Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2011 HSC ASSESSMENT TASK 3

Mathematics Extension 2

Examiner: S. Ireland

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on both sides of the paper (with lines) in the booklet(s) provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- This is a school assessment task.The task's content, format and mark scheme do not necessarily reflect that of the HSC.
- Attempt all questions
- Each new question is to be started on a **new page.**

Class Teacher:
(Please tick or highlight)

- O Ms Collins
- O Mr Fletcher
- O Mr Ireland

Student Number

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	/15	/15	/15	/15	/15	/15	/15	/15	/120	/100

Question 1 (15 marks) Start a new page. **Marks**

(a) Evaluate $\int_1^3 \frac{dx}{x(x+2)}$ 3

(b) Evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx$ 3

(c) Find $\int \frac{x}{x^2+2x+5} dx$ 3

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ 4

(e) Find $\int \frac{dx}{\sqrt{x^2-6x+8}}$ 2

Question 2 (15 marks) Start a new page.

(a) If $z = \frac{1+7i}{3-4i}$, then

(i) Write z in $a + ib$ form (where a and b are real) 1

(ii) Find $|z|$ 1

(iii) Find arg z 1

(iv) Calculate z^8 1

(b) On separate Argand diagrams sketch the locus of a point which satisfies:

(i) $\arg\,(z+1+i) = \frac{\pi}{4}$ 1

(ii) $\text{Re}(z) + \text{Im}(\bar{z}) = 1$ 1

(c) Express the square root of $-2i$ in the form $a + ib$ 2

(d) If z is a complex number such that $z + \frac{1}{z}$ is real, prove that either z is real or $|z| = 1$. 3

(e) The equation $z^3 + az^2 + bz + 6 = 0$, where a and b are real numbers, has $1 + i$ as a root. 4

Find a and b, and solve the equation completely.

Question 3 (15 marks) Start a new page. **Marks**

(a) (i) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity r, then k is a zero of multiplicity $r-1$ of $P'(x)$. 1

(ii) Given that the polynomial $P(x) = x^4 + 5x^3 + 9x^2 + 7x + 2$ has a zero of multiplicity 3, factorise $P(x)$. 3

(b) The equation $x^3 - 4x^2 + 5x + 2 = 0$ has roots α, β, γ.

Find:

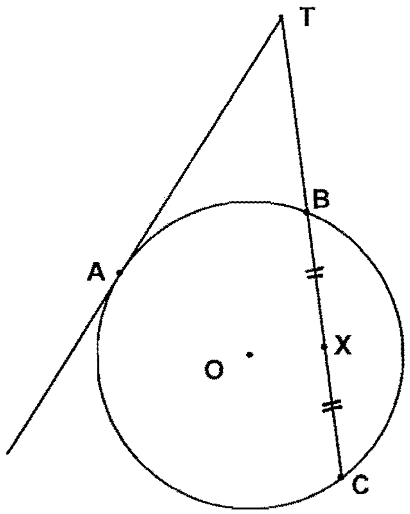
(i) $\alpha^2 + \beta^2 + \gamma^2$ 2

(ii) $\alpha^3 + \beta^3 + \gamma^3$ 2

(c) If $\alpha, \beta,$ and γ are roots of $8x^3 - 4x^2 + 6x - 1 = 0$, find the equation whose roots are $\frac{1}{1-\alpha}, \frac{1}{1-\beta}$ and $\frac{1}{1-\gamma}$. 3

(d)

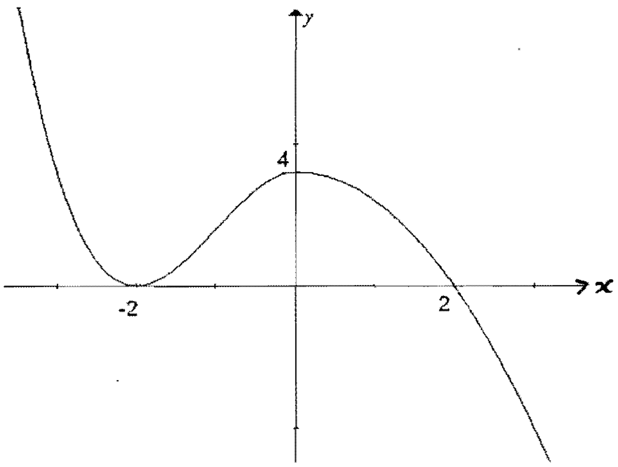
(i) Copy the diagram into your answer booklet:



(ii) A, B, C are three points on the circumference of a circle, centre O. The tangent at A meets BC produced at T. X is the midpoint of BC.
Prove that $\angle AOT = \angle AXT$. 4

Question 4 (15 marks) Start a new page. **Marks**

(a) The diagram shows $y = f(x)$.



Draw separate one-third page sketches of the following:

(i) $y = \frac{1}{f(x)}$ 2

(ii) $y = |f(x)|$ 1

(iii) $|y| = f(|x|)$ 2

(iv) $y^2 = f(x)$ 2

(v) $y = e^{f(x)}$ 2

(b) Find the equation of the tangent to $x^3 + \mathrm{xy} - \mathrm{y}^3 = 1$ at the point $(1, 1)$. 3

(c) Sketch on the same number plane $y = |x| - 2$ and

$y = 4 + 3x - x^2$.

Hence, or otherwise, solve $\frac{|x|-2}{4+3x-x^2} > 0$ 3

Question 5 (15 marks) Start a new page. **Marks**

(a) Let $P(x_1, y_1)$ be a point on the ellipse $16x^2 + 25y^2 = 400$.

(i) Draw the ellipse, showing all intercepts. 1

(ii) Write down the eccentricity. 1

(iii) Show that the normal at P has equation

$25y_1x - 16x_1y - 9x_1y_1 = 0$ 2

(iv) The normal at P meets the major axis at N. Using the focus-directrix definition of an ellipse, or otherwise,

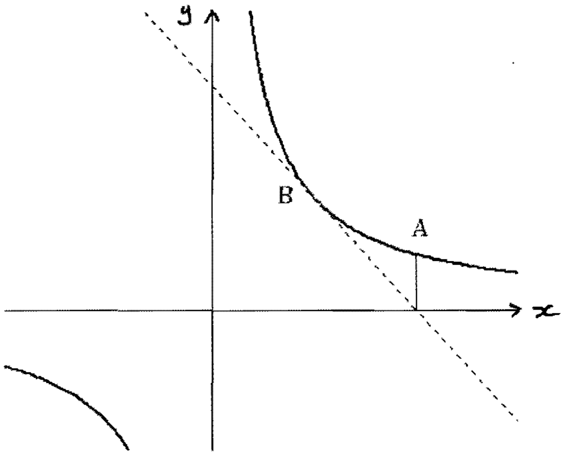
prove that $\frac{NS}{NS'} = \frac{PS}{PS'}$ 3

(b) The area enclosed by the parabola $y = (x-3)^2$ and the straight line $y = 9$ is rotated about the y-axis. Use the method of cylindrical shells to find the exact volume of the resulting solid. 4

(c) The region bounded by the curve $y = x^2$ and the straight line $y = 4$ is rotated about the line $x = 2$. Use a slicing method – with slices perpendicular to the axis of rotation - to find the exact volume of the resulting solid. 4

Question 6 (15 marks) Start a new page. **Marks**

(a) A and B are variable points on the rectangular hyperbola $xy = c^2$.



The tangent at B passes through the foot of the ordinate (y–value) of A.

(i) If A and B have parameters t_1 and t_2 show that $t_1 = 2t_2$. 3

(ii) Hence prove that the locus of the midpoint of AB is also a rectangular hyperbola. 2

(b) (i) Show that the locus specified by

$$|z - 2| = 2\left(\text{Re}\,(z) - \tfrac{1}{2}\right)$$

is a branch of the hyperbola $\frac{x^2}{1} - \frac{y^2}{3} = 1$, and indicate why it must be that particular branch. 3

(ii) Sketch the locus, and find the possible set of values of each of $|z|$ and $\arg z$ for a point on the locus. 3

(c) The region bounded by the parabolas $y = 6 - x^2$ and $y = \frac{1}{2}x^2$ forms the base of a solid. Cross-sections by planes perpendicular to the y-axis are semi-circles, with their diameters in the base of the solid.

(i) Find the points of intersection of the two parabolas. 1

(ii) Find the volume of the solid. 3

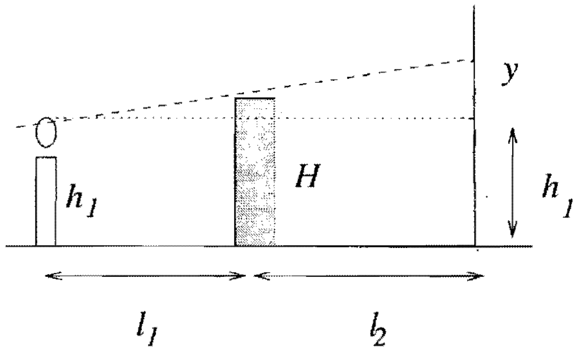
Question 7 (15 marks) Start a new page. **Marks**

(a) A castle with walls 30 metres high is surrounded by a moat 20 metres wide. An archer, kneeling at the edge of the moat, attempts to shoot over the wall.

(i) Taking g as 10 ms^{-2} and the speed of an arrow as it leaves the bowstring as 40 ms^{-1} - and disregarding air resistance - derive the equations of motion for an arrow. 2

(ii) Calculate the range of angles through which the archer must fire in order to clear the top of the wall. 4

(b) A candle is placed a distance l_1 from a thin block of wood of height H. The block is a distance l_2 from a wall, as shown in the diagram:



The candle burns down so that the height of the flame h_1 decreases at the rate of 3 cm/h. Find the rate at which the length of the shadow, y, cast by the block on the wall increases.

(Your answer will be in terms of the constants l_1 and l_2). 3

(c) For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution? 3

(d) The function f is defined by

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}.$$

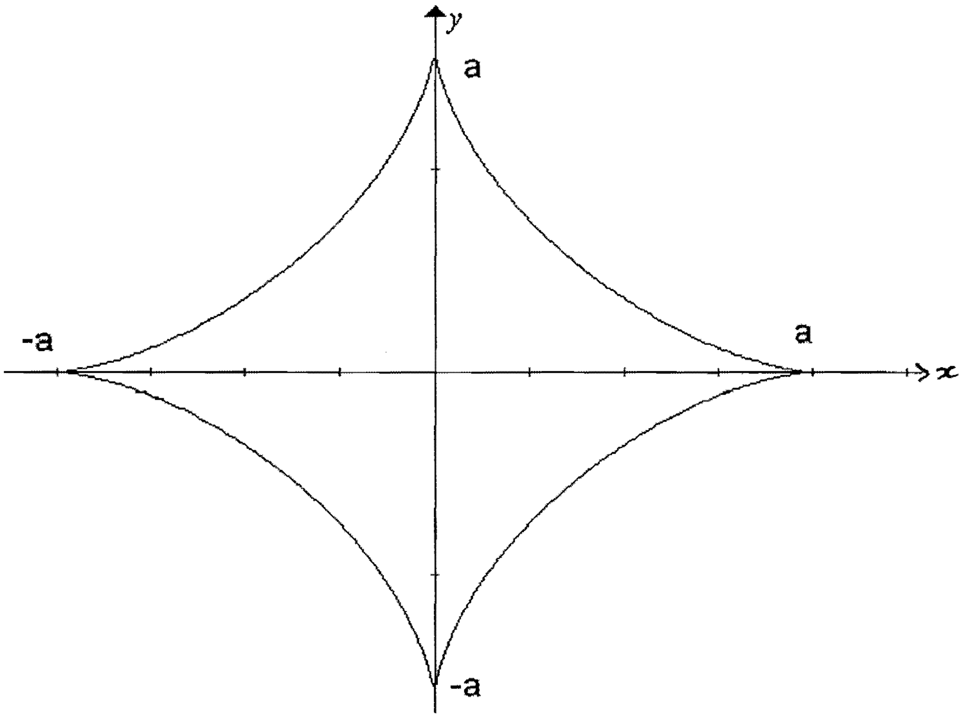
Find the maximum value of $f(x)$ using a graphical method. 3

Question 8 (15 marks) Start a new page. **Marks**

(a) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$, show that $I_n = \frac{n-1}{n} I_{n-2}$.

Hence evaluate I_4 and I_6. 5

(b) The asteroid curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is sketched below:



(i) Show that a parametric representation of the curve is given by $x = a\cos^3\theta,\ y = a\sin^3\theta$ 1

(ii) Show that the gradient of the tangent to the asteroid at any point $P(a\cos^3\phi, a\sin^3\phi)$ on it is equal to $\frac{dy}{dx} = -\tan\phi$. 1

(iii) Show that the length of a tangent line to the asteroid at any point $P(a\cos^3\phi, a\sin^3\phi)$ on it, cut off by the coordinate axes, is constant. 3

(iv) Find the area enclosed by the asteroid curve. [The substitution $x = a\cos^3\theta$ and the results of Question 8 part (a) may be useful.] 5

END OF EXAMINATION

Q1 (a)

Let $\dfrac{1}{x(x+2)} = \dfrac{a}{x} + \dfrac{b}{x+2}$

$\therefore 1 = (a+b)x + 2a$

$\therefore \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{2} \end{cases}$

$\therefore \displaystyle\int_1^3 \frac{dx}{x(x+2)} = \frac{1}{2}\int_1^3 \left(\frac{1}{x} - \frac{1}{x+2}\right) dx$

$= \frac{1}{2}\left[\ln x - \ln(x+2)\right]_1^3$

$= \frac{1}{2}\left[(\ln 3 - \ln 5) - (\ln 1 - \ln 3)\right]$

$= \frac{1}{2}\left[2\ln 3 - \ln 5\right]$

$\left(= \frac{1}{2}\ln\left(\frac{9}{5}\right)\right)$

Correct Solution with working : 3

Applies partial fractions correctly but does not get answer correctly : 2

Reasonable attempt to use partial fractions : 1

1 (b)

$\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx = \int_0^{\frac{\pi}{4}} (1+\tan^2 x)\cdot\tan^4 x \cdot \sec^2 x \, dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} (\tan^4 x + \tan^6 x)\sec^2 x \, dx$

$= \left[\dfrac{\tan^5 x}{5} + \dfrac{\tan^7 x}{7}\right]_0^{\pi/4}$

$= \left(\frac{1}{5} + \frac{1}{7}\right) - (0+0)$

$= \dfrac{12}{35}$

✓ recognises + uses trig. identity correctly

✓ correct primitive

✓ correct answer

Q1 ctd

(c) $\displaystyle\int \frac{x}{x^2+2x+5}\,dx = \int \frac{\frac{1}{2}(2x+2)-1}{x^2+2x+5}\,dx$

$= \dfrac{1}{2}\displaystyle\int \frac{2x+2}{x^2+2x+5}\,dx - \int \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

ALT: Complete square in denominator & use $u = x+1$ substitution.

✓ formats numerator appropriately

✓ correctly split into 2 integrals

✓ correct answer

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$.

Let $t = \tan\frac{x}{2}$ $\quad\therefore \dfrac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$\therefore dx = \dfrac{2\,dt}{1+t^2}$

Also, when $x=0$, $t=0$

& $x = \frac{\pi}{2}$, $t = 1$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x} = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$

$= \displaystyle\int_0^1 \frac{2\,dt}{1+t^2+2t+1-t^2}$

$= \displaystyle\int_0^1 \frac{1}{1+t}\,dt$

$= \left[\ln(1+t)\right]_0^1$

$= \ln 2$.

✓ Correctly sets up t substitution

✓ Correct substitution in integral

✓✓ Correct working and answer

1 (e) ctd.

$\displaystyle\int \frac{dx}{\sqrt{x^2-6x+8}} = \int \frac{dx}{\sqrt{(x-3)^2-1}}$

$= \ln\left|x-3+\sqrt{x^2-6x+8}\right| + C$

✓ completes square

✓ correct answer

[no penalty for omitting "C" or absolute value signs.]

Q2

a) (i) $\dfrac{1+7i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{3+4i+21i-28}{3^2+4^2}$

$= \dfrac{-25+25i}{25}$

$= -1+i$

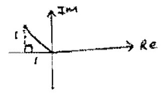
✓ correct answer

(ii) $|z| = \sqrt{1^2+1^2}$

$= \sqrt{2}$

✓ correct answer

(iii)



$\therefore \arg z = \dfrac{3\pi}{4}$

✓ correct answer

(iv) $z^8 = \left[\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}\right]^8$

$= (\sqrt{2})^8\,\text{cis}\left(8\cdot\frac{3\pi}{4}\right)$

$= 16\,\text{cis}\,6\pi$

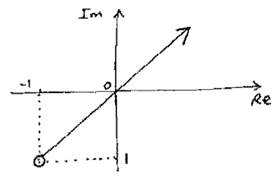
$= 16$

✓ correct answer

Q2 ctd

(b) (i) $\arg\left(z-(-1-i)\right) = \frac{\pi}{4}$

$\therefore$



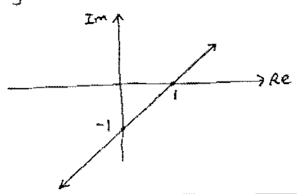
✓ Correct sketch

(ii) $Re(z) + Im(\bar{z}) = 1$

Let $z = x+iy$ $\quad\therefore \bar{z} = x-iy$

$\therefore x - y = 1$

$\therefore$



✓ Correct Sketch.

(c) $(a+ib)^2 = -2i$

$(a^2-b^2)+(2ab)i = 0 - 2i$

$\therefore \begin{cases} a^2-b^2 = 0 \\ ab = -1 \end{cases}$

$b = \frac{-1}{a}$ $\quad\therefore a^2 - \frac{1}{a^2} = 0$

$a^4 - 1 = 0$

$(a^2-1)(a^2+1) = 0$

$\therefore a = -1$ or 1 $\quad$ (a is real)

$\therefore b = 1$ or -1

Thus square roots are $1-i$ and $-1+i$

(i.e. $\pm(1-i)$)

✓

✓

ctd **2(d)**

Let $z = x + iy$

$\therefore z + \frac{1}{z} = x + iy + \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$

$= x + iy + \frac{x-iy}{x^2+y^2}$

$= \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$. ✓ Simplifies $z+\frac{1}{z}$ correctly

, if $z + \frac{1}{z}$ is real, then $y = \frac{y}{x^2+y^2}$

$\therefore y(x^2+y^2) = y$ ✓ uses assumption correctly

$y(x^2+y^2-1) = 0$

thus $y = 0$ or $x^2+y^2-1 = 0$ ✓ Correct conclusion

ie z is real or $|z| = 1$.

(e) $z^3 + az^2 + bz + 6 = 0$.

Since $1+i$ is a root, so is $1-i$ ✓ Correct conjugate root

(as coefficients are real).

Let the third root be γ.

Then $(1+i)(1-i)(\gamma) = -6$ [product of roots]

$\therefore 2\gamma = -6$ ✓ Correct third root

$\gamma = -3$

Also, since $(1+i) + (1-i) + (-3) = -a$ [Sum of roots]

$\therefore -1 = -a$

$\therefore a = 1$ ✓ correct a

Finally, we have:

$(1+i)(-3) + (1-i)(-3) + (1+i)(1-i) = b$

$\therefore -3-3i-3+3i+2 = b$

$\therefore b = -4$ ✓ correct b

Q3

(a) (i) Let k be a zero of multiplicity r (>1) for $P(x)$.

Then $P(x) = (x-k)^r \cdot Q(x)$ [and k is not a zero of $Q(x)$]

$\therefore P'(x) = Q(x) \cdot r(x-k)^{r-1} + (x-k)^r \cdot Q'(x)$

$= (x-k)^{r-1}\left[rQ(x) + (x-k)\cdot Q'(x)\right]$

$= (x-k)^{r-1} \cdot Q_1(x)$

, where $Q_1(k) \neq 0$, as $Q(k) \neq 0$.

$\therefore k$ is a zero of multiplicity $r-1$ of $P'(x)$. ✓ Essentially correct proof.

(ii) $P(x) = x^4 + 5x^3 + 9x^2 + 7x + 2$

$\therefore P'(x) = 4x^3 + 15x^2 + 18x + 7$

$\therefore P''(x) = 12x^2 + 30x + 18$

$= 6(2x^2 + 5x + 3)$

$= 6(2x+3)(x+1)$

Thus, by (i), either $-\frac{3}{2}$ or -1 is a zero of multiplicity 3 of $P(x)$. ✓ identifies possible zeros of multiplicity 3.

Since $P'(-1) = P(-1) = 0$, it must be -1. ✓ determines it is -1

$\therefore$ (by inspection) $P(x) = (x+1)^3(x+2)$. ✓ Final factorisation

Q3 ctd.

(b) $x^3 - 4x^2 + 5x + 2 = 0$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ ✓ Correct start

$= 4^2 - 2(5)$

$= 6$ ✓ correct answer

$\alpha^3 - 4\alpha^2 + 5\alpha + 2 = 0$

$+ \quad \beta^3 - 4\beta^2 + 5\beta + 2 = 0$

$\gamma^3 - 4\gamma^2 + 5\gamma + 2 = 0$ ✓ correct idea of subbing & adding

$(\alpha^3+\beta^3+\gamma^3) - 4(\alpha^2+\beta^2+\gamma^2) + 5(\alpha+\beta+\gamma) + 6 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 4(6) - 5(4) - 6$

$= -2$ ✓ correct answer

(c) $8x^3 - 4x^2 + 6x - 1 = 0$ has roots α, β, γ.

Let $y = \frac{1}{1-x}$

$\therefore 1 - x = \frac{1}{y}$

$\therefore x = 1 - \frac{1}{y}$

$\therefore x = \frac{y-1}{y}$ ✓

Thus required equation is:-

$8\left(\frac{y-1}{y}\right)^3 - 4\left(\frac{y-1}{y}\right)^2 + 6\left(\frac{y-1}{y}\right) - 1 = 0$ ✓

$\therefore 8(y-1)^3 - 4y(y-1)^2 + 6y^2(y-1) - y^3 = 0$

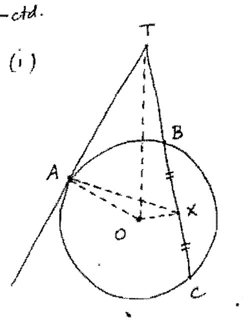
$\therefore 8y^3 - 24y^2 + 24y - 8 - 4y^3 + 8y^2 - 4y + 6y^3 - 6y^2 - y^3 = 0$

$\therefore 9y^3 - 22y^2 + 20y - 8 = 0$ ✓

(ie. $9x^3 - 22x^2 + 20x - 8 = 0$, or any multiple of this)

Q3 ctd.

(d) (i)



(ii) Join OX, OA, OT, AX.

Then:

$\angle OXB = 90°$ [Centre to midpoint of chord is perpendicular to it] ✓

$\angle OAT = 90°$ [tangent is ⊥ to radius at point of contact] ✓

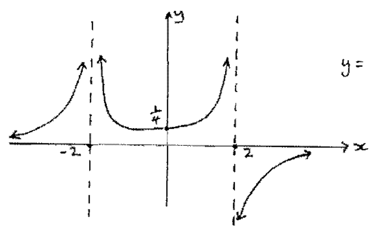
$\therefore$ OATX is a cyclic quadrilateral [opposite angles supplementary] ✓

$\therefore \angle AOT = \angle AXT$ [angles standing on same arc are equal] ✓

[needs accurate reasons for full marks].

Q4

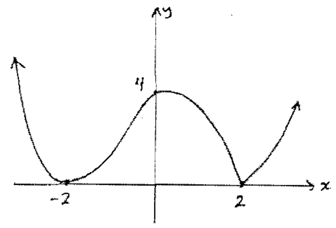
(a) (i)



$y = \frac{1}{f(x)}$ ✓✓

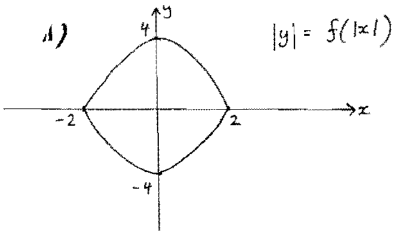
[asymptotes & intercept needed for full marks]

(ii)



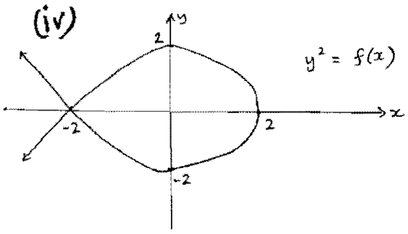
$y = |f(x)|$ ✓

iii)



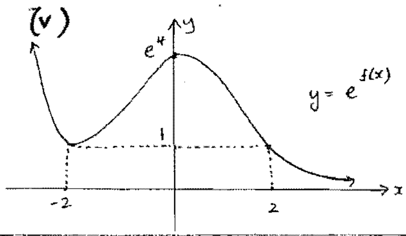
✓✓ (note: is 'pointy' at $x = \pm 2$)

(iv)



✓✓ (note: vertical tangent at $x = 2$)

(v)



✓ basic shape + intercept

✓ correct behavior at $x = \pm 2$

(b) $x^3 + xy - y^3 = 1$

$\therefore \quad 3x^2 + y \cdot 1 + x \cdot \frac{dy}{dx} - 3y^2 \cdot \frac{dy}{dx} = 0$ ✓ Correct implicit differentiation

Thus at (1,1) we have :-

$3 + 1 + \frac{dy}{dx} - 3\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = 2$ ✓ correct value of $\frac{dy}{dx}$ at (1,1)

$\therefore \quad y - 1 = 2(x-1)$

$\therefore \quad y = 2x - 1$

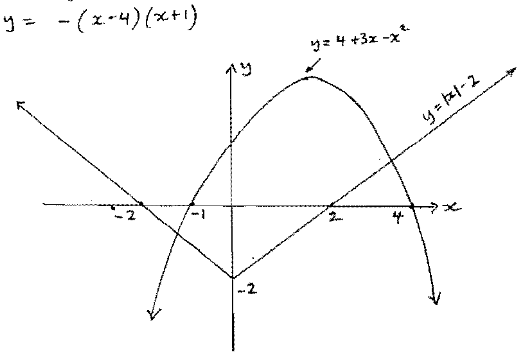
ie $\quad 2x - y - 1 = 0$ ✓ correct final answer

Q4 - ctd

(c) $y = 4 + 3x - x^2$

$= -(x^2 - 3x - 4)$

$\therefore y = -(x-4)(x+1)$



✓ correct graphs.

The expression $\frac{|x| - 2}{4 + 3x - x^2} > 0$

when either $y = |x| - 2$ and $y = 4 + 3x - x^2$ are both positive, or they are both negative.

From the graphs we see that this is when:

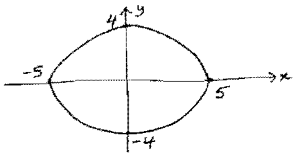
$-2 < x < -1 \quad$ or $\quad 2 < x < 4$. ✓✓

Q5

(a) (i) $16x^2 + 25y^2 = 400$

$\therefore \quad \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\therefore$



✓

(ii) $b^2 = a^2(1 - e^2)$

$\therefore 1 - e^2 = \frac{b^2}{a^2}$

$= \frac{16}{25}$

$\therefore e^2 = \frac{9}{25}$

$e = \frac{3}{5}$ ✓

(iii) $16x^2 + 25y^2 = 400$

$\therefore \quad 32x + 50y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{32x}{50y}$

$= -\frac{16x_1}{25y_1}$ at $P(x_1, y_1)$

$\therefore \quad m_N$ at $P = \frac{25y_1}{16x_1}$ ✓ for m_N at P

Thus normal is :

$y - y_1 = \frac{25y_1}{16x_1}(x - x_1)$

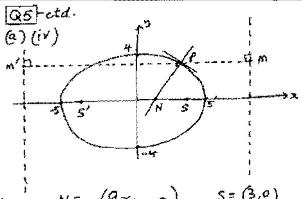
$16x_1y - 16x_1y_1 = 25y_1x - 25x_1y_1$

$\therefore \quad 25y_1x - 16x_1y - 9x_1y_1 = 0$

as required. ✓ correct substitution and expansion

Q5 - ctd.

(a) (iv)



$N = \left(\frac{9x_1}{25}, 0\right) \qquad S = (3,0)$

$S' = (-3,0)$

Directrices are $x = \pm\frac{25}{3}$

Thus $\frac{NS}{NS'} = \frac{3 - \frac{9x_1}{25}}{\frac{9x_1}{25} + 3}$

$= \frac{25 - 3x_1}{25 + 3x_1}$

Now $\frac{PS}{PM} = \frac{PS'}{PM'} = e$ (by definition)

$\therefore \quad \frac{PS}{PS'} = \frac{PM}{PM'}$

$= \frac{\frac{25}{3} - x_1}{x_1 + \frac{25}{3}}$

$= \frac{25 - 3x_1}{25 + 3x_1}$

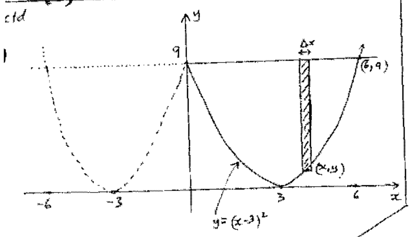
$= \frac{NS}{NS'}$, as required.

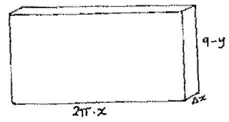
3 marks Correct solution

2 marks Uses focus directrix definition or other correct method AND obtains coords of N, S, S', $x = \pm\frac{25}{3}$.

1 mark Uses focus directrix definition or other correct method OR obtains coords of N, S, S', $x = \pm\frac{25}{3}$

5(b) ctd





$A(x) = 2\pi x\cdot(9-y)$

$\therefore \Delta V \doteqdot 2\pi x\cdot(9-y)\Delta x$

But $9-y = 9-(x-3)^2$
$= 6x - x^2$

$\therefore V = \lim_{\Delta x\to 0}\sum_0^6 2\pi x\,(6x-x^2)\,\Delta x$

$= 2\pi\int_0^6 (6x^2-x^3)\,dx$

$= 2\pi\left[2x^3 - \frac{x^4}{4}\right]_0^6$

$= 2\pi(432-324)$

$\therefore V = 216\pi$ units³

(b) **4 marks**: Correct solution

3 marks:
- $\Delta V = 2\pi x(9-y)\Delta x$
- $9-y = 6x - x^2$
- $V = 2\pi\int_0^6 (6x^2-x^3)\,dx$

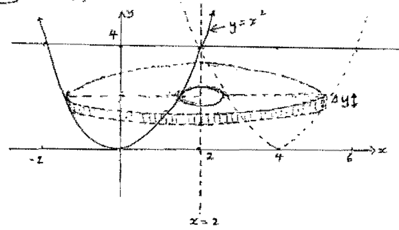
or Correctly uses $y=(x-3)^2$ to change $y\to x$ but makes ONE error in the subsequent working.

2 marks: 2 errors

1 mark:
- Correctly finds definite integral from incorrect expression for V

or 1 step of correct working

Q5 ctd. (c)



Consider a slice in the form of a washer perpendicular to the line $x=2$.

It has outer radius $x+2$, and inner radius $2-x$, and thickness Δy.

Its volume is thus:

$\Delta V = \pi\left[(2+x)^2 - (2-x)^2\right]\cdot\Delta y$

Now $y = x^2 \quad \therefore x = \sqrt{y}$

$\therefore \Delta V = \pi\left[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\right]\cdot\Delta y$

$= \pi\,(8\sqrt{y})\,\Delta y$

$\therefore V = \pi\int_0^4 8\sqrt{y}\,dy$

$= 8\pi\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^4$

$= 8\pi\cdot\frac{2}{3}\cdot 8$

$\therefore V = \frac{128\pi}{3}$ units³.

Marks: (c) **4 marks**

Correct solution leading to answer of $\frac{128\pi}{3}$ units³

3 marks

Correct $\Delta V = \pi\left[(2+x)^2-(2-x)^2\right]\Delta y$
Correct change of $x\to y$
Correct limits
Correct $V = \pi\int_0^4 8\sqrt{y}\,dy$.

or Incorrect ΔV involving annulus with $R_1 = 2+x$ and $R_2 = 2-x$
Correct change of $x \to y$
Correct limits
Correct answer from previous error

2 marks
- Incorrect ΔV not involving an annulus following through with no further errors.

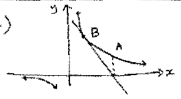
or - Incorrect ΔV involving an annulus with $R_1, R_2 \neq 2+x$ or $2-x$ following through with no further errors.

1 mark
- Correct change of $x \to y$

or - Correct definite integral from their incorrect expression for V.

or - Annulus with R_1, R_2.

Q6 (a)



i) We have $B = \left(ct_2, \frac{c}{t_2}\right)$ & $A = \left(ct_1, \frac{c}{t_1}\right)$.

As $y = c^2x^{-1}$

$\therefore \frac{dy}{dx} = \frac{-c^2}{x^2}$

$\therefore m_T = \frac{-c^2}{c^2t_2^2}$

$= -\frac{1}{t_2^2}$

$\therefore$ Tangent at B is: $y - \frac{c}{t_2} = \frac{-1}{t_2^2}(x - ct_2)$

$\therefore \quad t_2^2y - ct_2 = -x + ct_2$

At the foot of A, $y=0 \quad \therefore x = 2ct_2$

But this equals the x-coordinate of A.

ie. $ct_1 = 2ct_2$

$\therefore \quad t_1 = 2t_2$ as required.

ii) midpoint of $AB = \left[\frac{c}{2}(t_1+t_2), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right]$

$\therefore$ From part (i), midpoint $= \left[\frac{c}{2}(3t_2), \frac{c}{2}\left(\frac{1}{2t_2}+\frac{1}{t_2}\right)\right]$

$= \left(\frac{3ct_2}{2}, \frac{3c}{4t_2}\right)$

Eliminating the parameter t_2, we get:

$xy = \frac{3ct_2}{2}\cdot\frac{3c}{4t_2}$

$\therefore xy = \frac{9c^2}{8}$,

which represents a rectangular hyperbola.

(i) **3 marks**: Correct solution

2 marks: Finds equation of tangent at B

or Proves true for $A\left(ct, \frac{1}{t}\right)$, $B\left(ct_2, \frac{1}{t_2}\right)$

1 mark: Finds gradient of tangent at B

or recognizes x intercept = x coord of A

or $A\left(ct, \frac{c}{t}\right)$ $B\left(ct, \frac{c}{t_2}\right)$

(ii) **2 marks**: correct solution

1 mark: Correct midpoint in terms of t_2

Q6 ctd.

(b) (i) $|z-2| = 2\left(\text{Re } z - \frac{1}{2}\right)$

Let $z = x+iy$

Since $|z-2| \geq 0$, $\therefore \text{Re } z \geq \frac{1}{2}$, ie. $x \geq \frac{1}{2}$

We have:

$|(x-2)+iy| = 2\left(x-\frac{1}{2}\right)$

$\sqrt{(x-2)^2+y^2} = 2\left(x-\frac{1}{2}\right)$

$(x-2)^2 + y^2 = 4\left(x-\frac{1}{2}\right)^2$

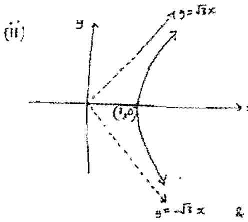
$\therefore x^2 - 4x + 4 + y^2 = 4x^2 - 4x + 1$

$\therefore 3x^2 - y^2 = 3$

$\therefore \frac{x^2}{1} - \frac{y^2}{3} = 1 \quad (x \geq 1)$

[Alternatively, candidates could use the focus-directrix definition with:
$e=2$, $S=(2,0)$, directrix $x=\frac{1}{2}$]

(ii)



$\frac{x^2}{1} - \frac{y^2}{3} = 1 \quad (x\geq 1)$

[Note: asymptotes are $y = \pm\frac{b}{a}x$
ie $y = \pm\sqrt{3}x$.]

Thus:

$|z| \geq 1$

& $-\frac{\pi}{3} < \arg z < \frac{\pi}{3}$

6(b)(i) **3 marks**: Correct soln

2 marks: Correct derivation of equation

or correct branch and correct expansion of LHS

or correct branch and correct expansion of RHS

1 mark: correct branch

or correct expansion $\Rightarrow 3x^2 - y^2 = 3$

or correct expansion of RHS or LHS.

(ii) **3 marks**: correct solution

1 mark for each of
- correct sketch
- must show asymptote(s) or indicate or other clear indication graph is a hyperbola

Q6 (c)

At intersections, $\frac{1}{2}x^2 = 6 - x^2$.

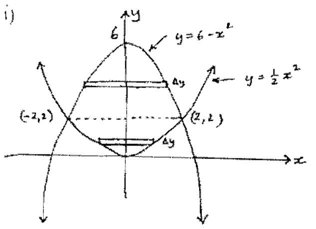
$x^2 = 12 - 2x^2$

$3x^2 = 12$

$x = \pm 2$

Thus intersections are $(-2, 2)$ and $(2, 2)$

i)



Area of semi-circular disks $= \frac{1}{2}\cdot\pi x^2$

But we must express this in terms of y, but the expression will depend on whether disks are above or below the line $y = 2$.

Above: $x^2 = 6 - y$, Below: $x^2 = 2y$

Thus: $V = \frac{\pi}{2}\int_0^2 2y\,dy + \frac{\pi}{2}\int_2^6 (6-y)\,dy$

$= \frac{\pi}{2}\left[y^2\right]_0^2 + \frac{\pi}{2}\left[6y - \frac{y^2}{2}\right]_2^6$

$= \frac{\pi}{2}\left[4-0\right] + \frac{\pi}{2}\left[(36-18)-(12-2)\right]$

$= \frac{\pi}{2}(4 + 18 - 10)$

$\therefore V = 6\pi$ units^3.

⑥(c)(ii) 1 mark: correct solution

(ii) 3 marks: correct solution

2 marks: correct integrals with limits

or wrong limits with no further errors

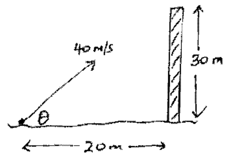
1 mark: Correct ΔV

or 1 correct integral with limits

or uses $A(x) = \frac{\pi x^2}{2}$ ~~...~~ with no further errors.

Q7

(a) (i)



$\ddot{x} = 0$

$\therefore \dot{x} = c$

at $t=0$, $\dot{x} = 40\cos\theta$ $\therefore c = 40\cos\theta$

$\therefore \boxed{\dot{x} = 40\cos\theta}$

$\therefore x = 40\cos\theta\, t + C$

at $t=0$, $x = 0$ $\therefore C = 0$

$\therefore \boxed{x = 40\cos\theta \cdot t}$ ✓ derives x equation

$\ddot{y} = -10$

$\therefore \dot{y} = -10t + c$

at $t=0$, $\dot{y} = 40\sin\theta$

$\therefore c = 40\sin\theta$

$\therefore \boxed{\dot{y} = -10t + 40\sin\theta}$

$\therefore y = -5t^2 + 40\sin\theta\, t + c$

at $t=0, y=0$ $\therefore c = 0$

$\therefore \boxed{y = -5t^2 + 40\sin\theta \cdot t}$ ✓ derives y equation

(ii) Cartesian equation of path is:

$y = -5\left(\frac{x}{40\cos\theta}\right)^2 + 40\left(\frac{x}{40\cos\theta}\right)\cdot\sin\theta$

at $x = 20$, $y > 30$:

$\therefore -5\left(\frac{20}{40\cos\theta}\right)^2 + 40\left(\frac{20}{40\cos\theta}\right)\sin\theta > 30$ ✓ correct trajectory & use of data

$\therefore -\frac{5}{4}\sec^2\theta + 20\tan\theta - 30 > 0$

$\therefore -(1 + \tan^2\theta) + 16\tan\theta - 24 > 0$ ✓ Correct simplification

$\therefore \tan^2\theta - 16\tan\theta + 25 < 0$

$(\tan^2\theta - 16\tan\theta + 64) - 39 < 0$

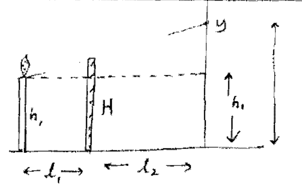
$(\tan\theta - 8)^2 < 39$

$\therefore -\sqrt{39} < \tan\theta - 8 < \sqrt{39}$

$\therefore -\sqrt{39} + 8 < \tan\theta < \sqrt{39} + 8$ ✓ correct tan θ

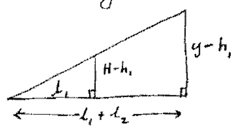
$\therefore 60°19' < \theta < 85°59'$ ✓ correct angles range

Q7 ctd (b)



The goal is to find $\frac{dy}{dt}$. (Note: H, l_1, l_2 are all constants)

using similar triangles, we have :-



Thus $\frac{y - h_1}{l_1 + l_2} = \frac{H - h_1}{l_1}$ ✓ equation relating y to h_1, l_1, l_2, H.

$\therefore y - h_1 = \frac{l_1 + l_2}{l_1}(H - h_1)$

$\therefore y = \frac{l_1 + l_2}{l_1}(H - h_1) + h_1$

Thus $\frac{dy}{dt} = \frac{l_1 + l_2}{l_1}\cdot -1\cdot\frac{dh_1}{dt} + \frac{dh_1}{dt}$ ✓ correct derivative

But $\frac{dh_1}{dt} = -3$

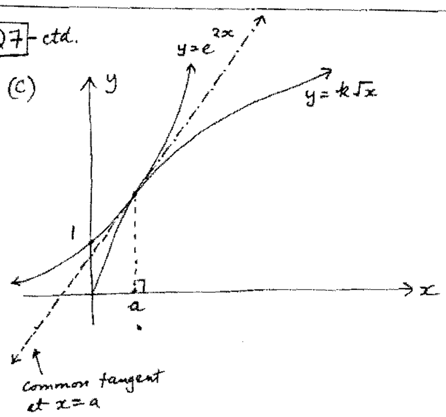
$\therefore \frac{dy}{dt} = 3\left(\frac{l_1 + l_2}{l_1}\right) - 3$ ✓ final correct relation (any correct form is OK)

(ie. $\frac{dy}{dt} = 3\left[\frac{l_1 + l_2}{l_1} - 1\right]$

$= 3\cdot\frac{l_2}{l_1}$)

Q7 ctd.

(c)



If the curves $y = e^{2x}$ and $y = k\sqrt{x}$ intersect just once, at $x = a$ (say) then we know two facts:

(1) they have the same y-value at $x = a$

and, (2) they share a tangent line at $x = a$ ✓ correct reasoning / realisation they share a tangent line.

Thus: (1) $e^{2a} = k\sqrt{a}$

and (2) $2e^{2a} = \frac{k}{2\sqrt{a}}$

Solving simultaneously,

$2\cdot k\sqrt{a} = \frac{k}{2\sqrt{a}}$

$\therefore a = \frac{1}{4}$ ✓ correct working

Thus $e^{2(\frac{1}{4})} = k\sqrt{\frac{1}{4}}$

$\therefore e^{\frac{1}{2}} = k\cdot\frac{1}{2}$

$\therefore k = 2\sqrt{e}$ ✓ correct answer

7 ctd.

i)

$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}$

$= \sqrt{16-(x-4)^2} - \sqrt{1-(x-7)^2}$

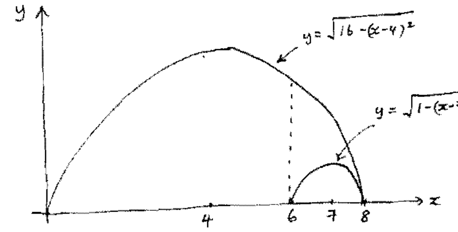
Thus $f(x)$ is the difference of two functions: $y = \sqrt{16-(x-4)^2}$ & $y = \sqrt{1-(x-7)^2}$.

We can graph these on the same number plane:

$y = \sqrt{16-(x-4)^2}$ is the top half of the circle $(x-4)^2 + y^2 = 16$, and has $D: 0 \le x \le 8$

$y = \sqrt{1-(x-7)^2}$ is the top half of the circle $(x-7)^2 + y^2 = 1$, and has $D: 6 \le x \le 8$.

Graphing them, we see:



$\therefore$ In the domain of $f(x)$, $D: 6 \le x \le 8$, the maximum difference occurs at $x=6$.

Thus maximum value of $f(x)$ is equal to $f(6) = \sqrt{12}$.

Correct answer with correct working, & using graph(s) : 3

Substantially correct (max. of 1 error), & including graph(s) : 2

Some correct reasoning (and including a correct graph) : 1

Q8 (a)

$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \frac{d}{dx}(-\cos x)\, dx$

$= \left[-\cos x \cdot \sin^{n-1} x\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (-\cos x)\cdot(n-1)\sin^{n-2} x \cdot \cos x \, dx$ ✓ Correct Integration by parts.

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin^{n-2} x \, dx$

$= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\cdot \sin^{n-2} x \, dx$ ✓ correct simplification

$= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2} x - \sin^n x)\, dx$

$\therefore I_n = (n-1)\left[I_{n-2} - I_n\right]$

$\therefore I_n + (n-1) I_n = (n-1) I_{n-2}$ ✓ correctly simplified

$\therefore I_n = \frac{n-1}{n} \cdot I_{n-2}$ as required.

Now, $\therefore I_4 = \frac{3}{4} I_2$

$= \frac{3}{4}\cdot\frac{1}{2}\cdot I_0$

$= \frac{3}{4}\cdot\frac{1}{2}\cdot \int_0^{\pi/2} dx$

$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$\therefore I_4 = \frac{3\pi}{16}$ ✓ correct I_4

So $\therefore I_6 = \frac{5}{6}\cdot I_4$

$\therefore I_6 = \frac{15\pi}{96}$

$\therefore I_6 = \frac{5\pi}{32}$. ✓ correct I_6

Q8 ctd. (b)

i) If $x = a\cos^3\theta$ & $y = a\sin^3\theta$, then

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\theta + a^{\frac{2}{3}}\sin^2\theta$

$= a^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta)$

$= a^{\frac{2}{3}}$

So $\therefore$ it is a correct parametrisation. ✓ confirms parametrisation

ii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$= \frac{3a\sin^2\theta\cdot\cos\theta}{3a\cos^2\theta\cdot -\sin\theta}$

$= -\frac{\sin\theta}{\cos\theta}$

$\therefore \frac{dy}{dx} = -\tan\theta$.

So at P, $\frac{dy}{dx} = -\tan\phi$. ✓ derives $\frac{dy}{dx}$ correctly

iii) The tangent at P is given by:

$y - a\sin^3\phi = -\tan\phi\,(x - a\cos^3\phi)$

$= -\frac{\sin\phi}{\cos\phi}(x - a\cos^3\phi)$

$\therefore \cos\phi . y - a\sin^3\phi\cos\phi = -\sin\phi\, x + a\sin\phi\cos^3\phi$ ✓ A correct equation of the tangent

$\therefore \sin\phi\cdot x + \cos\phi\cdot y = a\sin\phi\cos\phi$

Thus for y-intercept $(x=0)$, $y = a\sin\phi$ & for x-intercept $(y=0)$, $x = a\cos\phi$. ✓ correct intercepts

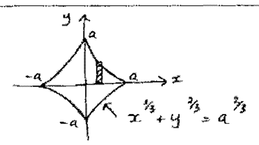
Thus length cut off by axes is given by $\sqrt{(a\sin\phi)^2 + (a\cos\phi)^2}$

$= \sqrt{a^2(\sin^2\phi + \cos^2\phi)}$

$= a$, which is a constant. ✓ Shows the intercepted length is constant.

Q8 ctd.

(b) (iv)



Let A = the area enclosed in Quadrant 1.

Then $A = \int_0^a y\, dx$

$= \int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx$ ✓ correct Cartesian integral

Using a substitution $x = a\cos^3\theta$, we see:

$dx = 3a\cos^2\theta . -\sin\theta . d\theta$

Also, when $\begin{cases} x=0, & \cos^3\theta = 0 \therefore \theta = \frac{\pi}{2} \\ x=a, & \cos^3\theta = 1 \therefore \theta = 0 \end{cases}$

Thus, $A = \int_{\frac{\pi}{2}}^{0} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\cos^2\theta\right)^{\frac{3}{2}} \cdot -3a\cos^2\theta\sin\theta\, d\theta$ ✓ correct substitution

$= 3a\int_0^{\frac{\pi}{2}} a(1-\cos^2\theta)^{\frac{3}{2}} . \cos^2\theta\sin\theta\, d\theta$

$= 3a^2\int_0^{\frac{\pi}{2}} \sin^3\theta\cdot\cos^2\theta\sin\theta\, d\theta$ ✓ correct simplifications

$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta(1-\sin^2\theta)\, d\theta$

$= 3a^2\int_0^{\frac{\pi}{2}} (\sin^4\theta - \sin^6\theta)\, d\theta$.

But from part (a) of Q8,

$\int_0^{\frac{\pi}{2}} (\sin^4\theta - \sin^6\theta)\, d\theta = I_4 - I_6$ ✓ correct evaluation of integrals

$= \frac{3\pi}{16} - \frac{5\pi}{32}$

$= \frac{\pi}{32}$

$\therefore A = 3a^2 \cdot \frac{\pi}{32} = \frac{3a^2\pi}{32}$

$\therefore$ area of asteroid $= 4A = \frac{3a^2\pi}{8}$ units^2. ✓ correct final answer.