Question 1 (3,3,3,3,3 marks)

Find

a) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{\tan^2 x + 4}$

b) $\displaystyle\int_0^3 xe^x \, dx$

c) $\displaystyle\int \frac{dx}{1 + \cos 2x}$ Use the substitution $t = \tan x$

d) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 8x + 25}}$

e) Use partial fractions to find $\displaystyle\int \frac{10}{x^2(x+5)} \, dx$

START A NEW PAGE

Question 2 (2, 2 + 1 + 2, 3, 2 + 2 + 1 marks)

a) Express $\dfrac{(3-i)^2}{2+i}$ in the form $a + ib$

b) Express $z = 2 - 2\sqrt{3}i$ in modulus argument form.

Hence i) evaluate z^5

ii) solve $z^2 = 2 - 2\sqrt{3}i$

c) Sketch on an argand diagram the locus $z^2 - \bar{z}^2 = 16i$

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d) If z satisfies $|z - 2i| = 1$ and the point P represents z on an Argand Diagram.

i) Sketch the locus of P as z varies.

ii) Find the maximum and minimum values of $\arg z$, where $-\pi < \arg z < \pi$.

iii) Find the value of z when $\arg z$ takes this minimum value, and mark on your sketch the position P_0 for this value of z.

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Question 3 (1 + 5 + 2 + 2, 3, 2 marks)

a) A function $y = f(x)$ is given in parametric form by

$$x = \tan\theta \quad \text{and} \quad y = 4\sin 2\theta \qquad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

i) Show that $y = \dfrac{8x}{1+x^2}$

ii) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any stationary points and the equations of any asymptotes.

iii) Use the graph of $y = f(x)$ to sketch on separate axes the graphs of:

1. $y = \{f(x)\}^2$

2. $y^2 = f(x)$

b) Show that the curves $x^2 + cxy + y^2 = c + 2$ ($c \neq -2$) for various values of c touch at the point (1,1)

c) A tennis match between two players consists of a number of sets. The match continues until the player who first wins three sets wins it. Whenever Lance and Olaf play tennis against each other, the probability that Lance will win a set is $\frac{2}{3}$ and the probability that Olaf wins a set is $\frac{1}{3}$. If Lance and Olaf play a tennis match, show that the probability that Lance wins the match is $\frac{64}{81}$.

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Question 4 (2, 2, 3 + 1, 2 + 2, 1 + 2 marks)

a) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

i) Find the equation of the tangent to the ellipse at P($a\cos\theta$, $b\sin\theta$).

ii) Find the equation of the normal to the ellipse at P($a\cos\theta$, $b\sin\theta$)

iii) 1. The tangent at P cuts the X and Y axes at A and B respectively.

Find the area of ΔAOB.

2. Deduce the minimum area of ΔAOB.

iv) The normal cuts the X axis at N. If e is the eccentricity of the ellipse, show that:

1. $OA.ON = a^2e^2$

2. $\left(\frac{PA}{PN}\right)^2 = \frac{\tan^2\theta}{1-e^2}$

b) A local council has 5 Liberals, 6 Labour and 2 Green members from whom a committee of 5 is to be selected.

i) What is the probability of 1 Green member being on the committee?

ii) What is the probability that the Liberals have a majority on the committee?

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Question 5 (3, 2 + 1, 3, 3 + 3 marks)

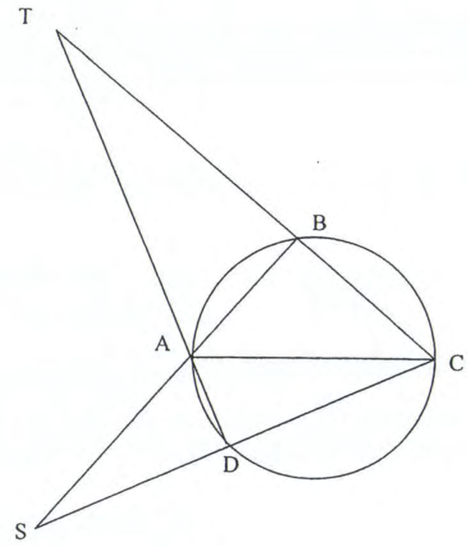
a) Show that $1-i$ is a zero of $P(x)=x^3-7x^2+12x-10$. Hence factor $P(x)$.

b) Find all the solutions to $z^5-1=0$ and show them on an argand diagram.

c) Consider the equation $x^3-6x^2+ax+10=0$ which has real roots that form an arithmetic sequence.

Find the three roots of the equation and hence solve for a

d) ABCD is a cyclic quadrilateral such that AC is a diameter. DC meets AB at S and BC meets AD at T.



Copy the diagram onto your paper and

i) Prove BTSD is a cyclic quadrilateral **and** that ST is a diameter.

ii) The tangent at A crosses BT at X and SD at Y.
Prove this tangent is parallel to ST.

START A NEW PAGE

Question 6 (5,5,5 marks)

a) The area between the curve $y=x(4-x)$ and the positive x axis is rotated about the y axis. Using the method of cylindrical shells, find the volume of the solid so formed.

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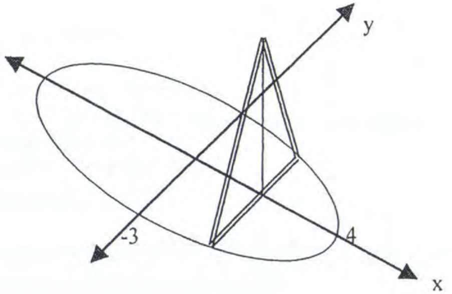
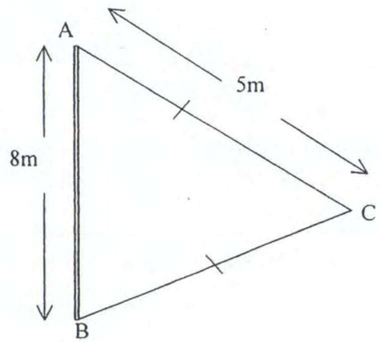
b) The base of a solid is the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. Each section of the solid by planes perpendicular to the x axis is an isosceles triangle whose height is twice its base. Find the volume of the solid so formed.



c) In the diagram below ABC is a triangular sheet of thin paper which is being rolled up around a circular rod AB of radius 0·01m at a rate of three turns per second. Show that the rate at which the **area** ABC is diminishing at the end of the first second, neglecting the increase of radius of the roller due to rolled up paper, is $\frac{12\pi}{25}(1-\frac{\pi}{50})$ m^2/s



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Question 7 (2 + 3 + 3, 3, 2, 2 marks)

a) A particle P_1 is projected from the origin with velocity V at an angle of elevation θ.

i) Assuming the usual equations of motion, show that the particle reaches a maximum height of $\frac{V^2\sin^2\theta}{2g}$.

ii) A second particle P_2 is projected from the origin with velocity $\frac{3V}{2}$ at an angle $\frac{\theta}{2}$ to the horizontal. The two particles reach the same maximum height.

1) Show that $\theta = \cos^{-1}(\frac{1}{8})$

2) Do the two particles take the same time to reach this maximum height? Justify your answer.

b) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\, dx \quad n \geq 0$

Show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2} \quad n \geq 2$

c) i) Show that

$$1+(1+x)+(1+x)^2+(1+x)^3+\ldots\ldots+(1+x)^n = \frac{(1+x)^{n+1}-1}{x}$$

ii) Hence by considering the coefficient of x^r on both sides of the identity, show that

$${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \ldots\ldots\ldots\, {}^rC_r = {}^{n+1}C_{r+1}$$

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Question 8 (6, 2 + 2, 3 + 2 marks)

a) The sequence known as the Fibonacci Sequence is defined by

$U_1 = 1,\ U_2 = 2$ and $U_n = U_{n-1} + U_{n-2}$ for $n > 2$.

Use the principal of Mathematical Induction to prove that

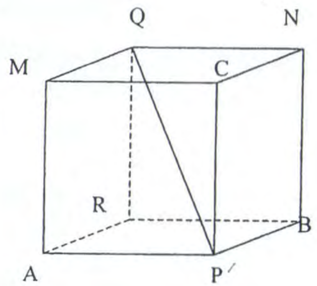
$U_1 + U_2 + U_3 + \ldots\ldots + U_n = U_{n+2} - 2$

b) If $a \geq 0$ and $b \geq 0$

i) Prove $\dfrac{a+b}{2} \geq \sqrt{ab}$

ii) Hence show that $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

c) In the diagram below, the diagonal PQ of the rectangular prism makes angles, α, β and γ with edges PA, PB and PC respectively.
(ie. $\angle QPA = \alpha$, $\angle QPB = \beta$ and $\angle QPC = \gamma$)



i) Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

ii) Consider the above result.
If the angles α, β, and γ are in the ratio such that $\alpha : \beta : \gamma = 1 : 1 : 2$, is it possible for the prism to be constructed?

Justify your answer.

{Hint: Let α, α and 2α be the angles and solve for α}

END OF EXAMINATION

This page may be removed for your convenience.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0.$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln x\ \{x + \sqrt{(x^2-a^2)}\},\ |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln x\ \{x + \sqrt{(x^2+a^2)}\}.$$

NOTE: $\ln x = \log_e x,\ x > 0.$

) let $u = \tan x$

$du = \sec^2 x\, dx$

$\int_0^{\pi/4} f(x) \Rightarrow \int_0^1 \frac{du}{u^2+4}$

$= \frac{1}{2}\left[\tan^{-1}\frac{u}{2}\right]_0^1$

$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$

) $\int_0^3 x e^x \, du$

$= \left[xe^x\right]_0^3 - \int_0^3 e^x dx$

$= 3e^3 - \left[e^x\right]_0^3$

$= 2e^3 + 1$

) $t = \tan\theta$

$d\theta = \frac{dt}{1+t^2}$

$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2}$

$= \int \frac{dt}{1+t^2+1-t^2}$

$= \int \frac{dt}{2}$

$= \frac{t}{2} = \frac{\tan x + c}{2}$

d) $\int \frac{dx}{\sqrt{x^2-8x+16+9}}$

$\int \frac{dx}{\sqrt{(x-4)^2+3^2}}$

let $u = x-4$

$du = dx$

$\int \frac{du}{\sqrt{u^2+3^2}}$

$= \log\left(u+\sqrt{u^2+3^2}\right)$

$= \log\left(x-4+\sqrt{(x-4)^2+9}\right) + C$

e)

$\int \frac{10\, dx}{x^2(x+5)} = \int \frac{a}{x+5} + \frac{bx+c}{x^2}\, dx$

$\therefore ax^2 + (bx+c)(x+5) = 10$

$x = 0 \Rightarrow 5c = 10 \therefore c = 2$

$x = -5 \Rightarrow 25a = 10 \therefore a = \frac{2}{5}$

$x = 1 \Rightarrow a + (b+c)6 = 10$

$\therefore \frac{2}{5} + 6b + 12 = 10$

$\therefore b = -\frac{2}{5}$

$\int \frac{2}{5}\times\frac{1}{x+5} - \frac{2}{5x} + 2x^{-2}\, dx = \frac{2}{5}\ln(x+5) - \frac{2}{5}\ln x - \frac{2}{x} + c$

16 Q2

a) $\frac{(3-i)^2}{2+i} = \frac{8-6i}{2+i}$ $\boxed{2}$

$= \frac{8-6i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{16-8i-12i-6}{5}$

$= 2-4i$

b) $z = 2-2\sqrt{3}i$ $\boxed{2}$

$|z| = 4$

$\arg z = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)$

$= \frac{-\pi}{3}$

$\therefore z = 4\,\text{cis}\left(\frac{-\pi}{3}\right)$

i) $z^5 = 4^5\,\text{cis}\left(\frac{-5\pi}{3}\right)$ $\boxed{1}$

ii) $z^2 = 4\,\text{cis}\left(\frac{-\pi}{3}\right)$ $\boxed{2}$

$z = \pm 2\,\text{cis}\left(\frac{-\pi}{6}\right)$

$= \pm 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

c) $z^2 - \bar{z}^2 = 16i$ $\boxed{3}$

let $z = x+iy$.

$z^2 = (x^2 + 2xyi - y^2)$

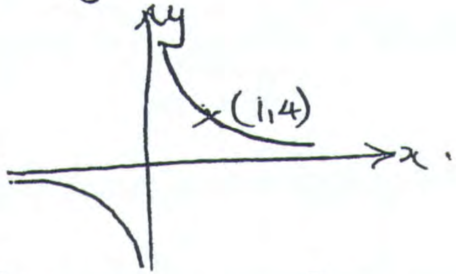
$\bar{z}^2 = x^2 - 2xyi - y^2$

$\therefore z^2 + \bar{z}^2$

$\Rightarrow x^2 + 2xyi - y^2 - (x^2 - 2xyi - y^2) = 16i$

$4xyi = 16i$

$xy = 4$



d)

$\boxed{2}$

ii) Max arg $= \frac{2\pi}{3}$ $\boxed{2}$

Min arg $= \frac{\pi}{3}$

ii) $P_0 = \sqrt{3}\,\text{cis}\frac{\pi}{3}$ $\boxed{1}$

$= \frac{\sqrt{3}}{2}, \frac{3i}{2}$

Q3

a) $x = \tan\theta$

$\boxed{1}$

$\therefore y = 4\sin 2\theta$

$= 8\sin\theta\cos\theta$

$= 8 \times \frac{x}{\sqrt{x^2+1}} \times \frac{1}{\sqrt{x^2+1}}$

$y = \frac{8x}{x^2+1}$.

i) $\frac{dy}{dx} = \frac{8(1+x^2) - 16x^2}{(1+x^2)^2}$ $\boxed{5}$

$= \frac{8-8x^2}{(1+x^2)^2} = \frac{8(1-x^2)}{(1+x^2)^2}$

$y' = 0 \Rightarrow x = \pm 1$

test

x	-1^-	-1	-1^+
y'	$-$	0	$+$

$\therefore$ Min at $x=-1$, $y=-4$

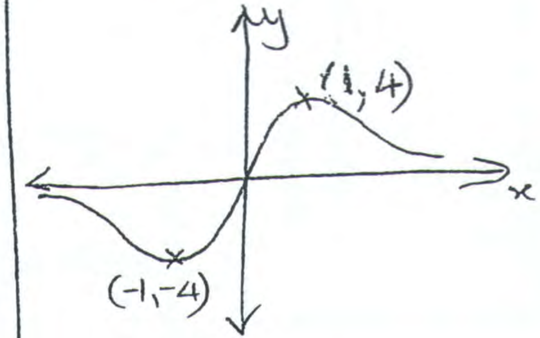
x	1^-	1	1^+
y'	$+$	0	$-$

$\therefore$ Max at $x=1$, $y=4$

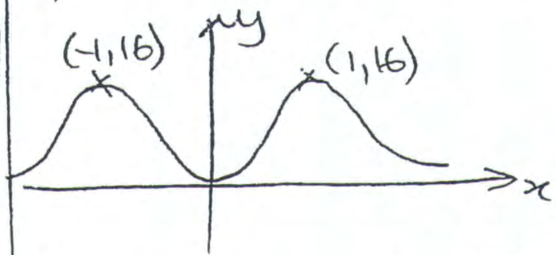
as $x \to \infty$ $y = 0^+$

as $x \to -\infty$ $y \to 0^-$

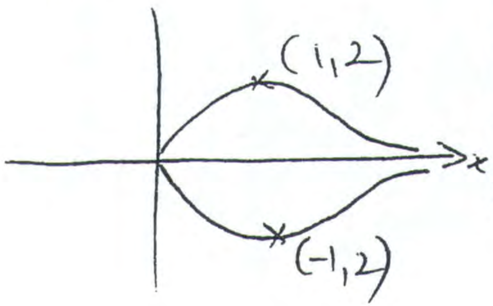
$\therefore$ asymptote at $y=0$



iii) 1. $y = (f(x))^2$. $\boxed{2}$



2. $y^2 = f(x)$ $\boxed{2}$



b) $x^2 + cxy + y^2 = c + 2$ $\boxed{3}$

d wrt x.

$2x + cxy' + cy + 2yy' = 0$

$\therefore \frac{dy}{dx} = -\frac{(2x+cy)}{(2y+cx)}$

at $(1,1)$ $\frac{dy}{dx} = -\frac{(2+c)}{2+c} = -1$

As they all have the same gradient at $(1,1)$ they must touch

c) The possibilities as follows $\boxed{2}$

LLL	$P = \left(\frac{2}{3}\right)^3$	$=$	a
LLOL	$P\left(\frac{2}{3}\right)^3 \times \frac{1}{3}$	$=$	b
LOLL	"	$=$	b
OLLL	"	$=$	b
OOLLL	$P\left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2$	$=$	c
OLOLL		$=$	c
OLLOL		$=$	c
LOOLL		$=$	c
LLOOL		$=$	c
LOLOL		$=$	c

$\therefore$

P(Lance wins)

$= a + 3b + 6c$

$= \left(\frac{2}{3}\right)^3\left[1 + 3\times\frac{1}{3} + 6\times\frac{1}{9}\right]$

$= \frac{8}{27}\left[2 + \frac{6}{9}\right]$

$= \frac{64}{81}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

D.w.r.t x

$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$

$y'=\frac{-b^2x}{a^2y}$

$\therefore m=\frac{-b\cos\theta}{a\sin\theta}$

t $(a\cos\theta, b\sin\theta)$

$y-b\sin\theta=-\frac{b^2a\cos\theta}{a^2b\sin\theta}(x-a\cos\theta)$ $\boxed{2}$

$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$

$ay\sin\theta+bx\cos\theta=ab(\sin^2\theta+\cos^2\theta)$

$\therefore \frac{y\sin\theta}{b}+\frac{x\cos\theta}{a}=1$

) Normal $m=\frac{a\sin\theta}{b\cos\theta}$ $\boxed{2}$

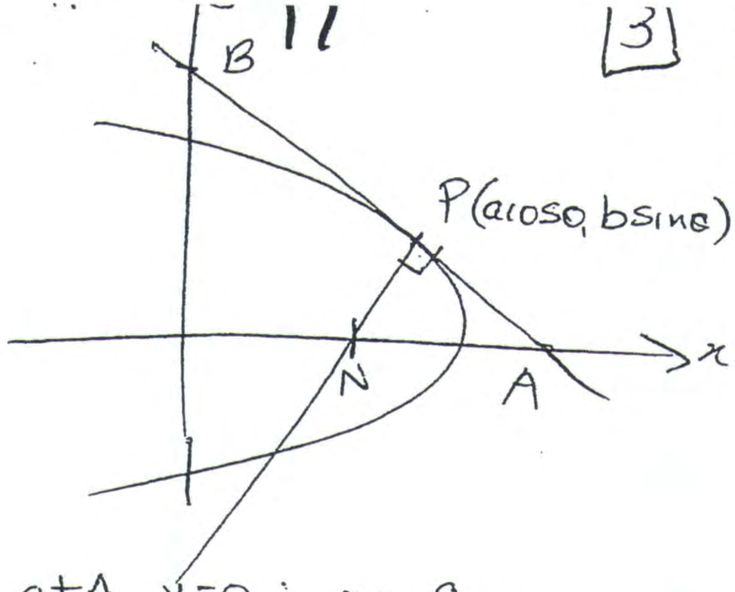
$y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$

$yb\cos\theta-b^2\cos\theta\sin\theta=xa\sin\theta-a^2\cos\theta\sin\theta$

$xa\sin\theta-yb\cos\theta=\cos\theta.\sin\theta(a^2-b^2)$

$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

11 $\boxed{3}$



at A $y=0 \therefore x=\frac{a}{\cos\theta}$

B $x=0 \therefore y=\frac{b}{\sin\theta}$

Area $=\frac{1}{2}\times\frac{a}{\cos\theta}\cdot\frac{b}{\sin\theta}$

$=\frac{ab}{\sin2\theta}$

2 $\therefore$ Min Area when $\sin2\theta=1$ $\boxed{1}$

$\therefore$ area $=ab$

iv) 1. at N $y=0 \therefore x=\frac{(a^2-b^2)\cos\theta}{a}$

$\boxed{2}$

$OA=\frac{a}{\cos\theta}$ $ON=\frac{(a^2-b^2)\cos\theta}{a}$

$OA\times ON=a^2-b^2$

$=a^2-a^2(1-e^2)$

$=(ae)^2.$

2. $\boxed{2}$

$\frac{PA}{PN}=\tan\angle PNA$

$=\tan\theta$

$=|\text{gradient } PN|$

$=\frac{a}{b}\frac{\sin\theta}{\cos\theta}.$

$\left(\frac{PA}{PN}\right)^2=\frac{a^2\tan^2\theta}{b^2}$

$=\frac{\tan^2\theta}{b^2/a^2}$

$=\frac{\tan^2\theta}{1-e^2}.$

)

3

If roots are

$\alpha - d, \alpha, \alpha + d$

$3\alpha = 6$

$\alpha = 2$

1

$\alpha(\alpha^2 - d^2) = -10$

$\therefore 8 - 2d^2 = -10$

$d^2 = 9$

$d = \pm 3$

Roots are

$\boxed{-1, 2, 5}$

1

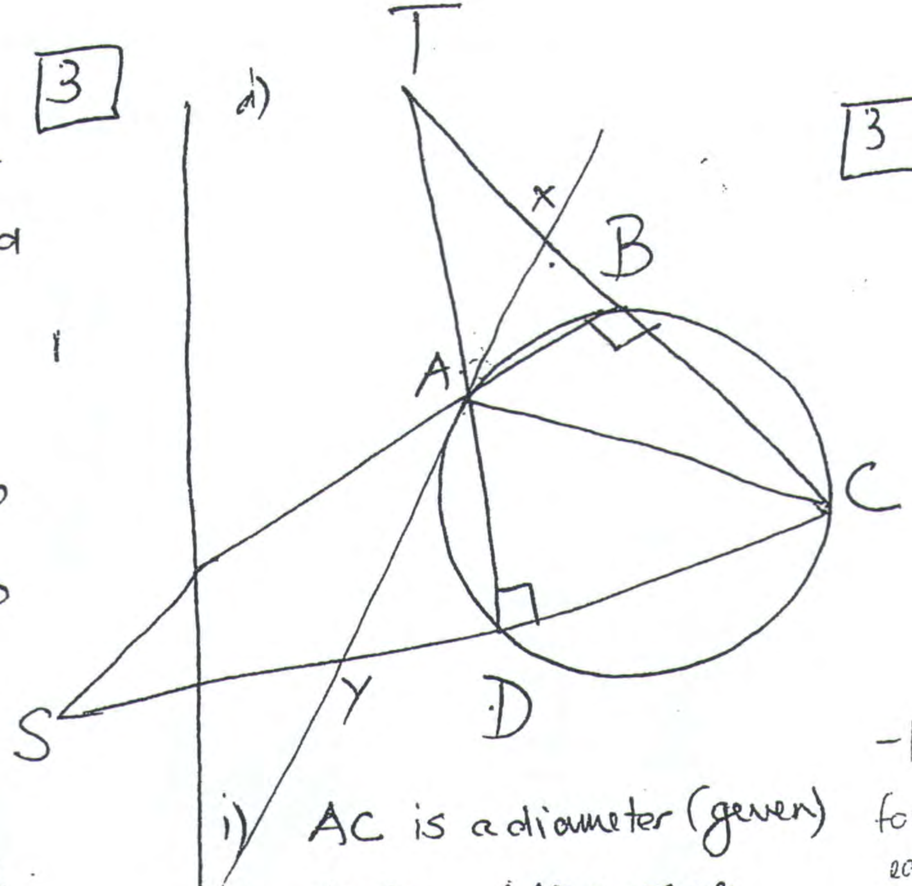
$P(-1) = 0 = -1 - 6 - a + 10$

$a = 3$

1

d)

3



-1 for each missing bit

i) AC is a diameter (given)

$\therefore \angle ABC = \angle ADC = 90°$ ($\angle$s in semi circle)

$\therefore \angle TBS = \angle TDS = 90°$ (Straight angle)

$\therefore$ TBDS is cyclic (2 eq $\angle$s on same arc TS)

$\therefore$ TS is diameter ($\angle$ in Semi Circle $= 90°$)

ii) let $\angle YAD = x$

3

$\therefore \angle XAT = x$ (Vert op)

$\therefore \angle ABD = x$ (Tangent, Alt Segment Thm)

$= \angle SBD$ (Same angle)

$= \angle STD$ ($\angle$s same arc)

$\therefore \angle STA = \angle TAX$

$\therefore TS \parallel YX$ (pr = alt $\angle$s)

-1 each missing bit

$\ldots) = \dfrac{{}^{2}C_1 \times {}^{11}C_4}{{}^{12}C_5}$

$= \dfrac{2 \times 330}{1287} = \dfrac{20}{39}$ $\boxed{1}$

) $P(\text{Lib Maj})$ $\boxed{2}$

$= \dfrac{C_5 + {}^{5}C_4 \times {}^{8}C_1 + {}^{5}C_3\,{}^{8}C_2}{{}^{13}C_5}$ 1

$= \dfrac{1 + 40 + 280}{1287}$

$= \dfrac{107}{429}$ 1

a) $P(1-i) = (1-i)^3 - 7(1-i)^2 + 12(1-i) - 10$ 18 $\boxed{3}$

$= 1 - 3i - 3 + i - 7(1 - 2i - 1) + 12 - 12i - 10$

$= 1 - 3i - 3 + i - 7 + 14i + 7 + 12 - 12i - 10$

$= 0$ 1

$\therefore 1 - i$ is a zero.

$P(x) = (x-(1-i))(x-(1+(1+i))(x-k)$ 1

Solve for k.

$(1-i)(1+i)k = 10$

$2k = 10$

$k = 5$ 1

$\therefore P(x) = (1-(1-i))(1-(1+i))(x-5)$ $\boxed{2}$

b) $z^5 = 1 \Rightarrow z_1 = 1$

$|z| = 1$

$z_2 = \text{cis}\,\frac{2\pi}{5}$

$z_3 = \text{cis}\,\frac{4\pi}{5}$

$z_4 = \text{cis}\left(-\frac{4\pi}{5}\right)$

$z_5 = \text{cis}\left(-\frac{2\pi}{5}\right.$ 1

$\arg z_1 = 0$

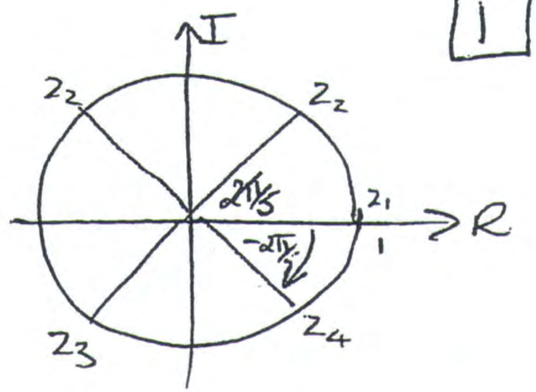
$\arg z_2 = \frac{2\pi}{5}$

$\arg z_3 = \frac{4\pi}{5}$

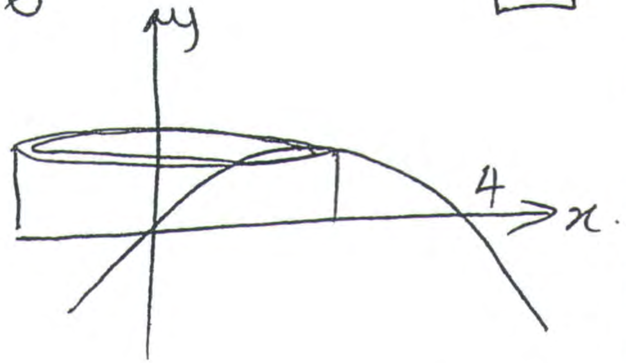
$\arg z_4 = -\frac{4\pi}{5}$

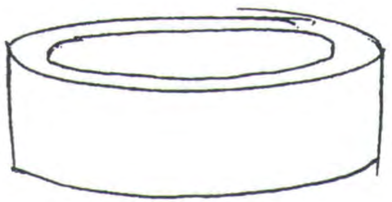
$\arg z_5 = -\frac{2\pi}{5}$ 1

$\boxed{1}$

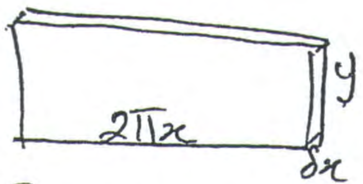


6 $\boxed{5}$





A cylindrical shell is approximately a slab



$\delta V = 2\pi xy\,\delta x$

$V = 2\pi\int_0^4 x^2(4-x)\,dx$

$= 2\pi\int_0^4 (4x^2 - x^3)\,dx$

$= 2\pi\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^4 = \frac{128\pi}{3}$ cubic units

b) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ $\boxed{5}$

$y^2 = \left(1 - \frac{x^2}{16}\right)\times 9$

Area of $\Delta = \frac{1}{2}\times 2y \times 4y$

$= 4y^2$

$\delta V = 4y^2\,\delta x$

$V = \int_{-4}^{4} 4y^2\,dx$

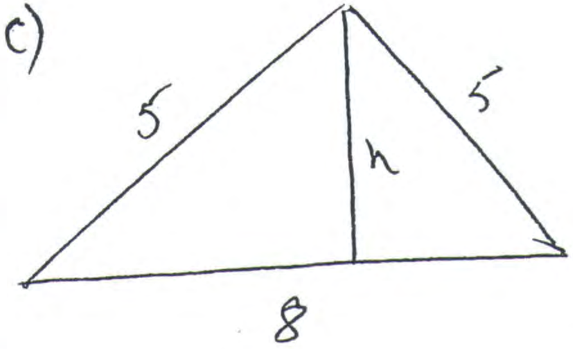
$= 72\int_0^4 \left(1 - \frac{x^2}{16}\right) dx$

$= 72\left[x - \frac{x^3}{48}\right]_0^4$

$= 72\left[\frac{12-4}{3}\right]$

$= 192$ cubic units

c) $\boxed{5}$



1 roll $= .02\pi$

$\therefore$ 3 rolls $= .06\pi$ m/s

$\therefore$ rate of decrease $\frac{dh}{dt} = -.06\pi$ m/s

$A = \frac{1}{2}bh$

$t=0 \quad h=3$ $\left(\frac{dA}{dh} = \frac{8h}{3}\right)$

$\therefore \frac{1}{2}$ base is $\frac{4}{3}h$

$\therefore$ Area $= \frac{4}{3}h^2$

$\frac{dA}{dt} = \frac{dA}{dh}\times\frac{dh}{dt} = \frac{8h}{3}\times\frac{dh}{dt}$

at $t = 1$ sec $h = 3 - .06\pi$

$\frac{dA}{dt} = \frac{8}{3}\left[3 - \frac{3\pi}{50}\right]\times\frac{3\pi}{50} = \frac{24\pi}{50}\left[1 - \frac{\pi}{50}\right]$

$= \frac{12\pi}{25}\left(1 - \frac{\pi}{50}\right) m^2/s$

a i) $\boxed{2}$

$\ddot{x}=0$ $\quad \ddot{y}=-g$

$\dot{x}=V\cos\theta$ $\quad \dot{y}=V\sin\theta-gt$

$x=Vt\cos\theta$ $\quad y=Vt\sin\theta-\frac{gt^2}{2}$

Max height when $\dot{y}=0$

ie $t=\frac{V\sin\theta}{g}$

$y_{max}=\frac{V^2\sin^2\theta}{g}-\frac{gV^2\sin^2\theta}{2g^2}$

$=\frac{V^2\sin^2\theta}{2g}$ $\boxed{3}$

ii) P_2 has Vel $\frac{3V}{2}$ + angle $\frac{\theta}{2}$

$y_{max}=\frac{V^2\sin^2\theta}{2g}=\frac{\left(\frac{3}{2}V\right)^2\sin^2\theta/2}{2g}$

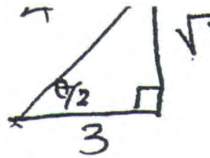
$\Rightarrow \sin^2\theta=\frac{9\sin^2\theta/2}{4}$

$4\sin^2\theta=9\sin^2\theta/2$

$2\sin\theta=3\sin\theta/2$

$4\sin\theta/2\cos\theta/2=3\sin\theta/2$ $\quad \sin\theta/2\neq0$

$\therefore \cos\theta/2=3/4$



$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\theta/2$

$=\frac{9}{16}-\frac{7}{16}$

$=\frac{1}{8}$

$\therefore \cos^{-1}\left(\frac{1}{8}\right)$

2. $\boxed{3}$

$t_1=\frac{V\sin\theta}{g}$

$t_2=\frac{\frac{3V}{2}\sin\theta/2}{g}$

If time is the same ie $t_1=t_2$ this will give $\theta=\cos^{-1}(1/8)$ as in pt 1.

$\therefore \frac{V\sin\theta}{g}=\frac{3V\sin\theta/2}{2g}$

$4\sin\theta/2\cos\theta/2=3\sin\theta/2$

$\div \sin\theta/2\neq0$

$\cos\theta/2=3/4$ as in pt 1 $\therefore \cos^{-1}(1/8)=\theta$ $\therefore$ They do take the same time

$\boxed{5}$

$I_n=\int_0^{\pi/2}x^n\cos x\,dx$.

$=\left[x^n\sin x\right]_0^{\pi/2}-\int_0^{\pi/2}nx^{n-1}\sin x\,dx$

$=\left(\frac{\pi}{2}\right)^n-n\int_0^{\pi/2}x^{n-1}\sin x\,dx$

$=\left(\frac{\pi}{2}\right)^n-n\left[\left[x^{n-1}\cdot-\cos x\right]_0^{\pi/2}-\int_0^{\pi/2}(n-1)x^{n-2}\cdot-\cos x\,dx\right]$

$=\left(\frac{\pi}{2}\right)^n-n\left(0+(n-1)\int_0^{\pi/2}x^{n-2}\cos x\,dx\right.$

$=\left(\frac{\pi}{2}\right)^n-n(n-1)I_{n-2}$

c) i) $a=1$ $\quad r=(1+x)$ $\quad n=n+1$ $\boxed{2}$

$\therefore S_n=\frac{1\left((1+x)^n-1\right)}{1+x-1}=\frac{(1+x)^{n+1}-1}{x}$.

Q7c $\boxed{2}$

LHS $= \dfrac{{}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + \dots {}^{n+1}C_{r+1} x^{r+1} \dots {}^{n+1}C_{n+1}}{x}$

$= {}^{n+1}C_1 + {}^{n+1}C_2 x + \dots {}^{n+1}C_{r+1} x^r + \dots$

$\therefore$ Coeff x^r is ${}^{n+1}C_{r+1}$.

LHS

The first term to have an x^r term will be $(1+x)^r$ $\therefore$ Coef ${}^rC_r$

Next $(1+x)^{r+1}$ Coef ${}^{r+1}C_r$ etc.

$\therefore$ LHS coef of x^r is

${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \dots {}^nC_r$

$\therefore$ ${}^nC_r + {}^{n-1}C_r + \dots {}^rC_r = {}^{n+1}C_{r+1}$.

Q8a. $\boxed{6}$

$U_1 = 1$ $\quad U_n = U_{n-1} + U_{n-2}$

$U_2 = 2$ $\quad \therefore U_3 = U_2 + U_1 = 3$

$U_4 = U_3 + U_2 = 5$

Step 1.

$n = 1$ $\quad S_1 =$

$\therefore S_n = U_{n+2} - 2 \Rightarrow U_3 - 2 = 3 - 2 = 1 = U_1$

True for $n=1$.

Assume true for n.

ie $U_1 + U_2 + U_3 + \dots + U_n = U_{n+2} - 2 = S_n$

Now $S_{n+1} = U_{n+3} - 2$

& $S_{n+1} = S_n + U_{n+1} = U_{n+2} - 2 + U_{n+1}$

$= U_{n+2} + U_{n+1} - 2$

$= U_{n+3} - 2$

$= S_{n+1}$ by formula.

By the process of M.I if it is true for $n = 2$ true for $n=3$ etc $\therefore$ true for integral values of n

20

Prove $\frac{a+b}{2} \geq \sqrt{ab}.$ $\boxed{2}$

Consider $(x-y)^2 = x^2 - 2xy + y^2 \geq 0$

$\therefore x^2 + y^2 > 2xy$

let $x^2 = a$ & $y^2 = b$

$\therefore a+b > 2\sqrt{ab}$

$\frac{a+b}{2} > \sqrt{ab}$ $\boxed{2}$

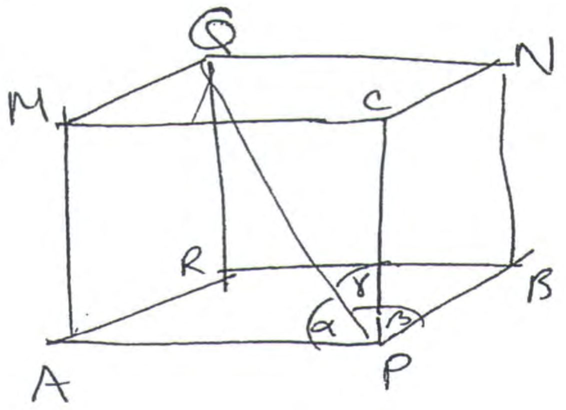
)

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geqslant \sqrt{\left(\frac{a+b}{2}\right) \times \left(\frac{c+d}{2}\right)}$$

$$\frac{a+b+c+d}{4} \geqslant \sqrt{\sqrt{ab}\sqrt{cd}}$$

$$\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}.$$

i)



$\boxed{3}$

① $\cos\alpha = \frac{AP}{QP}$

② $\cos\beta = \frac{BP}{QP}$

③ $\cos\gamma = \frac{CP}{QP}$

$QP^2 = QA^2 + AP^2 = PC^2 + PB^2 + AP^2$

$QP^2 = QB^2 + BP^2 = PC^2 + PA^2 + BP^2$

$QP^2 = QC^2 + PC^2 = PA^2 + BP^2 + PC^2$

$\therefore QP^2 = PA^2 + PB^2 + PC^2$ (7)

$PA^2 = QP^2\cos^2\alpha$ ④

$\Rightarrow PB^2 = QP^2\cos^2\beta^2$ ⑤

$PC^2 = QP^2\cos^2\gamma$ ⑥

④ + ⑤ + ⑥

$PA^2 + PB^2 + PC^2 = QP^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$ ⑧

(7) into ⑧

$\Rightarrow c^2\alpha + c^2\beta + c^2\gamma = 1.$

ii) From i) $\boxed{2}$

$\cos^2\alpha + \cos^2\alpha + \cos^2 2\alpha = 1.$

Solve for α.

$2\cos^2\alpha + \cos^2 2\alpha = 1.$

$2\cos^2\alpha + (2\cos^2\alpha - 1)^2 = 1$

$2c^2\alpha + 4c^4\alpha - 4c^2\alpha + 1 = 1$

$4c^4\alpha - 2c^2\alpha = 0$

$2c^2\alpha(2c^2\alpha - 1) = 0$

$\therefore \cos^2\alpha = 0 \Rightarrow \alpha = \frac{\pi}{2}.$

$\cos^2\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{4}.$

If $\alpha = \frac{\pi}{4}$ $2\alpha = \frac{\pi}{2}$

Not possible as prism will be flat with no height.