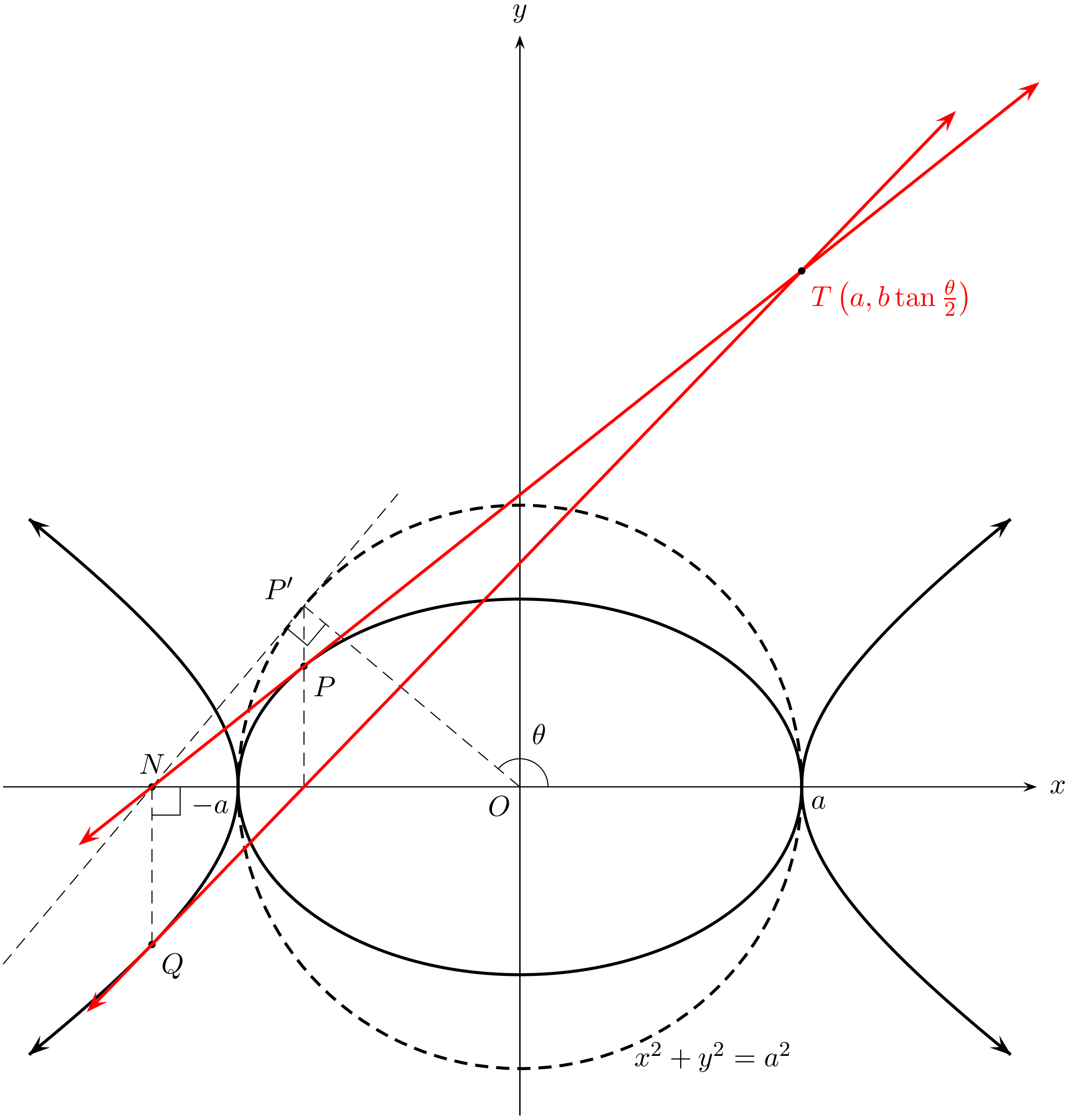
Substitute $x = a$ into (1):

$$\begin{aligned}\cos\theta + \frac{y\sin\theta}{b} &= 1 \\ y &= \frac{b}{\sin\theta}(1-\cos\theta) \\ &= b\left(\frac{1-\cos\theta}{\sin\theta}\right)\end{aligned}$$

As $\cos\theta = \cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\frac{\theta}{2}$ and $\sin\theta = \sin 2\left(\frac{\theta}{2}\right)$,

$$\begin{aligned}y &= b\left(\frac{1-\left(1-2\sin^2\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \\ &= b\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \\ &= b\tan\frac{\theta}{2}\end{aligned}$$

iv. (2 marks)



(b) i. (3 marks)

✓ [1] for successfully using integration by parts with the phantom term.

✓ [1] for further substantial progress in removing integrals and replacing with I_n of a lower order.

✓ [1] for final result.

$$I_n = \int \frac{1}{(1+x^2)^n}\, dx$$

Applying integration by parts with phantom term u being the current integrand and $dv = 1$:

$$\left|\begin{array}{ll} u = \left(1+x^2\right)^{-n} & v = x \\ du = -n\left(1+x^2\right)^{-n-1} \times 2x & dv = 1 \end{array}\right.$$

$$\begin{aligned} I_n &= \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 -2nx^2\left(1+x^2\right)^{-n-1}\, dx \\ &= 2^{-n} + 2n\int_0^1 \left(1+x^2-1\right)\left(1+x^2\right)^{-n-1}\, dx \\ &= 2^{-n} + 2n\int_0^1 \left(\left(1+x^2\right)^{-n} - \left(1+x^2\right)^{-n-1}\right)\, dx \\ &= 2^{-n} + 2n\left(\underbrace{\int_0^1 \frac{1}{(1+x^2)^n}\, dx}_{=I_n} - \underbrace{\int_0^1 \frac{1}{(1+x^2)^{n+1}}\, dx}_{=I_{n+1}}\right) \\ &= 2^{-n} + 2nI_n - 2nI_{n+1} \end{aligned}$$

$$\therefore 2nI_n - I_n = 2nI_{n+1} - 2^{-n}$$

$$2nI_{n+1} = (2n-1)I_n + 2^{-n}$$

$$I_{n+1} = \frac{2n-1}{2n}I_n + \frac{1}{2n\times 2^n} = \frac{2n-1}{2n}I_n + \frac{1}{n\times 2^{n+1}}$$

ii. (2 marks)

$$I_1 = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$$

$$\begin{aligned} I_2 &= \frac{2(1)-1}{2(1)}I_1 + \frac{1}{1\times 2^2} \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4} \\ &= \frac{\pi}{8} + \frac{1}{4} \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{2(2)-1}{2(2)}I_2 + \frac{1}{2\times 2^3} \\ &= \frac{3}{4}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{16} \\ &= \frac{3\pi}{32} + \frac{1}{4} \end{aligned}$$

Question 16(Lam)

(a) i. (4 marks)

✓ [1] for correctly evaluating the original integral in terms of dv and dt.

✓ [1] for obtaining t in terms of v expression.

✓ [1] for correctly evaluating the second integral

✓ [1] for final result required.

As Particle A is projected upwards:

- Force due to gravity: $-mg$.
- Force due to air resistance: $-mkv$.

$$\begin{aligned} F = m\ddot{x} &= -mg - mkv \\ \therefore \ddot{x} &= -g - kv \\ \frac{dv}{dt} &= -(g + kv) \\ \frac{dv}{g + kv} &= -dt \end{aligned}$$

Integrating,

$$\begin{aligned} \frac{1}{k}\int \frac{k\,dv}{g + kv} &= -\int dt \\ \frac{1}{k}\ln(g + kv) &= -t + C_1 \end{aligned}$$

At $t = 0$, $v = u$:

$$\begin{aligned} C_1 &= \frac{1}{k}\ln(g + ku) \\ \therefore t &= \frac{1}{k}\ln(g + ku) - \frac{1}{k}\ln(g + kv) \\ &= \frac{1}{k}\ln\left(\frac{g + ku}{g + kv}\right) \end{aligned}$$

(Use log law to switch the fraction's numerator/denominator, and exponentiate:)

$$\begin{aligned} e^{-kt} &= \frac{g + kv}{g + ku} \\ g + kv &= (g + ku)e^{-kt} \\ v = \frac{dx}{dt} &= \frac{(g + ku)e^{-kt}}{k} - \frac{g}{k} \\ x &= \frac{-\frac{1}{k}(g + ku)e^{-kt}}{k} - \frac{gt}{k} + C_2 \end{aligned}$$

When $t = 0$, $x = 0$:

$$\begin{aligned} C_2 &= \frac{g + ku}{k^2} \\ \therefore x &= \frac{g + ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k} \end{aligned}$$

ii. (3 marks) The particles meet when

$$\frac{g+ku}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}=h-\frac{gt}{k}-\frac{ge^{-kt}}{k^2}+\frac{g}{k^2}$$
$$\cancel{\frac{g}{k^2}}-\cancel{\frac{ge^{-kt}}{k^2}}+\frac{u}{k}-\frac{ue^{-kt}}{k}-\cancel{\frac{gt}{k}}=h-\cancel{\frac{gt}{k}}-\cancel{\frac{ge^{-kt}}{k^2}}+\cancel{\frac{g}{k^2}}$$
$$\therefore \frac{u}{k}\left(1-e^{-kt}\right)=h$$
$$1-e^{-kt}=\frac{hk}{u}$$
$$-kt=\ln\left(\frac{u-kh}{u}\right)$$
$$t=\frac{1}{k}\ln\left(\frac{u}{u-hk}\right)$$

(b) i. (3 marks) Let $P(n)$ be the proposition:

$$P(n): f(x)=e^{2x} \text{ then } f^{(n)}(x)=2^n e^{2x}$$

- Base case: $P(1)$

$$f(x)=e^{2x}$$
$$f'(x)=2e^{2x}=2^1e^{2x}$$

Hence $P(1)$ is true.

- Inductive hypothesis: assume $P(k)$ is true for some $k\in\mathbb{Z}^+$, i.e.

$$P(k): f(x)=e^{2x} \text{ then } f^{(k)}(x)=2^k e^{2x}$$

- Examine $P(k+1)$:

$$f''(x)=2\times f'(x)=2^2e^{2x}$$
$$f^{(3)}(x)=2\times f''(x)=2^3e^{2x}$$
$$f^{(k+1)}(x)=\frac{d}{dx}\left(f^{(k)}(x)\right)=\frac{d}{dx}\left(2^ke^{2x}\right)$$
$$=2^k\times 2e^{2x}=2^{k+1}e^{2x}$$

Hence $P(k+1)$ is also true and $P(n)$ is true by induction.

ii. (2 marks)

- Let $f(x)=e^{2x}$.
- $f(0)=e^0=1$

Using results from above,

- $f'(0)=2^1e^0=2$
- $f''(0)=2^2e^0=2^2$
- $f^{(3)}(0)=2^3e^0=2^3$

Hence

$$f(x)=e^{2x}$$
$$=1+2x+\frac{x^2}{2!}\times 2^2+\frac{x^3}{3!}\times 2^3+\cdots$$
$$=1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\cdots$$

iii. (3 marks)

✓ [1] for correct cancellation of the relevant polynomial terms after substitution.

✓ [1] for correct factorisation.

✓ [1] for taking limit to obtain -4.

$$\begin{aligned}\frac{1+2x+2x^2-e^{2x}}{\tan x - x} &= \frac{\cancel{1+2x+2x^2} - \left(\cancel{1+2x+\frac{4x^2}{2!}} + \frac{8x^3}{3!} + \cdots\right)}{\left(\cancel{x} + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right) - \cancel{x}} \\ &= \frac{-\frac{8x^3}{6} - \frac{16x^4}{24} - \cdots}{\frac{x^3}{3} + \frac{2x^5}{15} + \cdots} \\ &= \frac{\cancel{x^3}\left(-\frac{4}{3} - \frac{2x}{3} - \cdots\right)}{\cancel{x^3}\left(\frac{1}{3} + \frac{2x^2}{15} + \cdots\right)}\end{aligned}$$

As $x \to 0$,

$$\frac{1+2x+2x^2-e^{2x}}{\tan x - x} = \frac{\left(-\frac{4}{3} - \frac{2x}{3} - \cdots\right)}{\left(\frac{1}{3} + \frac{2x^2}{15} + \cdots\right)} = \frac{-\frac{4}{3}}{\frac{1}{3}} = -4$$