NORTH SYDNEY BOYS HIGH SCHOOL

2004
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Examiner: G. Rezcallah

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Ee
- O Mr Rezcallah
- O Mr Barrett

Student Number:______________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{120}$	$\overline{100}$

QUESTION 1 Begin a new page.

Marks

(a) Find: (i) $\displaystyle\int \frac{x^4}{\sqrt{x^5 - 7}}\,dx$. 2

(ii) $\displaystyle\int \frac{1}{e^x + e^{-x}}\,dx$ 2

(b) Evaluate $\displaystyle\int_2^6 x\sqrt{6-x}\,dx$ Using the substitution $u^2 = 6 - x$. 3

(c) (i) Find the constants A and B such that 2

$$\frac{1}{\cos x} = \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$$

(ii) Hence, find the exact value of the integral $\displaystyle\int_0^{\frac{\pi}{6}} \sec x\,dx$ 2

(d) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

Hence, evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}}\,dx$. 2

QUESTION 2 Begin a new page.

Marks

(a) Consider the complex numbers $Z_1 = \sqrt{2}\,(1 + i\sqrt{3})$ and $Z_2 = 2\sqrt{6}\,(1 + i)$

(i) Express $z = \dfrac{Z_1}{Z_2}$ exactly in the form $x + iy$, where x and y are real. 2

(ii) Write Z_1 , Z_2 and z in modulus/argument form. 2

(iii) Hence, find the exact value of $\cos\dfrac{\pi}{12}$. 1

(iv) On an Argand diagram draw the vectors $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OS}$, to represent Z_1 , Z_2 and $Z_1 - Z_2$ respectively. 2

(b) Indicate on an Argand diagram the region which contains the point P representing z when :

(i) $\mathrm{Re}(z + iz) \geq 2$ 2

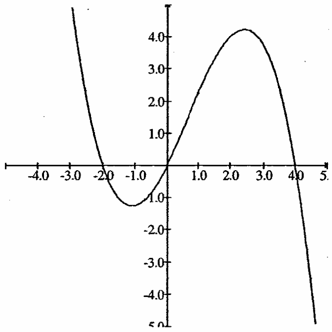
(ii) $1 \leq |z - 1 - i| \leq 3$ where $z = x + iy$ 2

(c) By applying De Moivre's theorem and by also expanding $(\cos\theta + i\sin\theta)^5$, express $\sin 5\theta$ as a polynomial in $\sin\theta$. 4

QUESTION 3 Begin a new page.

Marks

(a) The diagram shows the graph of $y = f(x)$ which passes through the origin and cuts the x axis at $x = -2$ and $x = 4$. The point $(1\ ,\ 2\frac{1}{4})$ belongs to the curve.



(i) Write down the equation of $y = f(x)$. 2

On separate diagrams, sketch each of the following: 10

(ii) $y = -f(x)$

(iii) $y = f(-x)$

(iv) $y^2 = f(x)$

(v) $y = |f(|x|)|$

(vi) $y = \dfrac{1}{1 - f(x)}$

(b) Consider in the set of complex numbers C:

w the cubic root of unity, $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$.

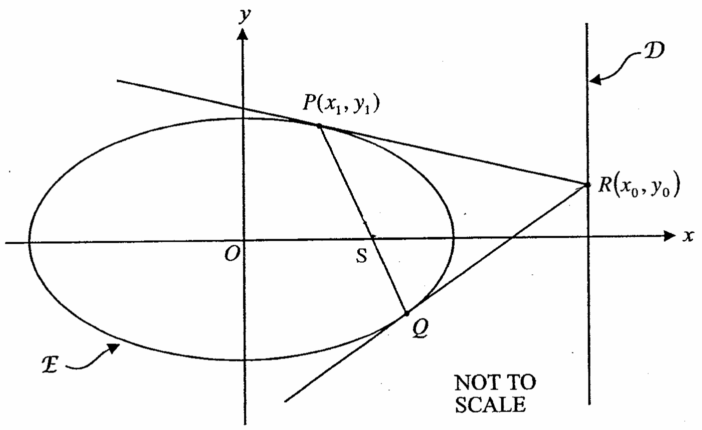
(i) Show that $1 + w + w^2 = 0$ 1

(ii) Prove that $x^2 + y^2 + z^2 = 6ab$ 2

QUESTION 4 Begin a new page.

Marks

(a)



The ellipse E with equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ has a directrix D as shown in the diagram.
Point $R(x_0, y_0)$ lies on D.
PQ is the chord of contact from R where P is the point (x_1, y_1).

(i) Write down the equation of D and the coordinates of the focus S. 2

(ii) Show that the tangent at a point P (x_1, y_1) has an equation of

$$\frac{xx_1}{16}+\frac{yy_1}{9}=1.$$ 2

(iii) Write down the equation of chord PQ and show that the focus S lies on PQ. 2

(iv) Show that the angle subtended by PR at the focus S is 90°. 3

(v) Hence, deduce that the points P, S, and R are concyclic.. 1

QUESTION 4 (Continued)

Marks

(b)

(i) Sketch the graph of $f(x)=\sqrt{(x+1)^2}+\sqrt{(x-1)^2}$. 2

(ii) Hence, solve the equation $-2 \leq \sqrt{(x+1)^2}+\sqrt{(x-1)^2} < 2$ 1

(c) The equation $x^3+kx+r=0$ has roots $\alpha, \beta,$ and γ.

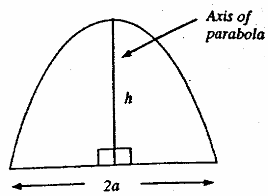
Find the value of the expression $\alpha^3+\beta^3+\gamma^3$ 2

QUESTION 5 Begin a new page.

Marks

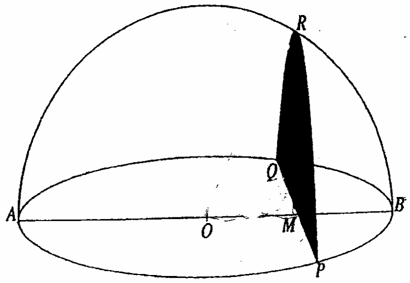
(a) (i) A parabolic segment has height h and width 2a.

Use Simpson's rule with three function values to show that the exact area of this segment is $\frac{4}{3}ah$.



2

In the diagram below, a tent has a circular base with centre O and radius a, and AOB is a diameter of the base. - The shaded area PMQR is a typical cross section of the tent perpendicular to AB, and meets AB at a point M distant x from O. The curve PRQ is a parabola with axis RM and QM = RM.



(ii) Use part (i) to show that the shaded area PMQR *is* $\frac{4}{3}(a^2 - x^2)$. 2

(iii) Find the volume of the tent. 2

(b) Factorise the polynomial $P(z) = z^4 - 2z^2 + 8z - 3$ fully over C
Given that $P(1 - \sqrt{2}i) = 0$. 4

(c) (i) By considering the perfect square $(\sqrt{x} - \sqrt{y})^2$ where x and y are positive, prove that $\frac{x+y}{2} \geqslant \sqrt{xy}$. 2

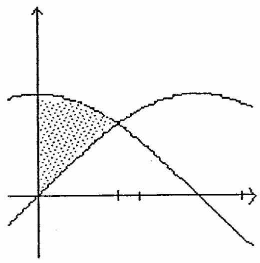
(ii) Hence, if a, b, c and d are positive numbers, prove that :

$$4(ab + bc + cd + da) \leq (a + b + c + d)^2$$ 3

QUESTION 6 Begin a new page.

Marks

(a) In the diagram, the shaded region is bounded by the y axis and the curves $y = \cos x$ and $y = \sin x$.



(i) Show that the curves intersect at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ 1

(ii) The shaded region is rotated about the y axis.
Find the exact value of the volume obtained by this rotation, using the method of cylindrical shells. 5

(b) By considering the binomial expansion of $(1+i)^n$

Show that $1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots\ldots = 2^{\frac{n}{2}} \cos\frac{n\pi}{4}$ 2

(c) The normal at a point $P\left(x_1, \frac{9}{x_1}\right)$ on the rectangular hyperbola $xy = 9$ meets the curve again at another point A.

(i) Prove that the equation of intersection of the normal at P and the rectangular hyperbola is $x_1^3\, x^2 + (81 - x_1^4)x - 81x_1 = 0$ 2

(ii) Hence, prove that the coordinates of A are $\left(-\frac{81}{x_1^3}, -\frac{x_1^3}{9}\right)$ 2

(iii) Let M be the midpoint of AP. Derive the cartesian equation of the locus of M. 3

QUESTION 7 Begin a new page.

Marks

(a) Consider the word **SOCCER**.

How many:

(i) six-letter different arrangements 1

(ii) selections of 4 letters 2

can be made from the letters in the word **SOCCER**?

(b) A particle of mass m is projected vertically upward under gravity in a medium in which the resistance is proportional to square of the velocity (mkv^2), where k is a constant.

(i) Show that the terminal speed V in the medium is $\sqrt{\dfrac{g}{k}}$ 1

If the speed of projection is equal to the terminal velocity V in the medium, show that:

(ii) the particle reaches a maximum height of $\dfrac{V^2}{2g}\ln 2$ above the point of projection. 3

(iii) the time taken to reach its maximum height is $\dfrac{\pi V}{4g}$ 4

(iv) the time t in the downward motion as a function of v is given by

$$t = \frac{1}{2\sqrt{gk}}\ln\left(\frac{V+v}{V-v}\right)$$ 4

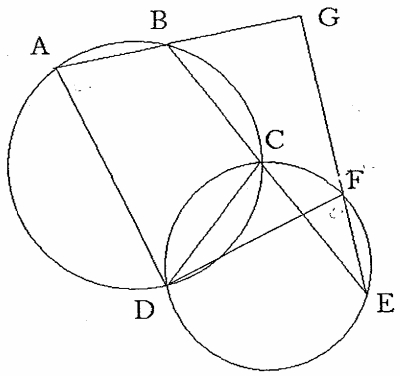
QUESTION 8 Begin a new page.

Marks

(a) If $I_n = \int_0^1 (1-x^2)^n \, dx$ for $n \geq 0$, show that $I_n = \frac{2n}{1+2n} I_{n-1}$ for $n \geq 1$. 3

Hence, find an expression for I_n in terms of n for $n \geq 1$. 2

(b)



Two circles intersect at C and D. ABCD is a cyclic quadrilateral in one circle.

BC produced meets the other circle at E. C, F, E and D are concyclic points.

AB produced meets EF produced at G.

Prove that GFDA is a cyclic quadrilateral. 4

(c) A sequence is defined by the recurrence relationship:

$U_1 = 1$ and $U_{n+1} = \frac{1}{2}\left[U_n + \frac{2}{U_n}\right]$ when $n \geq 1$, n a positive integer

(i) Prove by mathematical induction: $\frac{U_n - \sqrt{2}}{U_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$ 4

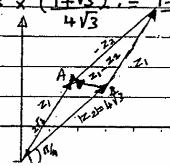
(ii) Hence, show that for n sufficiently large, U_n is very close to $\sqrt{2}$ 2

Solution of NSBHS Ext. II - 2004 HSC Trial.

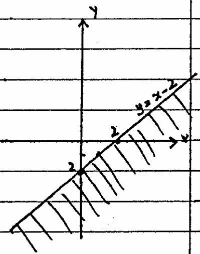
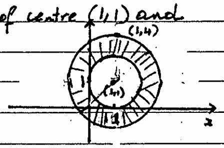
Question 1.	Suggested Marks.
(a) (i) $\int \frac{x^4\,dx}{\sqrt{x^5-7}}$ let $u = x^5 - 7$, $du = 5x^4\,dx$.	
$\frac{1}{5}\int \frac{5x^4\,dx}{\sqrt{x^5-7}} = \frac{1}{5}\int \frac{du}{\sqrt{u}} = \frac{1}{5}\int u^{-\frac{1}{2}}\,du$	1 for correct substitution or correct modified primitive
$= \frac{1}{5} 2u^{\frac{1}{2}} + C = \frac{2}{5}\sqrt{x^5-7} + C.$	1 for correct answer with C
(ii) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{e^x\,dx}{e^{2x}+1}.$	1 for correct start and substitution
let $u = e^x$, $du = e^x dx$ $= \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1} e^x + C.$	1 for correct answer in terms of x.
(b) $\int_2^6 x\sqrt{6-x}\,dx.$ $u^2 = 6 - x \Rightarrow x = 6 - u^2$; $2u\,du = -dx \Rightarrow dx = -2u\,du$	
$\int (6-u^2)\sqrt{u^2}\cdot(-2u\,du)$	1 for substitution to obtain $\int$ in terms of u.
$= \int (6-u^2)(-2u^2)\,du = -2\int_2^0 (6u^2 - u^4)\,du$	
$= -2\left[\frac{6u^3}{3} - \frac{u^5}{5}\right]_2^0 = 2\left[2u^3 - \frac{u^5}{5}\right]_0^2$	1 for finding primitive with correct bounds.
$= 2\left[2(2)^3 - \frac{2^5}{5}\right] = 2\left[16 - \frac{32}{5}\right] = \frac{96}{5}$	1 for correct answer.
(c) (i) $\frac{1}{\cos x} = \frac{A\cos x(1+\sin x) + B\cos x(1-\sin x)}{1-\sin^2 x}$	Award:
$= \frac{A\cancel{\cos x}(1+\sin x) + B\cancel{\cos x}(1-\sin x)}{\cos^{\cancel{2}} x}$	1 mark for one correct answer or correct 2 equations in terms of A and B.
$= \frac{A(1+\sin x) + B(1-\sin x)}{\cos x}$	
$= \frac{(A+B) + (A-B)\sin x}{\cos x}$	2 marks for both correct answers.
$\left.\begin{array}{l} A + B = 1 \\ A - B = 0 \Rightarrow A = B \end{array}\right\} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$ ✓, $B = \frac{1}{2}$ ✓	

NSBHS - Ext II Trial 2004 Solutions.	Marks.
(ii) $\int_0^{\pi/6}\sec x\,dx = \int_0^{\pi/6}\frac{1}{\cos x}\,dx = \frac{1}{2}\int_0^{\pi/6}\frac{\cos x\,dx}{1-\sin x}+\frac{1}{2}\int_0^{\pi/6}\frac{\cos x\,dx}{1+\sin x}$	
$= -\frac{1}{2}\int_0^{\pi/6}\frac{-\cos x}{1-\sin x}\,dx + \frac{1}{2}\int_0^{\pi/6}\frac{\cos x}{1+\sin x}\,dx$	1 for correct substitution to get the correct primitive in either
$= -\frac{1}{2}\left[\ln(1-\sin x)\right]_0^{\pi/6} + \frac{1}{2}\left[\ln(1+\sin x)\right]_0^{\pi/6}$.	
$= \frac{1}{2}\left[\ln\frac{(1+\sin x)}{1-\sin x}\right]_0^{\pi/6} = \frac{1}{2}\ln\frac{(1+\frac{1}{2})}{(1-\frac{1}{2})}$	1 mark for correct answer.
$= \frac{1}{2}\ln\frac{\frac{3}{2}}{\frac{1}{2}} = \frac{1}{2}\ln 3. = \ln\sqrt{3}$.	
(d) (i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.	Award:
let $u = a-x \Rightarrow \int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du$. ✓	2 marks for correct proof.
$du = -dx$	
$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx$.	1 mark for any minor step not shown in the proof.
or: $\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du = \int_0^a f(a-u)\,du$	
$= \int_0^a f(a-x)\,dx$.	
$I = \int_0^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx = \int_0^{\pi/2}\frac{e^{\sin(\frac{\pi}{2}-x)}}{e^{\sin(\frac{\pi}{2}-x)}+e^{\cos(\pi/2-x)}}\,dx$	
$= \int_0^{\pi/2}\frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}\,dx$.	1 mark for correct method using property.
$I+I = \int_0^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx + \int_0^{\pi/2}\frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx$	1 for correct answer.
$= \int_0^{\pi/2}\frac{(e^{\sin x}+e^{\cos x})}{(e^{\sin x}+e^{\cos x})}\,dx = \int_0^{\pi/2}dx = \left[x\right]_0^{\pi/2}$	
$\therefore 2I = \frac{\pi}{2} \Rightarrow I = \pi/4$.	

NSBHS Trial Ext 2 Solutions.	
Question 2.	Marks.
(a) $z_1 = \sqrt{2}(1+i\sqrt{3})$. $z_2 = 2\sqrt{6}(1+i)$.	Award:
(i) $z = \frac{z_1}{z_2} = \frac{\sqrt{2}(1+i\sqrt{3})}{2\sqrt{6}(1+i)} \times \frac{(1-i)}{(1-i)}$	2 marks for correct z answer.
$= \frac{\sqrt{2}(1-i+i\sqrt{3}-i^2\sqrt{3})}{2\sqrt{6}(1-i^2)}$ ✓	1 mark for:
$= \frac{\sqrt{2}(1-i+i\sqrt{3}+\sqrt{3})}{2\sqrt{6}(2)}$ but $\frac{\sqrt{2}}{2\sqrt{6}} = \frac{1}{2\sqrt{3}}$.	correct method, wrong answer or
$= \frac{(1+\sqrt{3}+i\sqrt{3}-i)}{4\sqrt{3}}$.	1 correct part of z showing $\frac{(1-i)}{(1-i)}$.
$= \frac{1+\sqrt{3}}{4\sqrt{3}} + \frac{i(\sqrt{3}-1)}{4\sqrt{3}}$. ✓	
(ii) $z_1 = \sqrt{2}(1+i\sqrt{3})$ $r = \sqrt{1^2+(\sqrt{3})^2} = 2$	
$= 2\sqrt{2}\left(\frac{1+i\sqrt{3}}{2}\right) = 2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{3}$.	1 mark for correct z_1 or z_2
$z_2 = 2\sqrt{6}(1+i) = 2\sqrt{6}\times\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\times\sqrt{2}$	
$= 4\sqrt{3}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 4\sqrt{3}\,\mathrm{cis}\,\frac{\pi}{4}$.	
$z = \frac{z_1}{z_2} = \frac{2\sqrt{2}\,\mathrm{cis}\,\pi/3}{4\sqrt{3}\,\mathrm{cis}\,\pi/4} = \frac{\sqrt{2}}{2\sqrt{3}}\,\mathrm{cis}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$	1 mark for correct z according to z_1 and z_2.
$= \frac{\sqrt{2}}{2\sqrt{3}}\,\mathrm{cis}\,\frac{\pi}{12}$.	
(iii) $z = \frac{\sqrt{2}}{2\sqrt{3}}\,\mathrm{cis}\,\frac{\pi}{12} = \frac{\sqrt{2}}{2\sqrt{3}}\left[\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right]$	
$\therefore \left[\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right] = \frac{2\sqrt{3}}{\sqrt{2}}z = \frac{2\sqrt{3}}{\sqrt{2}}\left[\frac{1+\sqrt{3}}{4\sqrt{3}}+\frac{i(\sqrt{3}-1)}{4\sqrt{3}}\right]$.	1 mark for the correct
$\therefore \cos\frac{\pi}{12} = \frac{2\sqrt{3}}{\sqrt{2}}\times\frac{(1+\sqrt{3})}{4\sqrt{3}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$. ✓	expression (even if unsimplified).
(iv) $\vec{BA} = \vec{z_1} - \vec{z_2}$	1 mark for correct representation of $\vec{OA}$ and $\vec{OB}$
	1 mark for $\vec{BA}$.



NSBHS Ext 2 Solutions.

	Marks.
(b)(i) $\text{Re}(z+iz) \geq 2$.	
$z+iz = x+iy+i(x+iy)$	
$= x+iy+ix-y$	1 mark for the correct line
$= (x-y)+i(x+y)$.	
$\text{Re}(z+iz) = x-y \geq 2$.	
or $y \leq x-2$.	1 mark for the correct region.
	
(ii) $\lvert z-(1+i)\rvert = 3$ is a circle of centre $(1,1)$ and $r=3$.	1 mark for identifying
$\lvert z-(1+i)\rvert = 1$ is a circle of centre $(1,1)$ and radius $= 1$.	and showing the 2 circles with their correct centres and radii
	1 mark for the shaded region.
(c) $(\cos\theta + i\sin\theta)^5$. Let $c=\cos\theta$. $s=\sin\theta$.	
$(c+is)^5$	
$= c^5+5c^4 s i + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$	
$= c^5 + 5c^4 s i - 10c^3 s^2 - 10c^2 s^3 i + 5cs^4 - i s^5$	1 mark for correct expansion.
$= c^5 - 10c^3s^2 + 5cs^4 + i(5c^4 s - 10c^2s^3 + s^5)$ ✓	
But $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. (De Moivre) ✓	1 mark for De Moivre's thm
$\therefore \sin 5\theta = 5c^4 s - 10c^2s^3 + s^5$	
$= 5c^4 s - 10(1-s^2)s^3 + s^5$ ✓	1 for correct $\sin 5\theta$ in terms of c and s.
$= 5(1-s^2)^2 s - 10s^3 + 10s^5 + s^5$	
$= 5(s^4-2s^2+1)s - 10s^3 + 11s^5$	
$= 5s^5 - 10s^3 + 5s - 10s^3 + 11s^5$	
$= 16s^5 - 10s^3 - 10s^3 + 5s$	
$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. ✓	1 for correct answer.
$\therefore \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.	

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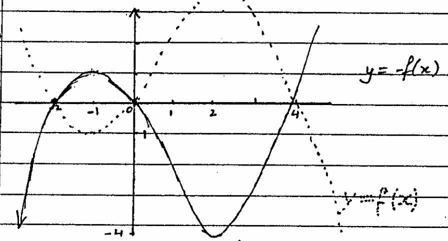
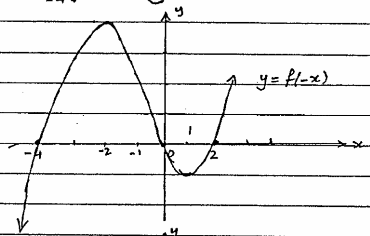
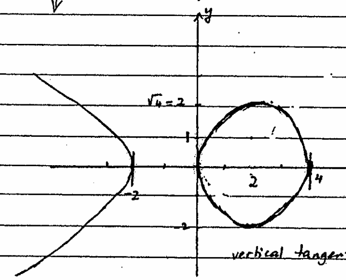
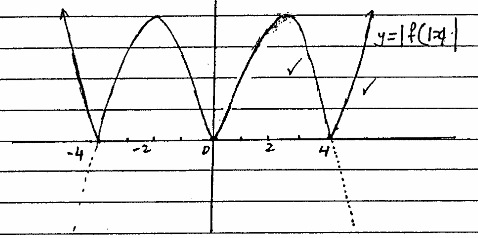
	Marks.
Question 3	
(a)(i) $y = ax(x-4)(x+2)$ ✓	1 mark for quartic showing factors.
$2\frac{1}{4} = a\,1\,(1-4)(1+2)$	
$\frac{9}{4} = -9a \Rightarrow a = -\frac{1}{4}$.	
$y = -\frac{1}{4}x(x-4)(x+2)$ or $-\frac{1}{4}(x^3-2x^2-8x)$ ✓	1 for correct expression. either factored or simplified form.
(ii)  ✓	correct graph: ✓
(iii)  ✓ ✓	Award: Correct graph ✓✓ substantially correct, but missing feature ✓
(iv)  ✓ ✓	Award: 1 mark for either: correct loop or correct $y=\sqrt{f(x)}$. 2 marks for correct graph including vertical tangents.
vertical tangents at $x=0,4,-2$	

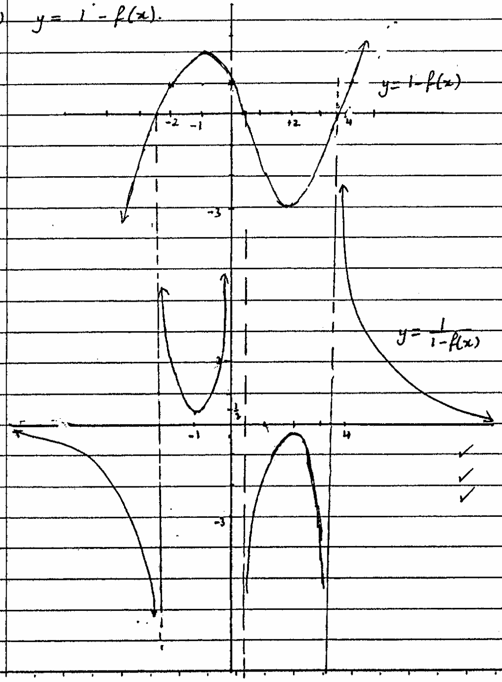
NSBHS Ext 2 - Solutions 2004 | Marks.

(v) First graph $f(|x|)$ where the function is reflected about y axis is reflected about y axis. for $x \geq 0$. Then reflect any parts under x axis about x axis.



Marks: 2 marks for correct graph awarded so that:
✓ 1 mark for $y=f(|x|)$
✓ 1 mark for reflection of $y=f(|x|)$ about x axis.

(vi) $y = 1 - f(x)$.



Award:

3 marks for correct graph $y=\frac{1}{1-f(x)}$.

2 marks for substantially correct, but missing one of asymptotes.

1 mark for correct $y=1-f(x)$ or 2 correct parts of the 4 parts shown.

NSBHS Ext 2 - Solutions 2004 | Marks.

(b) $x = a+b \qquad y = aw + bw^2 \qquad z = aw^2 + bw$.

(i) $w^3 = 1 \Rightarrow w^3 - 1 = 0$

$(w-1)(w^2+w+1) = 0$. ✓ for correct method.

$\therefore \; w^2 + w + 1 = 0$.

(ii) $x^2 = (a+b)^2 = a^2 + b^2 + 2ab$.

$y^2 = (aw+bw^2)^2 = a^2w^2 + b^2w^4 + 2abw^3$

$= a^2w^2 + b^2w + 2ab \qquad \begin{bmatrix} w^3=1 \\ w^4=w \end{bmatrix}$.

$z^2 = (aw^2+bw)^2 = a^2w^4 + 2abw^3 + b^2w^2$

$= a^2w + 2ab + b^2w^2$.

$x^2 + y^2 + z^2$

$= a^2+b^2+2ab + a^2w^2+b^2w+2ab+a^2w+2ab+b^2w^2$ ✓ for correct expansion.

$= a^2+a^2w^2+a^2w+b^2+b^2w+b^2w^2+6ab$.

$= a^2\underbrace{(1+w^2+w)}_{=0} + b^2\underbrace{(1+w+w^2)}_{=0} + 6ab$. ✓ for showing and proving the given answer.

$= 6ab$.

NSBHS Extension 2 - 2004 Trial Solution — Marks.

Question 4.

(1)(i) $b^2 = a^2(1-e^2)$ $\qquad \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

$9 = 16(1-e^2) \Rightarrow \dfrac{9}{16} = 1-e^2 \Rightarrow e^2 = 1 - \dfrac{9}{16} = \dfrac{7}{16}$.

$e = \dfrac{\sqrt{17}}{4}$.

Directrix $x = \dfrac{a}{e} = \dfrac{4}{\frac{\sqrt{17}}{4}} = \dfrac{16}{\sqrt{7}}$ or $\dfrac{16\sqrt{7}}{7}$. — ✓ for correct directrix.

$S(ae, 0) = S\left(\dfrac{4\sqrt{7}}{4}, 0\right) = S(\sqrt{7}, 0)$ — ✓ for correct focus.

(ii) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

$\dfrac{2x}{16} + \dfrac{2yy'}{9} = 0 \Rightarrow \dfrac{\cancel{2}yy'}{9} = -\dfrac{\cancel{2}x}{16}$.

$y' = -\dfrac{9x}{16y} \Rightarrow m = -\dfrac{9x_1}{16y_1}$. — ✓ for correct m

$y - y_1 = m(x - x_1)$

$y - y_1 = -\dfrac{9x_1}{16y_1}(x - x_1)$

$16yy_1 - 16y_1^2 = -9x_1x + 9x_1^2$

$9x_1x + 16yy_1 = 9x_1^2 + 16y_1^2 \qquad \div 9 \times 16$ — ✓ for the correct manipulation of the ~~one-pt formula~~.

$\dfrac{x_1x}{16} + \dfrac{yy_1}{9} = \dfrac{x_1^2}{16} + \dfrac{y_1^2}{9}$ $\qquad$ But $\dfrac{x_1^2}{16} + \dfrac{y_1^2}{9} = 1$.

$\dfrac{x_1x}{16} + \dfrac{yy_1}{9} = 1$.

(iii) $\dfrac{x_0x}{16} + \dfrac{y_0y}{9} = 1$ is the chord PQ equation. — ✓ for PQ equation in general.

$R(x_0, y_0)$ lies on D $\therefore x_0 = \dfrac{16}{\sqrt{7}}$.

$\dfrac{\cancel{16}}{\sqrt{7} \times \cancel{16}}x + \dfrac{y_0y}{9} = 1$.

$\dfrac{x}{\sqrt{7}} + \dfrac{yy_0}{9} = 1$ is equation of PQ. — ✓ for showing $S \in PQ$.

sub $S(\sqrt{7}, 0)$: LHS $= \dfrac{\sqrt{7}}{\sqrt{7}} + 0 = 1 =$ RHS. $\therefore S \in PQ$.

(iv) $\dfrac{x_1x}{16} + \dfrac{yy_1}{9} = 1$ is the tangent.

At $x = \dfrac{16}{\sqrt{7}}$; $\dfrac{\cancel{16}}{\sqrt{7} \times \cancel{16}}x_1 + \dfrac{yy_1}{9} = 1 \Rightarrow \dfrac{yy_1}{9} = 1 - \dfrac{x_1}{\sqrt{7}}$.

$y = \dfrac{9}{y_1}\left(1 - \dfrac{x_1}{\sqrt{7}}\right) = \dfrac{9}{\sqrt{7}y_1}(\sqrt{7} - x_1)$ — ✓ For finding y_R in terms of y_1

$R\left(\dfrac{16}{\sqrt{7}}, \dfrac{9}{\sqrt{7}y_1}(\sqrt{7} - x_1)\right)$

NSBHS Extension 2 Trial HSC - 2004. — Marks.

$m_{SR} = \dfrac{\frac{9}{\sqrt{7}y_1}(\sqrt{7} - x_1) - 0}{\frac{16}{\sqrt{7}} - \sqrt{7}} = \dfrac{\frac{9(\sqrt{7} - x_1)}{\cancel{\sqrt{7}}y_1}}{\frac{16-7}{\cancel{\sqrt{7}}}} = \dfrac{9(\sqrt{7} - x_1)}{\frac{y_1}{9}}$ — ✓ for gradient SR or PS

$m_{SR} = \dfrac{(\sqrt{7} - x_1)}{y_1}$

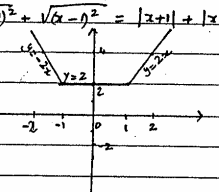
$m_{PS} = \dfrac{y_1 - 0}{x_1 - \sqrt{7}} = \dfrac{y_1}{x_1 - \sqrt{7}}$

$m_{PS} \times m_{SR} = \dfrac{(\sqrt{7} - x_1)}{\cancel{y_1}} \times \dfrac{\cancel{y_1}}{(x_1 - \sqrt{7})} = -\dfrac{(x_1 - \sqrt{7})}{(x_1 - \sqrt{7})} = -1$. — ✓ for proving the product of gradients is $= -1$.

$\therefore PS \perp SR \Rightarrow \angle PSR = 90°$.

(v) P, S, R are concyclic as the $\angle$ in a semi-circle is $90°$ (with PR as diameter).

(b)(i) $f(x) = \sqrt{(x+1)^2} + \sqrt{(x-1)^2} = |x+1| + |x-1|$



✓ ✓

Award:
2 marks for correct graph
0 for non Linear curve.
1 mark for part of the graph correct ~~showing intercepts~~. or for the correct equations.

(ii) Impossible no real solution. ✓ — 1 for correct (ii) answer according to graph

(c) $x^3 + kx + r = 0$

$\alpha^3 + k\alpha + r = 0$ $\qquad \alpha + \beta + \gamma = -\dfrac{b}{a} = 0$.

$\beta^3 + k\beta + r = 0$

$\gamma^3 + k\gamma + r = 0$. — 1 for correct method

$\alpha^3 + \beta^3 + \gamma^3 + k(\alpha + \beta + \gamma) + 3r = 0$. ✓

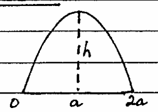
$\alpha^3 + \beta^3 + \gamma^3 = -3r - k(\alpha + \beta + \gamma)$

$= -3r$. ✓ — 1 for correct answer.

NSBHS Extension 2 Trial HSC. - 2004 — Marks.

Question 5:

(a) (i)



x	0	a	$2a$
y	0	h	0

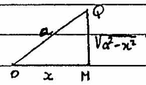
$H = a =$ spacing

$$\text{Area} = \frac{H}{3}\left[y_1 + 4y_2 + y_3\right]$$ ✓ for correct setting

$$= \frac{a}{3}\left[0 + 4h + 0\right]$$ ✓ for correct application of method leading to given answer.

$$= \frac{4ah}{3}.$$

(ii)



$QM = \sqrt{a^2 - x^2}$ (Pythagoras theorm)

$RM = QM = \sqrt{a^2 - x^2}$. ✓ for QM or RM

A_x = Area of parabola $= \frac{4}{3} \times QM \times RM$ (from i).

$$= \frac{4}{3}\sqrt{a^2-x^2}\sqrt{a^2-x^2}.$$ ✓ for correct method using (i) to get the given answer.

$$= \frac{4}{3}(a^2 - x^2).$$

(iii) $$V = \lim_{\Delta x \to 0} \sum_{x=-a}^{x=a} A_x \, \Delta x$$

$$= \int_{-a}^{a} \frac{4}{3}(a^2 - x^2)\,dx \checkmark = 2\int_{0}^{a} \frac{4}{3}(a^2 - x^2)\,dx$$ ✓ for correct integral

$$= \frac{8}{3}\left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{8}{3}\left[a^3 - \frac{a^3}{3}\right] = \frac{8}{3} \times \frac{2a^3}{3} = \frac{16}{9}a^3.$$ ✓ for correct answer

(b) $z^4 - 2z^2 + 8z - 3$.

$P(1-\sqrt{2}i) = 0 \Rightarrow$ the conjugate is a root $\Rightarrow P(1+\sqrt{2}i) = 0$

$[z - (1-\sqrt{2}i)][z - (1+\sqrt{2}i)] = 0$ ✓ — ✓ for conjugate factor

Sum $= 1 - \sqrt{2}i + 1 + \sqrt{2}i = 2$.

Product $= (1-\sqrt{2}i)(1+\sqrt{2}i) = 1 - 2i^2 = 3$.

$\Rightarrow z^2 - Sz + P = 0 \Rightarrow z^2 - 2z + 3 = 0$ is factor.

$$
\begin{array}{r|l}
 & z^2 + 2z - 1 \\
\hline
z^2 - 2z + 3 & z^4 \qquad - 2z^2 + 8z - 3 \\
 & \ominus z^4 \oplus 2z^3 \ominus 3z^2 \\
\hline
 & 2z^3 - 5z^2 + 8z - 3 \\
 & \ominus 2z^3 \oplus 4z^2 \ominus 6z \\
\hline
 & -z^2 + 2z - 3 \\
 & \oplus z^2 \ominus 2z \oplus 3 \\
\hline
 & 0
\end{array}
$$

✓ — ✓ for correct answer of division.

NSBHS Extension 2 Trial Solutions - 2004 — Marks.

Solving: $z^2 + 2z - 1 = 0$

$$z = \frac{-2 \pm \sqrt{4 - 4(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} \checkmark$$

$z = -1 \pm \sqrt{2}$. ✓ for correct real factors.

$$\therefore P(z) = \left(z - (1-\sqrt{2})i\right)\left(z - (1+\sqrt{2}i)\right)\left(z - (-1+\sqrt{2})\right)\left(z - (-1-\sqrt{2})\right) \checkmark$$

$$= (z - 1 + \sqrt{2}i)(z - 1 - \sqrt{2}i)(z + 1 - \sqrt{2})(z + 1 + \sqrt{2}).$$ ✓ for factors over $\mathbb{C}$

(c) (i) $(\sqrt{x} - \sqrt{y})^2 \geq 0$

$x + y - 2\sqrt{xy} \geq 0$. ✓ for correct start.

$x + y \geq 2\sqrt{xy}$. ✓ for correct method.

$\frac{x+y}{2} \geq \sqrt{xy}$.

(ii) let $x = \frac{a+b}{2}$. $y = \frac{c+d}{2}$.

From (i) $\frac{x+y}{2} \geq \sqrt{xy}$.

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}.$$ ✓ for applying (i)

$$\frac{a+b+c+d}{4} \geq \sqrt{\frac{ac+ad+bc+bd}{4}}$$ ✓ for manipulation

Square both sides: $\frac{(a+b+c+d)^2}{16} \geq \frac{ac+ad+bc+bd}{4}$ $\times 16$

$(a+b+c+d)^2 \geq 4(ac+ad+bc+bd)$. ✓ for getting the result.

$\therefore 4(ac+ad+bc+bd) \leq (a+b+c+d)^2$.

N.B: some students' solutions may be:

$a+b \geq 2\sqrt{ab}$
$b+c \geq 2\sqrt{bc}$
$+ \quad c+d \geq 2\sqrt{cd}$
$a+d \geq 2\sqrt{ad}$

$2(a+b+c+d) \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{cd} + \sqrt{ad})$

$(a+b+c+d) \geq (\sqrt{ab} + \sqrt{bc} + \sqrt{cd} + \sqrt{ad})$ ← Award 1 mark for this.

NSBHS Extension 2 - Maths Trial 2004. | Marks.

Question 6.

(a)(i) $y = \sin x = \cos x$ — or: sub $x = \pi/4$ and $y = \frac{1}{\sqrt{2}}$ in both.

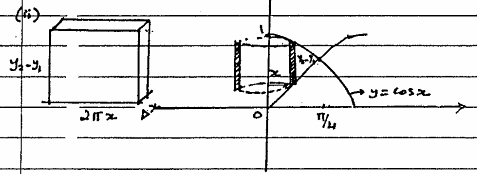
$\frac{\sin x}{\cos x} = 1$

$\tan x = 1.$

$\therefore x = \pi/4.$ — ✓ for showing.

$y = \sin \pi/4 = \frac{1}{\sqrt{2}}$. $\therefore$ pt of $\cap$ is $(\pi/4, \frac{1}{\sqrt{2}})$

(ii)



$A_x = 2\pi x\,(y_2 - y_1)$

$= 2\pi x\,(\cos x - \sin x)$ — ✓ for correct area.

$V_x = 2\pi x\,(\cos x - \sin x)\,\Delta x.$

$V = \lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{\pi/4} 2\pi x\,(\cos x - \sin x)\,\Delta x.$ — ✓ for correct V. expression.

$= \int_0^{\pi/4} 2\pi x\,(\cos x - \sin x)\,dx.$

$= 2\pi \int_0^{\pi/4} \underbrace{x\,(\cos x - \sin x)\,dx}_{I}.$

Let $u = x$, $du = dx$; $v = \cos x - \sin x$, $dv = (\sin x + \cos x)\,dx$ — ✓ for correct method start of Integration by parts.

$= uv - \int v\,du.$

$= x(\sin x + \cos x) - \int (\cos x - \sin x)\,dx$

$= x(\sin x + \cos x) - [-\sin x + \cos x].$

$= x(\sin x + \cos x) + \sin x - \cos x.$

$V = 2\pi \left\{ \left[x(\sin x + \cos x) \right]_0^{\pi/4} + \left[\sin x - \cos x \right]_0^{\pi/4} \right\}$

$= 2\pi \left\{ \frac{\pi}{4}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] + \left[\cancel{\frac{\sqrt{2}}{2}} - \cancel{\frac{\sqrt{2}}{2}} - (1)\right] \right\}$ — ✓ for correct integration process.

$= 2\pi \left[\frac{\pi}{4} \times \sqrt{2} - 1\right] = 2\pi \left[\frac{\sqrt{2}\pi - 4}{4}\right].$

$= \pi \left[\frac{\sqrt{2}\pi - 4}{2}\right].$ cubic unit. — ✓ for correct answer.

{Accept : $\frac{2\pi^2}{4}\left(\frac{2}{\sqrt{2}} - 1\right)$}

NSBHS Extension 2 - Maths Trial 2004 | Marks.

2nd Approach to Q6 (a)(ii) — Marks to Q6 a(ii) again.

(ii) After obtaining:

$V = 2\pi \int_0^{\pi/4} x(\cos x - \sin x)\,dx$ — 1 mark for correct A

$= 2\pi \left[\int_0^{\pi/4} x \cos x\,dx - \int_0^{\pi/4} x \sin x\,dx\right].$ — 1 mark for correct V.

$I_1 = \int_0^{\pi/4} x \cos x\,dx$

$= \left[x \sin x\right]_0^{\pi/4} - \int_0^{\pi/4} \sin x\,dx.$ — 1 mark for correct Integration by parts

$= \left[x \sin x + \cos x\right]_0^{\pi/4} = \frac{\pi}{4} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1.$

$I_2 = \int_0^{\pi/4} x \sin x\,dx$

$= \left[-x \cos x\right]_0^{\pi/4} + \int_0^{\pi/4} \cos x\,dx$

$= \left[-x \cos x + \sin x\right]_0^{\pi/4} = -\frac{\pi}{4} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}.$ — 1 mark for correct process.

$V = 2\pi\left[I_1 - I_2\right]$

$= 2\pi \left[\frac{\pi}{4} \times \frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} - 1 + \frac{\pi}{4} \times \frac{1}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}}\right].$

$= 2\pi \left[\frac{2\pi}{4\sqrt{2}} - 1\right] = \frac{2\pi^2}{2\sqrt{2}} - 2\pi$ — 1 mark for correct answer.

$= \frac{\pi^2}{\sqrt{2}} - 2\pi.$

— or —

3rd approach to $\int$:

(ii) to get the volume $\int$ in one line:

$2\pi \left[(x \sin x + \cos x) + (x \cos x - \sin x)\right]_0^{\pi/4}$ — ✓

$= 2\pi \left[\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 + \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right].$ — ✓

$= 2\pi \left[\frac{2\pi}{4\sqrt{2}} - 1\right] = \frac{\pi^2}{\sqrt{2}} - 2\pi.$ — ✓

Note:

N.B: Some students start wrong to get:

$2\pi \int_0^{\pi/4} \cos x\,(\cos x - \sin x)\,dx$

and get the correct answer to that $\int$ which is $\frac{\pi^2}{4}$.

$\rightarrow$ to get $\pi^2/4$, students will get 3 marks.

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(b) $(1+i)^n = \left[\sqrt{2}\,\text{cis}\,\pi/4\right]^n$ De Moivre's thm.

Expanding $(1+i)^n$

$= 1 + ({}^nC_1)i + {}^nC_2 i^2 + {}^nC_3 i^3 + {}^nC_4 i^4 + {}^nC_5 i^5 + {}^nC_6 i^6 \ldots$

$= 1 + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i + \binom{n}{4} + \binom{n}{5}i - \binom{n}{6} + \ldots$

(knowing $i^2=-1,\ i^3=-i\ ;\ i^4=1\ ;\ i^5=i\ ;\ i^6=-1 \ldots$)

$= 1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots + i\left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} - \cdots\right]$

✓ for expansion.

But: $(1+i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$ as $\sqrt{2} = 2^{1/2}$

$= 2^{n/2}\cos\frac{n\pi}{4} + i\,2^{n/2}\sin\frac{n\pi}{4}$.

✓ for De Moivre's expansion.

Equating real sides:

$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = 2^{n/2}\cos\frac{n\pi}{4}$.

(c) (i) $xy = 9 \Rightarrow y = \frac{9}{x} = 9x^{-1}$.

$y' = -9x^{-2} = -\frac{9}{x^2}$.

tangent: $m_1 = -\frac{9}{x_1^2} \Rightarrow$ normal $m = \frac{x_1^2}{9}$.

$y - y_1 = \frac{x_1^2}{9}(x - x_1)$

$y = \frac{x_1^2}{9}x - \frac{x_1^3}{9} + y_1$ is normal.

✓ for normal equation

To intersect hyperbola: $y = y$.

$\frac{x_1^2}{9}x - \frac{x_1^3}{9} + y_1 = \frac{9}{x}$. but $y_1 = \frac{9}{x_1}$

$\frac{x_1^2 x}{9} - \frac{x_1^3}{9} + \frac{9}{x_1} = \frac{9}{x}$ $\times 9xx_1$.

$x_1^3x^2 - x_1^4x + 81x = 81x_1$.

$x_1^3x^2 + (81 - x_1^4)x - 81x_1 = 0$.

✓ for equating + manipulating to get the given relation.

(ii) quadratic equations whose roots should be x of P and A. i.e. $x_P = x_1$ and $x_A = \alpha$.

Use Sum $= -\frac{b}{a}$. or $P = -\frac{81x_1}{x_1^3}$

$x_1 + \alpha = \frac{-81 + x_1^4}{x_1^3}$ | $\alpha \cdot x_1 = -\frac{81x_1}{x_1^3}$

$\cancel{x_1} + \alpha = -\frac{81}{x_1^3} + \cancel{x_1}$. | $\alpha = -\frac{81}{x_1^2}$

$\alpha = -\frac{81}{x_1^3}$. $y_A = +\frac{9}{x_A} = +\frac{9}{-\frac{81}{x_1^3}} = -\frac{x_1^3}{9}$

✓ for showing x_A.

[Accept substitution if fully shown]

(OR) $(x - x_1)(x_1^3x + 81) = 0$ (Factoring)

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(iii) $A\left(\frac{-81}{x_1^3}, \frac{-x_1^3}{9}\right)$ $P\left(x_1, \frac{9}{x_1}\right)$.

$X = x_M = \frac{x_A + x_P}{2} = \frac{-\frac{81}{x_1^3} + x_1}{2} = \frac{-81 + x_1^4}{2x_1^3}$. .... (1)

$Y = y_M = \frac{y_A + y_P}{2} = \frac{-\frac{x_1^3}{9} + \frac{9}{x_1}}{2} = \frac{\frac{-x_1^4 + 81}{9x_1}}{2}$

$Y = \frac{81 - x_1^4}{18x_1}$. .... (2).

✓ for both coordinates of A.

$\frac{(2)}{(1)}$ $\frac{Y}{X} = \frac{81 - x_1^4}{18x_1} \div \frac{(x_1^4 - 81)}{2x_1^3}$.

$\frac{Y}{X} = \frac{(81 - x_1^4)}{18x_1} \times \frac{2x_1^3}{-(81 - x_1^4)} = -\frac{x_1^2}{9}$

$\Rightarrow x_1^2 = -\frac{9Y}{X}$. (3).

✓ getting x_1 in terms of X and Y.

Using (2) again: $Y = \frac{81 - x_1^4}{18x_1}$. .... (3)

Square both sides: $Y^2 = \frac{(81 - x_1^4)^2}{18^2 x_1^2}$

but $x_1^2 = -\frac{9Y}{X}$ $\Rightarrow Y^2 = \frac{\left(81 - \left(-\frac{9Y}{X}\right)^2\right)^2}{18^2\left(-\frac{9Y}{X}\right)}$.

✓ trying to get rid of x_1 in $x_1^2 = -\frac{9Y}{x}$.

$\therefore 18^2\left(-\frac{9Y}{X}\right)Y^2 = \left(81 - \frac{81Y^2}{X^2}\right)^2$.

$-\frac{18^2(9)Y^3}{X} = \frac{(81X^2 - 81Y^2)^2}{X^4}$ $\times X^4$.

$(-18^2)(9)(Y^3)X^3 = 81^2(X^2 - Y^2)^2$ $\div 9^3 = 729$

$-4Y^3X^3 = 9(X^2 - Y^2)^2$.

$\therefore 4X^3Y^3 = -9(X^2 - Y^2)^2$

or $4X^3Y^3 = 9(Y^2 - X^2)^2$

✓ 1 for getting the relation.

NSBHS Extension 2 - Maths Trial 2004.   Marks

Question 7:

(a) SOCCER

(i) 2 C's : $\frac{6!}{2} = 360$.

✓ for $\frac{6!}{2}$ or answer

(ii) 2C's $\quad {}^4C_2$

1C, 3 others $\quad {}^4C_1$

0C, 4 others $\quad {}^4C_0$

${}^4C_2 + {}^4C_1 + {}^4C_0 = 11$

or: 4 different letters from S, O, C, E, R: ${}^5C_4$

+ 2C's, 2 diff: ${}^4C_2$

${}^5C_4 + {}^4C_2 = 5 + 6 = 11$

✓ for correct method

✓ for answer.

N.B: If students wrote:

${}^4C_2 \times {}^2C_2 + {}^2C_1 \times {}^4C_3 + {}^2C_0 \times {}^4C_4 = 15 \rightarrow$ they get only 1 mark.

(b) (i) Downward Motion is the only motion to get the terminal velocity:

$\uparrow mkv^2$, $\downarrow mg$, $+\downarrow$

$m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

Terminal velocity: $\ddot{x} = 0$

$g - kV^2 = 0$

$\Rightarrow V^2 = \frac{g}{k}. \Rightarrow V = \sqrt{\frac{g}{k}}$ is Terminal velocity

✓ 1 mark to show the correct downward $\ddot{x} = 0$.

[No marks for upward motion $\ddot{x}$].

(ii) Upwards:

$+\uparrow$, $\downarrow mkv^2$, $\downarrow mg$

$M\ddot{x} = -Mkv^2 - Mg$.

$\ddot{x} = -kv^2 - g$.

$\ddot{x} = -(g + kv^2)$

$v\frac{dv}{dx} = -(g + kv^2)$.

Max ht: → ; Initial →

$$\therefore -\int_V^0 \frac{v\,dv}{g + kv^2} = \int_0^H dx$$

✓ for correct mtn $\ddot{x}$ or $\int$.

$$H - 0 = -\frac{1}{2}\int_V^0 \frac{2v\,dv}{g + kv^2} = -\frac{1}{2k}\int_V^0 \frac{2kv\,dv}{g + kv^2}$$

$$= -\frac{1}{2}\left[\ln(g + kv^2)\right]_V^0$$

✓ for correct integration

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$$H = -\frac{1}{2k}\left[\ln g - \ln(kV^2 + g)\right].$$

$$= \frac{1}{2k}\left[\ln(kV^2 + g) - \ln g\right]$$

$$= \frac{1}{2k}\ln\left(\frac{kV^2 + g}{g}\right)$$

$$= \frac{1}{2k}\ln\left(\frac{kV^2}{g} + 1\right) \quad \text{But } V^2 = \frac{g}{k}.$$

$$= \frac{1}{2k}\ln\left(\frac{k}{g}\times\frac{g}{k} + 1\right) = \frac{1}{2k}\ln 2.$$

✓ for correct manipulation to get the given answer.

Longer Approaches: is to find the general x equation and then sub $v = 0$ to get H:

OR

2nd approach:

(b-ii) $v\frac{dv}{dx} = -(g + kv^2)$

$$-\int_V^v \frac{v\,dv}{g + kv^2} = +\int_0^x dx$$

✓ for correct $\ddot{x}$ or $\int$

$$x = \frac{-1}{2k}\int_V^v \frac{2kv\,dv}{g + kv^2} = \frac{-1}{2k}\ln(g + kv^2)\Big]_V^v$$

$$= -\frac{1}{2k}\ln\left(\frac{g + kv^2}{g + kV^2}\right) = \frac{1}{2k}\ln\left(\frac{g + kV^2}{g + kv^2}\right).$$

✓ for correct $\int$ion.

at max ht. $v = 0$.

$$H = x = \frac{1}{2k}\ln\left(\frac{g + kV^2}{g}\right) \quad \text{But } V^2 = \frac{g}{k}.$$

$$H = \frac{1}{2k}\ln\left(1 + \frac{k}{g}\times\frac{g}{k}\right) = \frac{1}{2k}\ln 2.$$

✓ for manipulation to get correct given answer.

3rd approach: Finding constants.

Longer! (b-ii)

or

$$-x = \int \frac{v\,dv}{g + kv^2}$$

$$-x = \frac{1}{2k}\ln(g + kv^2) + C$$

✓ for correct x.

when $x = 0$, $V = \sqrt{\frac{g}{k}}$. $\quad 0 = \frac{1}{2k}\ln\left(g + k\frac{g}{k}\right) + C$

$C = -\frac{1}{2k}\ln 2.$ $\quad$ or $\quad C = -\frac{1}{2k}\ln(g + kV^2)$

$$+x = -\left[\frac{1}{2k}\ln(g + kv^2)\right] + \frac{1}{2k}\ln 2.$$

✓ for correct $\int$ion

$$x = \frac{1}{2k}\ln\left(\frac{2}{g + kv^2}\right) \quad \text{or} \quad x = \frac{1}{2k}\ln\left(\frac{g + kV^2}{g + kv^2}\right)$$

then showing when $x = H$, $v = 0$ that

✓ for correct manipulation.

(iii) Upward Mtn: $\ddot{x} = -(g + kv^2)$

$$\frac{dv}{dt} = -(g + kv^2).$$

max ht. → initial →

$$\Rightarrow \int_0^T dt = -\int_V^0 \frac{dv}{g + kv^2}$$

✓ for correct $\ddot{x}$ or $\int$

$$T = -\frac{1}{k}\int_V^0 \frac{dv}{\left(\frac{g}{k} + v^2\right)} = -\frac{1}{k}\sqrt{\frac{k}{g}}\left[\tan^{-1}\sqrt{\frac{k}{g}}\,v\right]_V^0$$

✓ for correct indefinite $\int$ answer.

$$T = -\frac{1}{k}\sqrt{\frac{k}{g}}\left[\tan^{-1}0 - \tan^{-1}\sqrt{\frac{k}{g}}\,V\right]$$

But $= -\frac{1}{k}\sqrt{\frac{k}{g}} = -\frac{1}{kV}$; $\tan^{-1}\sqrt{\frac{k}{g}}V = \tan^{-1}\frac{1}{V} \times V$

✓ for correct T equation

$$T = -\frac{1}{kV}\left[-\tan^{-1}1\right] = -\frac{1}{kV} \times \left(\frac{-\pi}{4}\right)$$

✓ for manipulation to get correct answer.

But $\frac{1}{k} = \frac{V^2}{g} \Rightarrow T = \frac{V^2}{g} \times \frac{1}{V} \times \frac{\pi}{4} = \frac{V}{g}\frac{\pi}{4}$.

2nd ☹ Longer Approach: get the time equation in general and then sub. in $v = 0$ (max ht).

OR

i.e. $\int_V^v \frac{dv}{g + kv^2} = -\int_0^t dt$

✓ for correct $\int$ or $\ddot{x}$

$$-t = \frac{1}{k}\int_V^v \frac{dv}{\frac{g}{k} + v^2}.$$

$$-t = \frac{1}{k}\sqrt{\frac{k}{g}}\,\tan^{-1}\left[\sqrt{\frac{k}{g}}\cdot v\right]_V^v$$

✓ for correct integration answer (indefinite)

$$t = -\frac{1}{k} \times \frac{1}{V}\left[\tan^{-1}\sqrt{\frac{k}{g}}\,v - \tan^{-1}\sqrt{\frac{k}{g}} \times V\right]$$

$$= -\frac{1}{kV}\left[\tan^{-1}\sqrt{\frac{k}{g}}\,v - \tan^{-1}\frac{1}{V} \times V\right].$$

✓ for correct definite $\int$ answer

at max ht., $v = 0$.

$T = -\frac{1}{kV}\left[-\tan^{-1}1\right] = \frac{1}{kV} \times \frac{\pi}{4}$ which could be shown as above

✓ for manipulation.

$$T = \frac{V^2}{g} \times \frac{1}{V} \times \frac{\pi}{4} = \frac{\pi V}{4g}.$$

(iv) Downward Motion:

$$\ddot{x} = \frac{dv}{dt} = g - kv^2.$$

$$\int_0^v \frac{dv}{g - kv^2} = \int_0^t dt$$

✓ for correct expression

$$t = \int_0^v \frac{dv}{(\sqrt{g} - \sqrt{k}v)(\sqrt{g} + \sqrt{k}v)}$$

$$\frac{1}{g - kv^2} = \frac{a}{\sqrt{g} - \sqrt{k}v} + \frac{b}{\sqrt{g} + \sqrt{k}v}$$

$$a = \lim_{v \to \sqrt{\frac{g}{k}}} \frac{1}{\sqrt{g} + \sqrt{k}v} = \frac{1}{\sqrt{g} + \sqrt{k}\cdot\frac{\sqrt{g}}{\sqrt{k}}} = \frac{1}{2\sqrt{g}}$$

$$b = \lim_{v \to -\sqrt{\frac{g}{k}}} \frac{1}{(\sqrt{g} - \sqrt{k}v)} = \frac{1}{\sqrt{g} + \sqrt{k}\cdot\frac{\sqrt{g}}{\sqrt{k}}} = \frac{1}{2\sqrt{g}}.$$

✓ for correct resolution of partial fractions.

$$\therefore t = \frac{1}{2\sqrt{g}}\left[\int \frac{dv}{\sqrt{g} - \sqrt{k}v} + \int \frac{dv}{\sqrt{g} + \sqrt{k}v}\right]$$

$$= \frac{1}{2\sqrt{g}}\left[\frac{1}{\sqrt{k}}\ln(\sqrt{g} - \sqrt{k}v) + \frac{1}{\sqrt{k}}\ln(\sqrt{g} + \sqrt{k}v)\right]$$

$$= \frac{1}{2\sqrt{g}} \times \frac{1}{\sqrt{k}}\ln\left(\frac{\sqrt{g} + \sqrt{k}v}{\sqrt{g} - \sqrt{k}v}\right)$$

✓ for integration expansion.

$$= \frac{1}{2\sqrt{gk}}\ln\left(\frac{\frac{\sqrt{g}}{\sqrt{k}} + v}{\frac{\sqrt{g}}{\sqrt{k}} - v}\right) = \frac{1}{2\sqrt{gk}}\ln\left(\frac{V + v}{V - v}\right).$$

✓ for manipulation to get given answer

OR

2nd approach: (Longer).

$$\frac{dv}{dt} = g - kv^2 = kV^2 - kv^2 = k(V^2 - v^2)$$

$$\therefore \int_0^v \frac{dv}{V^2 - v^2} = \int_0^t \frac{g}{V^2}\,dt$$

✓ for correct expression.

$$\frac{1}{V^2 - v^2} = \frac{1}{2V}\left[\frac{1}{V - v}\right] + \frac{1}{2V}\left[\frac{1}{V + v}\right]$$ partial fraction.

✓ for correct partial fractions.

$$\therefore \frac{gt}{V^2} = \int_0^v \frac{dv}{V^2 - v^2} = \frac{1}{2V}\left[-\ln(V - v) + \ln(V + v)\right]_0^v$$

✓ for correct

$$\frac{gt}{V^2} = \frac{1}{2V}\left[\ln\left(\frac{V + v}{V - v}\right)\right]_0^v = \frac{1}{2V}\left[\ln\left(\frac{V + v}{V - v}\right) - \ln 1\right]$$

$$\frac{gt}{V^2} = \frac{1}{2V}\ln\left(\frac{V + v}{V - v}\right)$$ But $V = \frac{\sqrt{g}}{\sqrt{k}}$.

✓ for correct manipulation.

$$t = \frac{V}{2g}\ln\left(\frac{V + v}{V - v}\right) = \frac{1}{2g}\frac{\sqrt{g}}{\sqrt{k}}\cdot\ln\left(\frac{V + v}{V - v}\right) = {}^{1}$$

$$t = \frac{1}{}\ln\left(\frac{V+v}{}\right).$$

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Question 8:

(a) $I_n = \int_0^1 (1-x^2)^n \, dx \, , \quad n \geq 0.$

$u = (1-x^2)^n \qquad dv = dx$

$du = n(1-x^2)^{n-1}(-2x) \qquad v = x.$

$I_n = uv - \int v \, du.$

$= \left[x(1-x^2)^n\right]_0^1 - n\int_0^1 x(-2x)(1-x^2)^{n-1} \, dx$ ✓ for correct start of integration of parts.

$= 0 + 2n\int x^2(1-x^2)^{n-1} \, dx.$

$= -2n\int (-x^2)(1-x^2)^{n-1} \, dx$ ✓ for answer

$= -2n\int (1-x^2-1)(1-x^2)^{n-1} \, dx.$

$= -2n\left[\int_0^1 (1-x^2)^1(1-x^2)^{n-1} dx - \int_0^1 (1-x^2)^{n-1} \, dx\right]$

$= -2n\left[\int_0^1 (1-x^2)^n \, dx - \int_0^1 (1-x^2)^{n-1} \, dx\right].$ ✓ for correct manipulation to get the relation.

$= -2n\left[I_n - I_{n-1}\right].$

$I_n = -2n\, I_n + 2n\, I_{n-1}.$

$I_n(1+2n) = 2n\, I_{n-1}$

$I_n = \dfrac{2n\, I_{n-1}}{1+2n} \quad$ for $n \geq 1.$

(ii) $I_0 = \int_0^1 (1-x^2)^0 \, dx = \int dx = [x]_0^1 = 1.$

$I_n = \dfrac{2n}{1+2n}\underbrace{\left[\dfrac{2(n-1)}{1+2(n-1)}\right]}_{I_{n-1}}\underbrace{\left[\dfrac{2(n-2)}{1+2(n-2)}\right]}_{I_{n-2}}\underbrace{\left[\dfrac{2(n-3)}{1+2(n-3)}\right]}_{I_{n-3}}\cdots\underbrace{\left(\dfrac{2\times3}{7}\right)}_{I_3}\times$ ✓ for correct resolution ~~to I_0~~

$\underbrace{\left(\dfrac{2\times2}{5}\right)}_{I_2}\times\dfrac{2}{3}\times 1.$

$I_n = \dfrac{2^n\left[n(n-1)(n-2)\ldots 3\times2\times1\right]}{(1+2n)(2n-1)(2n-3)(2n-5)\ldots 7\times5\times3\times1}$

✓ for relation.

$= \dfrac{2^n\, n!}{\underbrace{(2n+1)(2n-1)(2n-3)(2n-5)\cdots 7\times5\times3\times1}_{\text{odd Numbers}}}$

For the smarties odd numbers $= \dfrac{(2n+1)!}{2^n \cdot n!}$

$I_n = \dfrac{2^n\, n!}{\dfrac{(2n+1)!}{2^n \cdot n!}} = \dfrac{2^{2n}(n!)^2}{(2n+1)!}.$

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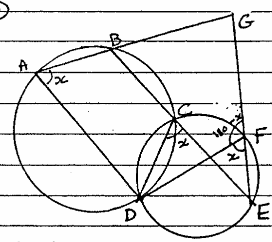
Q8 – (b)



Model Answer: | Award 1 mark for each of the reasons:

Let $\angle DAB = x.$

$\therefore \angle DCE = \angle DAB = x°$ (Exterior $\angle$ of cyclic quad. = opp. interior $\angle$) ✓

$\therefore \angle DFE = DCE = x$ ($\angle$s in same segment are =) ✓

$\therefore \angle DFG = 180° - x$ (GFE is a st. line). ✓

$\therefore \angle DAG + \angle DFG = x + (180 - x)$

$= 180°.$

[or: $\angle DFE = x = \angle DAG$ (Exterior $\angle$ theorem ...))

$\therefore$ GFDA is a cyclic quadrilateral ✓

since a pair of its opp. $\angle$s are supplementary

(c) $U_1 = 1 \; ; \; U_{n+1} = \frac{1}{2}\left[U_n + \frac{2}{U_n}\right]$ when $n \geq 1$: $\dfrac{U_n - \sqrt2}{U_n + \sqrt2} = \left(\dfrac{1-\sqrt2}{1+\sqrt2}\right)^{2^{n-1}}$

step 1: Prove true for $n = 1$. $S(n)$:

$U_1 = 1 \; ;$ LHS $= \dfrac{U_1 - \sqrt2}{U_1 + \sqrt2} = \dfrac{1-\sqrt2}{1+\sqrt2}$

RHS $= \left(\dfrac{1-\sqrt2}{1+\sqrt2}\right)^{2^0} = \left(\dfrac{1-\sqrt2}{1+\sqrt2}\right)^1 = \dfrac{1-\sqrt2}{1+\sqrt2}$ } $S(1)$ is true. ✓ for 1st step.

step 2: If $S(k)$ is true: $\dfrac{U_k - \sqrt2}{U_k + \sqrt2} = \left(\dfrac{1-\sqrt2}{1+\sqrt2}\right)^{2^{k-1}}$ then

Aim: prove $S(k+1) = \dfrac{U_{k+1} - \sqrt2}{U_{k+1} + \sqrt2} = \left(\dfrac{1-\sqrt2}{1+\sqrt2}\right)^{2^k}.$ | Award ✓ for each ~~important step as shown:~~

Proof: $\dfrac{U_{k+1} - \sqrt2}{U_{k+1} + \sqrt2} = \dfrac{\frac12\left[U_k + \frac{2}{U_k}\right] - \sqrt2}{\frac12\left[U_k + \frac{2}{U_k}\right] + \sqrt2} = \dfrac{\cancel{\frac12}\left[U_k + \frac{2}{U_k} - 2\sqrt2\right]}{\cancel{\frac12}\left[U_k + \frac{2}{U_k} + 2\sqrt2\right]}$ ✓

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$$\therefore \frac{U_{k+1}-\sqrt{2}}{U_{k+1}+\sqrt{2}} = \frac{U_k^2+2-2\sqrt{2}U_k}{U_k^2+2\sqrt{2}U_k+2}$$

$$= \frac{(U_k-\sqrt{2})^2}{(U_k+\sqrt{2})^2}$$ ✓

$$= \left(\frac{U_k-\sqrt{2}}{U_k+\sqrt{2}}\right)^2$$

from assumption

$$= \left[\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}}\right]^2$$

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{k-1}\cdot 2}$$ ✓

$$= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^k}.$$

$\therefore$ $S(k+1)$ is true whenever $S(k)$ is true.

$\Rightarrow$ $S(k)$ is true.

(ii) $-1 < \frac{1-\sqrt{2}}{1+\sqrt{2}} < 1$ Let $r = \frac{1-\sqrt{2}}{1+\sqrt{2}}$.

As $n \to \infty$: ratio $= \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}} \to 0$ ✓ for justification of why RHS $\to 0$

since $|r|<1$, $r^\infty \to 0$.

$$\therefore \frac{U_n-\sqrt{2}}{U_n+\sqrt{2}} \to 0.$$

✓ for showing the answer $U_n \to \sqrt{2}$.

$\therefore$ $U_n - \sqrt{2} \to 0$

$\therefore$ $U_n \to \sqrt{2}$.

Thus for n sufficiently Large, $U_n \to 2$.