Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2013 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Fletcher
- O Ms Beevers
- O Ms Ziaziaris

Student Number

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	15	16	Total	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100	/100

Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use multiple choice answer sheet for questions 1-10

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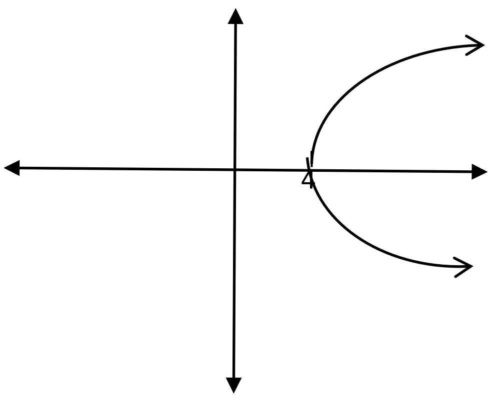
1. If $z = 5i$, find the value of $z\overline{z}$.
   (A) 25
   (B) -25
   (C) $5\sqrt{2}$
   (D) -5

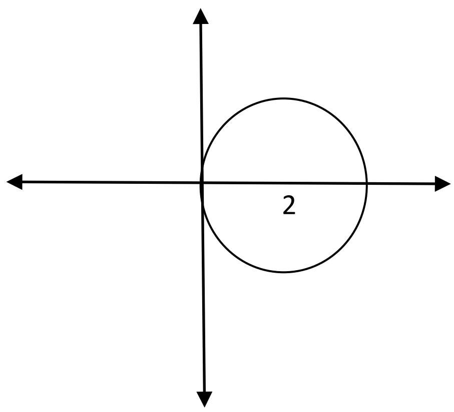
2. For the equation $2xy - x^2 + y^4 + 1 = 0$, the equation of the tangent at (-1,0) is given by,
   (A) $y = -x - 1$
   (B) $y = x - 1$
   (C) Does not exist
   (D) $y = x + 1$

3. The locus of all complex numbers for $|z-2| = \mathrm{Re}(z)$ is represented by which of the following diagrams.

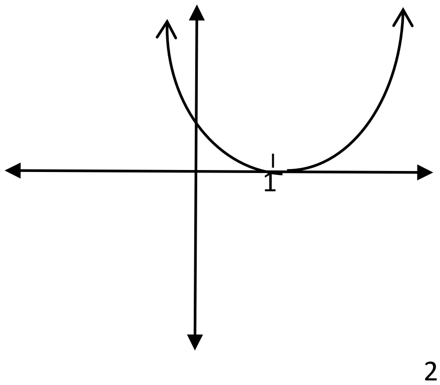
(A)



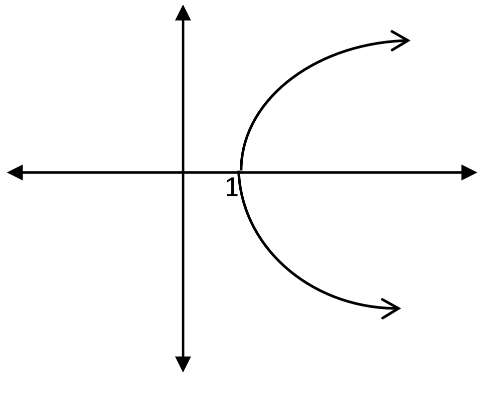
(B)



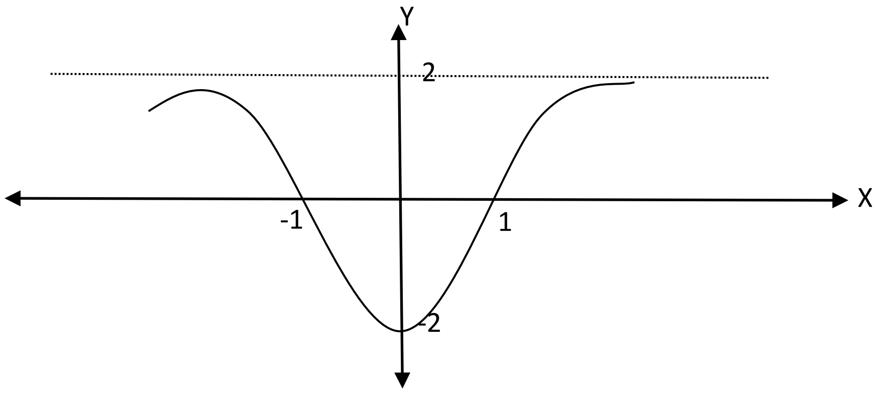
(C)



(D)

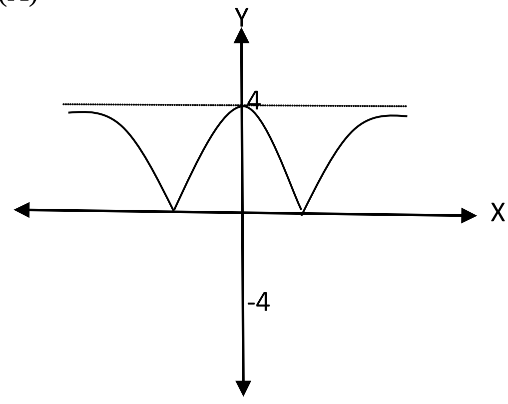


4. The graph of $y = f(x)$ is shown below.

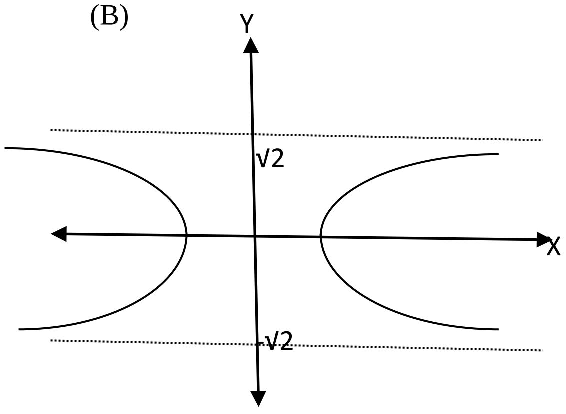


The sketch of $y^2 = f(x)$ is best represented by which of the following:

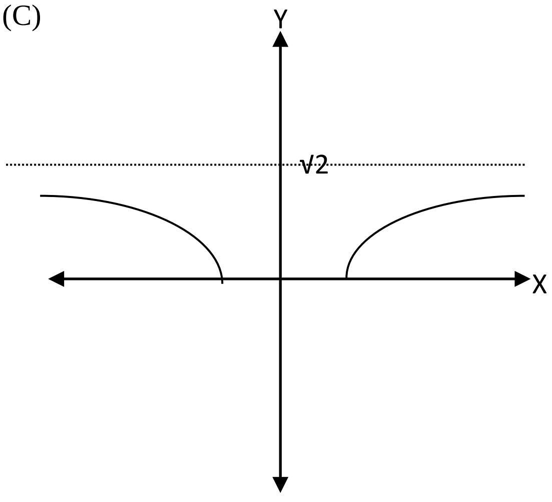
(A)



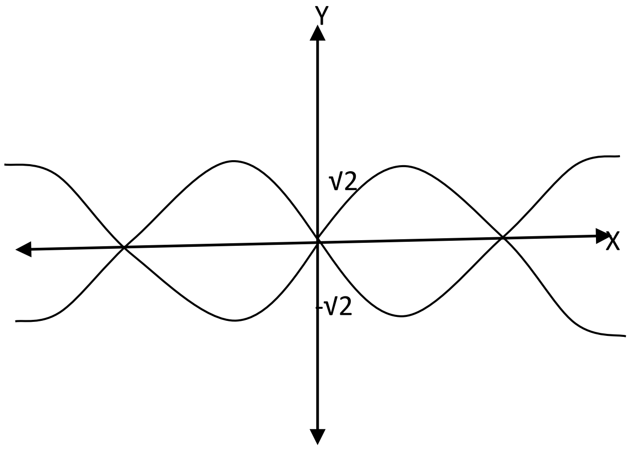
(B)



(C)



(D)



5. Find the coordinates of the foci for $xy = 8$.

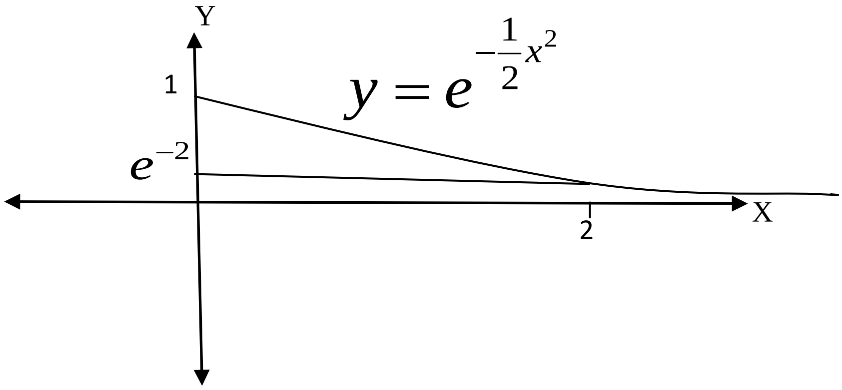
(A) $(4,4),(-4,-4)$

(B) $\left(2\sqrt{2},2\sqrt{2}\right),\left(-2\sqrt{2},-2\sqrt{2}\right)$

(C) $\left(8\sqrt{2},8\sqrt{2}\right),\left(-8\sqrt{2},-8\sqrt{2}\right)$

(D) $\left(4\sqrt{2},4\sqrt{2}\right),\left(-4\sqrt{2},-4\sqrt{2}\right)$

6. The volume of the solid obtained by revolving the region bounded by $y = e^{-\frac{1}{2}x^2}$, $y = e^{-2}$ and the lines $x = 0, x = 2$ about the Y-axis can be evaluated with which of the following integrals.



(A) $V = 2\pi \int_{e^{-2}}^{1} x\left(e^{-\frac{1}{2}x^2} - e^{-2}\right)dx$

(B) $V = 2\pi \int_{e^{-2}}^{1} x\left(e^{-\frac{1}{2}x^2}\right)dx$

(C) $V = 2\pi \int_{0}^{2} x\left(e^{-\frac{1}{2}x^2} - e^{-2}\right)dx$

(D) $V = 2\pi \int_{0}^{2} x\left(e^{-\frac{1}{2}x^2}\right)dx$

7. The square roots of $(-3-4i)$ in the form of $a+bi$ is

(A) $(-2-i),(2+i)$

(B) $(2-i),(-2+i)$

(C) $(1+2i),(-1+2i)$

(D) $(1-2i),(-1+2i)$

8. The polynomial $P(z)$ has real coefficients. Four of the roots of the equation $P(z)=0$ are $z=0$, $z=1+2i$, $z=1-2i$ and $z=3i$. The minimum number of roots that the equation $P(z)=0$ could have is

(A) 4
(B) 5
(C) 6
(D) 8

9. Using a suitable substitution, the definite integral $\int_0^{\frac{\pi}{24}} \tan 2x \sec^2 2x dx$ is equivalent to

(A) $\frac{1}{2}\int_0^{\frac{\pi}{24}} udu$

(B) $2\int_0^{\frac{\pi}{24}} udu$

(C) $2\int_0^{2-\sqrt{3}} udu$

(D) $\frac{1}{2}\int_0^{2-\sqrt{3}} udu$

10. Without evaluating the integrals which of the following will give an answer of zero.

(A) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^3\theta + 1}{\cos^2\theta} d\theta$

(B) $\int_{-1}^{1} (x^2 - 1)(1 - x^2)^3 dx$

(C) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \cos x dx$

(D) $\int_{-2}^{2} \left| x^2 - 4 \right| dx$

Section II

90 marks

Attempt Questions 11-16

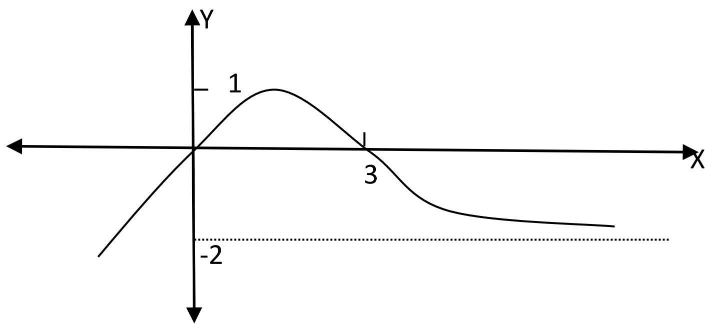
Allow about 2hours and 45 minutes for this section

Start each question on a NEW PAGE. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and /or calculations.

Question 11 (15 marks) Start a NEW PAGE.

(a) The graph of $y = f(x)$ is sketched below.



On separate axes sketch the graph of

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

(b) Consider the curve $y = \cos\left(\sqrt{x}\right),\ 0 \le x \le 4\pi^2$

(i) Find $\dfrac{dy}{dx}$ and the gradient of the limiting tangent as $x \to 0$. **2**

(ii) Find any stationary points and determine their nature. **3**

(iii) Sketch $y = \cos\left(\sqrt{x}\right),\ 0 \le x \le 4\pi^2$ and then complete the sketch to show the graph of $y = \cos\left(\sqrt{|x|}\right),\ -4\pi^2 \le x \le 4\pi^2$, showing any intercepts on the coordinate axes and the coordinates of any stationary points. **3**

(iv) Use the substitution $u^2 = x$, $u \ge 0$ to find the area bounded by the curve $y = \cos\left(\sqrt{|x|}\right)$ and the x axis between $x = -\dfrac{\pi^2}{4}$ and $x = \dfrac{\pi^2}{4}$. **3**

Question 12 (15 marks) Start a NEW PAGE.

(a) Express $\dfrac{(1+2i)^2}{2+i}$ in the form $a+ib$. **2**

(b) (i) Express $z=-3\sqrt{3}+3i$ in modulus argument form. **2**

(ii)Hence find the smallest positive integer n so that z^n is real. **1**

(c) Solve the equation $z^2+(z+1)^2=0$, where z is a complex number. **2**

(d) Sketch the following loci on separate argand diagrams.

(i) $z^2-\overline{z}^2=16i$. **2**

(ii) $\arg\left(\dfrac{z-i}{z-2}\right)=\dfrac{\pi}{2}$ **2**

(e) (i) Suppose z is any non-zero complex number.

Explain why $\dfrac{z}{\overline{z}}$ has modulus 1 and argument twice the argument of z. **2**

(ii) Find all complex numbers z so that $\dfrac{z}{\overline{z}}=i$.

Give your answer in the form $a+ib$, where a and b are real. **2**

Question 13 (15 marks) Start a NEW PAGE.

(a) $\displaystyle\int \frac{1}{x^2-6x+5}dx$ **3**

(b) $\displaystyle\int \frac{d\theta}{2-\sin\theta}$ **4**

(c) Use the substitution $u^6 = x$ to find $\displaystyle\int\left(\frac{1}{x^{\frac{1}{2}}-x^{\frac{1}{3}}}\right)dx$ **3**

(d) (i) Show that $\left(1-\sqrt{x}\right)^{n-1}\sqrt{x} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$ **1**

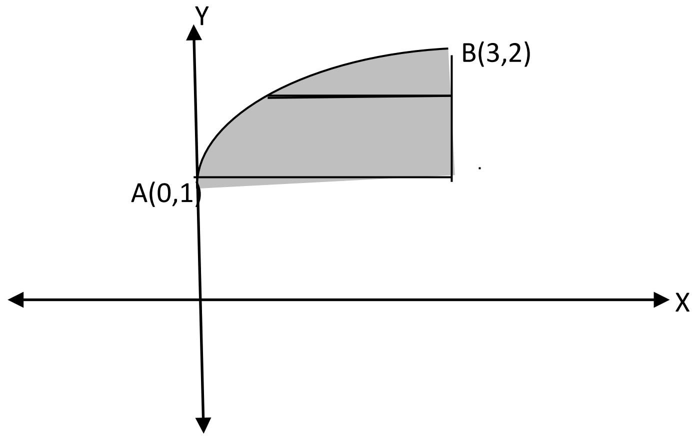
(ii) If $I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$ for $n \geq 0$

show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$. **2**

(iii) Deduce that $\dfrac{1}{I_n} = \dfrac{(n+2)!}{n!2!}$ **2**

Question 14 (15 marks) Start a NEW PAGE.

(a) $A(0,1)$ and $B(3,2)$ lie on the curve $y^2 = x+1$.
The shaded region in the diagram is bounded by the lines $y=1, x=3$ and arc AB.
A slice perpendicular to the line $x=4$ has been taken and the region is rotated about $x=4$.



(i) Show that the area of a slice is given by $A(y) = \pi\left[\left(5-y^2\right)^2 - 1\right]$ **2**

(ii) Find the volume of the solid generated. **2**

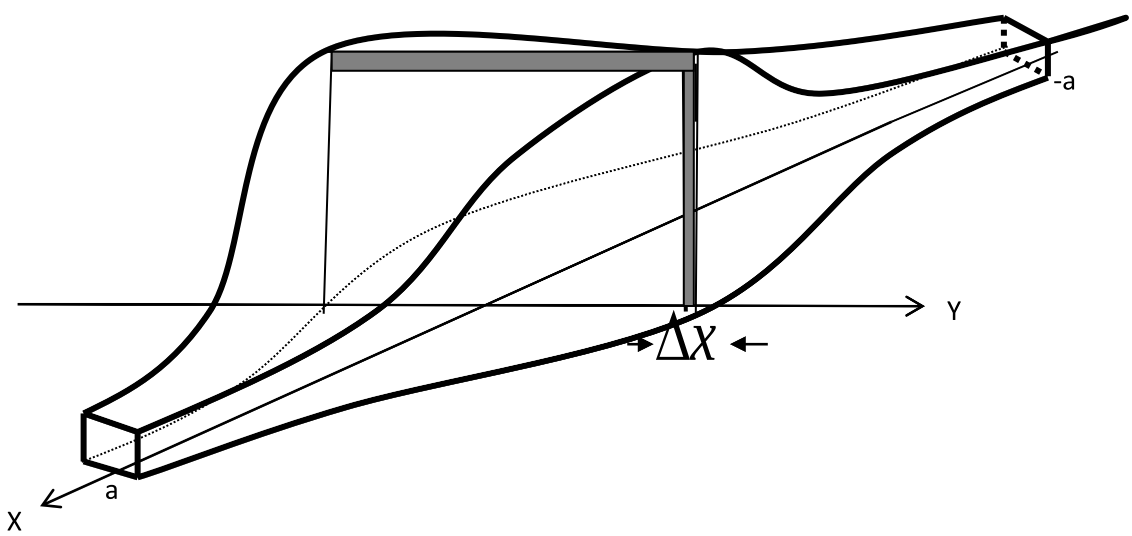
(b) (i) Sketch the ellipses $x^2 + 25y^2 = 100$ and $25x^2 + y^2 = 100$
on the same diagram, showing their intercepts with the coordinate axes.
You DO NOT need to show their foci or directrices. **2**

(iii) A child's spinning top is made by revolving the total area enclosed by each of these ellipses (ie. total area, not just common area) , around the vertical axis.
Let the two ellipses intersect at $x = a$ in the first quadrant.

By using **cylindrical shells**, find an expression for the volume in terms of a.
(Complete simplification is not necessary). **4**

Question 14 continues overleaf

(c)



A solid is formed with a square cross section at rightangles to the X-Y plane as shown in the diagram. The equation of the base is $4y^2 + (xy)^2 = 1$

(i) Show that the volume of a thin slice is approximately equal to $\dfrac{4}{4+x^2}\Delta x$, where Δx is the thickness of the slice. **2**

(ii) Hence calculate the volume of the solid bounded by $-a \le x \le a$. **2**

(iii) What is the limiting value of the volume of the solid as a approaches infinity. **1**

Question 15 (15 marks) Start a NEW PAGE.

(a) The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$ has a triple zero.

(i) Determine the value of the triple zero. **2**

(ii) Hence, find the value of C. **1**

(iii) Factorise $P(x)$ **1**

(b) The equation $x^3 - x^2 + 3 = 0$ has roots α, β and γ.

(i) Find the polynomial equation that has roots α^2, β^2 and γ^2.
Express with integral powers. **2**

(ii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

(c) The quartic polynomial $f(x) = x^4 + px^3 + qx^2 + rx + s$ has four zeroes α, β, γ and δ, such that the sum of α and β equals the sum of γ and δ.
Let $C = \alpha + \beta = \gamma + \delta$. Let $P = \alpha\beta$. Let $Q = \gamma\delta$.

(i) Find p, q, r and s in terms of C, P and Q. **3**

(ii) Show that the coefficients of $f(x)$ satisfy the condition $p^3 + 8r = 4pq$. **1**

(iii) It is given that the polynomial $g(x) = x^4 - 18x^3 + 79x^2 + 18x - 440$ has the property that the sum of two of the zeroes equals
the sum of the other two zeroes. Using the identities of part (i) or otherwise,
find all four zeroes of $g(x)$. **3**

Question 16 (15 marks) Start a NEW PAGE.

(a) Tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the points $P\left(x_{1,}y_1\right)$ and $Q\left(x_{2,}y_2\right)$ intersect at T. M is the midpoint of PQ.

(i) Given the tangent to the ellipse at P has equation $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$, write down the equation of the tangent to the ellipse at Q. **1**

(ii) Show that the line $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=\dfrac{xx_2}{a^2}+\dfrac{yy_2}{b^2}$ passes through T and M. **2**

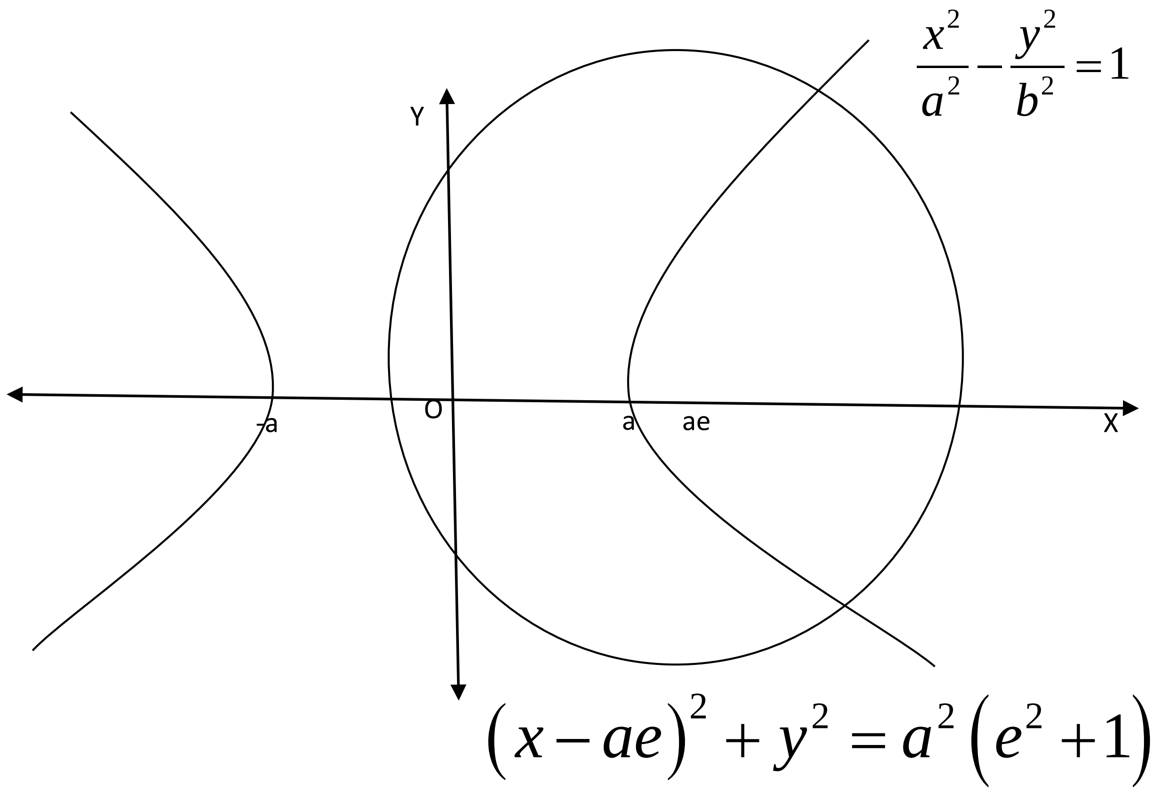
(iii) Deduce that the points O, T, M are collinear. **1**

(iv) Show that the product of the gradients of PQ and TM is a constant. **2**

(v) If PTQ is a right angle, show that $\dfrac{x_1x_2}{a^4}+\dfrac{y_1y_2}{b^4}=0$. **3**

Question 16 continues overleaf ⟶

(b)



(i) The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}-1=0$. (DO NOT PROVE THIS).

Show that if the tangent at P is also tangent to the circle with centre $\left(ae,0\right)$ and radius $a\sqrt{e^2+1}$, then $\sec\theta=-e$. **4**

(ii) Deduce that the points of contact P, Q on the hyperbola of the common tangents to the circle and hyperbola are the extremities of a latus rectum of the hyperbola, and state the co-ordinates of P and Q. **2**

END OF PAPER

NORTH SYDNEY BOYS HIGH SCHOOL 2013 HSC ASSESSMENT TASK 3 (TRIAL HSC)

MATHEMATICS EXTENSION 2

MULTIPLE CHOICE ANSWER SHEET

ANSWER QUESTIONS 1-10 on this sheet

NAME/NUMBER...

TEACHER'S NAME..

1. (A) (B) (C) (D)

2. (A) (B) (C) (D)

3. (A) (B) (C) (D)

4. (A) (B) (C) (D)

5. (A) (B) (C) (D)

6. (A) (B) (C) (D)

7. (A) (B) (C) (D)

8. (A) (B) (C) (D)

9. (A) (B) (C) (D)

10. (A) (B) (C) (D)

4UNIT SOLUTIONS 2013 TRIAL.

SECTION I

1. A
2. D
3. D
4. B.
5. A
6. C
7. D
8. B
9. D
10. C.

4. $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$

ie. $y = \sqrt{f(x)}$ plus reflection in x-axis.

where

$f(x) \geqslant 0$ $\quad \therefore D: x \geqslant 1, x \leqslant -1$

B.

5. $xy = 8.$

$e = \sqrt{2}$ $\quad$ Foci $(\pm c\sqrt{2}, \pm c\sqrt{2})$

$c^2 = 8$ $\quad = (\pm 4, \pm 4)$

$c = 2\sqrt{2}$

A.

1. $(5i)(-5i) = 25$

A.

2. $2y + 2x\frac{dy}{dx} - 2x + 4y^3\frac{dy}{dx} = 0$

$2y - 2x + (2x + 4y^3)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x - 2y}{2x + 4y^3}$

At $(-1, 0)$

$\frac{dy}{dx} = \frac{-2}{-2} = 1$

$m = 1$ $\quad (-1, 0)$

$y + 0 = x + 1$

$y = x + 1$ $\quad$ D.

3. $|x + iy - 2| = x$

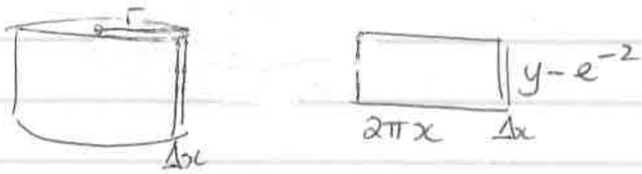
$\sqrt{(x-2)^2 + y^2} = x$

$x^2 - 4x + 4 + y^2 = x^2$

$y^2 = 4x - 4$

$y^2 = 4(x - 1)$ $\quad$ D.

6. Slice $\perp$ to x-axis.



$A(x) = 2\pi x\,(y - e^{-2})$

$= 2\pi x\left(e^{-\frac{1}{2}x^2} - e^{-2}\right)$

$V = 2\pi\int_0^2 \left(e^{-\frac{1}{2}x^2} - e^{-2}\right) dx.$

C.

7. $\sqrt{3 - 4i} = 2 - i, -2 + i$

B.

8. B. Roots occur in conjugate pairs

9. $\int_0^{\pi/24} \tan 2x \sec^2 2x\, dx$ $\quad u = \tan 2x$

$du = 2\sec^2 2x\, dx$

$= \frac{1}{2}\int_0^{2-\sqrt{3}} u\, du.$ $\quad x = \frac{\pi}{24}, u = 2 - \sqrt{3}$

$x = 0, u = 0$

D.

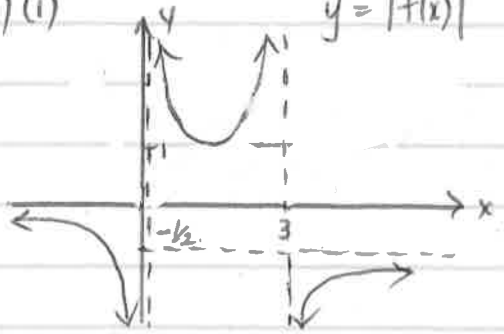
10. A even

B $-(1-x^2)(1-x^2)^3 = -1(1-x^2)^4$ even

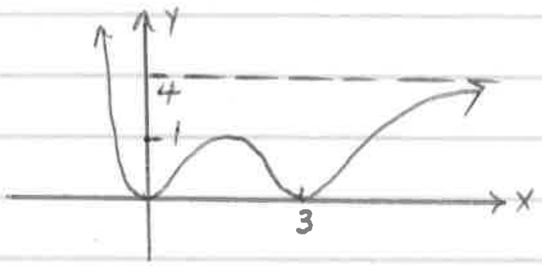
D even

C ODD $\therefore$ answer C.

11. a) (i) $y = |f(x)|$



(ii) $y = [f(x)]^2$



b) (i) $y = \cos\sqrt{x}$

$\frac{dy}{dx} = -\frac{1}{2} x^{-\frac{1}{2}} \sin\sqrt{x}$

$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

As $x \to 0$,

$\lim_{x\to 0} \frac{-\sin\sqrt{x}}{2\sqrt{x}} = -\frac{1}{2} \lim_{x\to 0} \frac{\sin\sqrt{x}}{\sqrt{x}}$

$= -\frac{1}{2} \times 1$

$= -\frac{1}{2}$

(ii) Stationary Pts : $\frac{dy}{dx} = 0$

$-\sin\sqrt{x} = 0$ But $x \neq 0$.

$\sin\sqrt{x} = 0$.

$\sqrt{x} = \cancel{0}, \pi, 2\pi$

$\therefore x = \cancel{0}, \pi^2, 4\pi^2$

$y = \cancel{1}, -1, 1$

Test $(\pi^2, 1)$

x	9	π^2	10
$\frac{dy}{dx}$	$-$	0	+

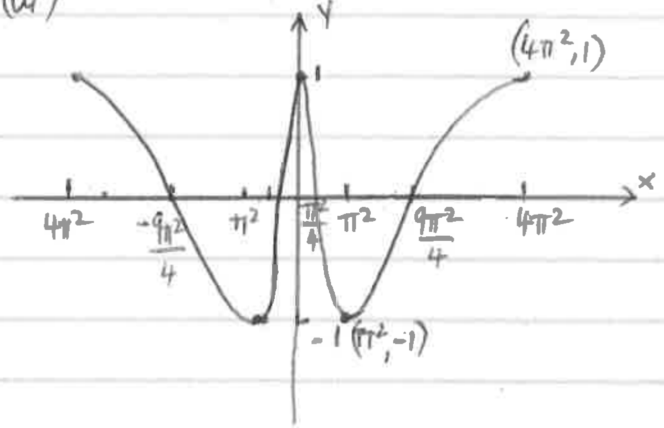
$\therefore \boxed{\min(\pi^2, -1)}$

Test $(4\pi^2, 1)$

x	36	$4\pi^2$	40
$\frac{dy}{dx}$	+	0	$-$

$\therefore \boxed{\max(4\pi^2, 1)}$

(iii)



X int : $\pm\frac{\pi^2}{4}, \pm\frac{9\pi^2}{4}$

(iv) $A = \int_{-\pi^2/4}^{+\pi^2/4} \cos\sqrt{|x|}\, dx$

$= 2\int_0^{\pi^2/4} \cos\sqrt{|x|}\, dx$ $u^2 = x$. $dx = du\,2u$.

$= 2\int_0^{\pi/2} \cos u \cdot 2u\,du$ $x = \pi^2/4,\ u = \frac{\pi}{2}$ $x = 0,\ u = 0$

$= 4\int_0^{\pi/2} u\cos u\,du$ $u = u$ $\frac{dv}{dx} = \cos u$ $du = dx$ $v = \sin u$

$= 4\left\{\left[u\sin u\right]_0^{\pi/2} - \int_0^{\pi/2} \sin u\,du\right\}$

$= 4\left\{\frac{\pi}{2} - \left[-\cos u\right]_0^{\pi/2}\right\}$

$= 4\left\{\frac{\pi}{2} + 0 - 1\right\}$.

$= (2\pi - 4)$ sq. units.

(12) (a) $\dfrac{1+4i-4}{2+i}$

$= \dfrac{-3+4i}{2+i} \times \dfrac{2-i}{2-i}$

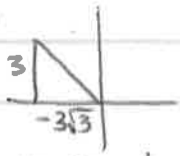
$= \dfrac{-6+3i+8i+4}{4+1}$

$= \boxed{\dfrac{-2}{5} + \dfrac{11i}{5}}$

(b) (i) $z = -3\sqrt{3}+3i$

$|z| = \sqrt{27+9}$

$|z| = 6.$



$\arg z = \dfrac{5\pi}{6}$

$\tan\theta = -\dfrac{1}{\sqrt{3}}$

$\theta = \dfrac{5\pi}{6}$

$\therefore \boxed{z = 6\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)}$

(ii) $z^n = 6^n\left(\cos\dfrac{5\pi n}{6} + i\sin\dfrac{5\pi n}{6}\right)$

For z^n to be real, $\sin\dfrac{5\pi n}{6} = 0$

which occurs when we have a multiple of π

$\therefore \dfrac{5\pi n}{6} = n\pi$

$\therefore$ First positive integer is $\boxed{n=6}$

(c) $z^2 + (z+1)^2 = 0$

$2z^2 + 2z + 1 = 0$

$z = \dfrac{-2 \pm \sqrt{4-8}}{4}$

$z = \dfrac{-2 \pm 2i}{4}$

$\boxed{z = -\dfrac{1}{2} \pm \dfrac{1}{2}i}$

d) (i) $z^2 - \bar{z}^2 = 16i$

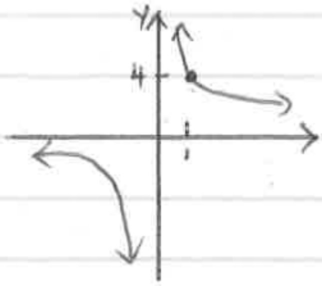
$(x+iy)^2 - (x-iy)^2 = 16i$

$x^2+2xyi-y^2-x^2+2xiy+y^2 = 16i$

$4xyi = 16i$

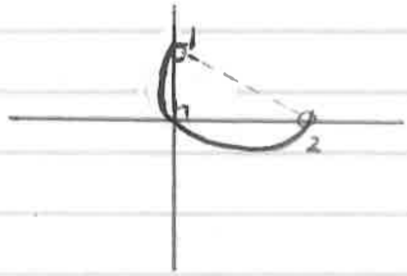
$xy = 4$

OR $z^2 - \bar{z}^2 = 4ixy$



(ii) $\arg\left(\dfrac{z-i}{z-2}\right) \dfrac{\pi}{2}$

ie. $\arg(z-i) - \arg(z-2) = \dfrac{\pi}{2}$



e) i) Let $z = r(\cos\theta + i\sin\theta)$

$\therefore \bar{z} = r(\cos\theta - i\sin\theta)$

$= r(\cos(-\theta) + i\sin(-\theta))$

$\therefore \dfrac{z}{\bar{z}} = \cos(2\theta) + i\sin(2\theta)$

(subtracting arguments)

$\therefore \left|\dfrac{z}{\bar{z}}\right| = 1$ and $\arg\dfrac{z}{\bar{z}} = 2\theta$

(ii) $\dfrac{z}{\bar{z}} = i$ From (i) modulus $= 1$

$\arg = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$

as Real part $= 0$.

$\therefore z$ will have $\arg \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\therefore z = a + ai$ for a any nonzero Real No.

(13) a) $$\int \frac{1}{x^2-6x+5}\,dx$$

$$= \int \frac{1}{(x-5)(x-1)}\,dx$$

$$= \int \frac{A}{(x-5)} + \frac{B}{(x-1)}\,dx$$

$A(x-1) + B(x-5) = 1$

$(A+B)x - A - 5B = 1$

$\therefore A+B=0$

$-A-5B = 1$

$B - 5B = 1$

$-4B = 1$

$B = -1/4$

$\therefore A = 1/4$

$$\therefore I = \int \frac{1/4}{(x-5)} - \frac{1/4}{(x-1)}\,dx$$

$$= \frac{1}{4}\ln|x-5| - \frac{1}{4}\ln|x-1| + C.$$

$$= \frac{1}{4}\ln\frac{|x-5|}{|x-1|} + C$$

c) $$\int \frac{1}{x^{1/2}-x^{1/3}}\,dx$$ $x = u^6$, $dx = 6u^5\,du$

$$\int \frac{6u^5}{u^3-u^2}\,du$$

$$= 6\int \frac{u^3}{u-1}$$

$$= 6\int\left(u^2+u+1 + \frac{1}{u-1}\right)du$$

$$\begin{array}{r} u^2+u+1 \\ u-1\,\overline{)\,u^3\qquad\quad} \\ \underline{u^3-u^2} \\ u^2 \\ \underline{u^2-u} \\ u \\ \underline{u-1} \\ 1 \end{array}$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right]$$

$$= \frac{6x^{1/2}}{3} + \frac{6x^{1/3}}{2} + 6x^{1/6} + 6\ln|x^{1/6}-1| + C$$

$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6}-1| + C.$$

b) $$\int \frac{d\theta}{2-\sin\theta}$$

Let $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$d\theta = \frac{2}{\sec^2\frac{\theta}{2}}\,dt$

$d\theta = 2\cos^2\frac{\theta}{2}\,dt$

$d\theta = \frac{2}{t^2+1}\,dt$

$$= \int \frac{1}{2-\frac{2t}{1+t^2}} \cdot \frac{2}{t^2+1}\,dt$$

$$= \int \frac{\cancel{1+t^2}}{2+2t^2-2t} \cdot \frac{2}{\cancel{t^2+1}}\,dt$$

$$= \int \frac{1}{t^2-t+1}\,dt$$

$$= \int \frac{dt}{(t-\frac{1}{2})^2+1-\frac{1}{4}}$$

$$= \int \frac{dt}{\frac{3}{4}+(t-\frac{1}{2})^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}(t-\tfrac{1}{2}) + C \quad = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}(\tan\tfrac{\theta}{2}-\tfrac{1}{2}) + C$$

d) (i) $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$

$$\begin{aligned}
RHS &= (1-\sqrt{x})^{n-1}\left(1-(1-\sqrt{x})^{1}\right)\\
&= (1-\sqrt{x})^{n-1}(1-1+\sqrt{x})\\
&= (1-\sqrt{x})^{n-1}\sqrt{x}\\
&= LHS.
\end{aligned}$$

(ii) $I_n = \int_0^1 (1-\sqrt{x})^n\,dx$

$u = (1-\sqrt{x})^n \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = n(1-\sqrt{x})^{n-1} \cdot -\frac{1}{2}x^{-1/2} \qquad v = x.$

$= \frac{-n}{2\sqrt{x}}(1-\sqrt{x})^{n-1}$

$$= \left[x(1-\sqrt{x})^n\right]_0^1 + \int \frac{nx}{2\sqrt{x}}(1-\sqrt{x})^{n-1}dx$$

$$= 0 + \frac{n}{2}\int \sqrt{x}\,(1-\sqrt{x})^{n-1}dx$$

$I_n = \frac{n}{2}\int (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ from part (i)

$= \frac{n}{2}\int (1-\sqrt{x})^{n-1} - \frac{n}{2}I_n$

$$\left(\frac{n}{2}+1\right)I_n = \frac{n}{2}I_{n-1}$$

$$I_n = \left(\frac{n}{2} \div \frac{n+2}{2}\right)I_{n-1}$$

$$I_n = \frac{n}{n+2}I_{n-1}$$

(iii) $I_n = \frac{n}{n+2} \times \frac{n-1}{n+1} \times I_{n-2}$.

$$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \dots \times I_1 \times I_0.$$

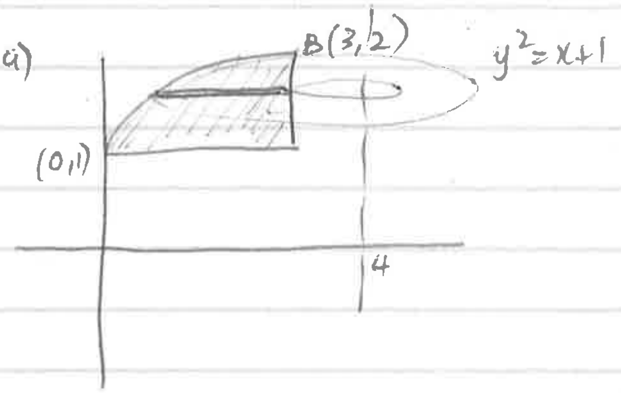
$$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \dots \times \frac{1}{3} \times 1$$

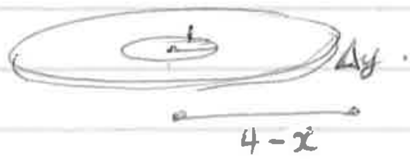
$$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \dots \times \frac{1}{3} \times \left(\frac{2}{2}\right) \times 1$$ (multiply top & bottom by 2)

$$= \frac{n! \times 2}{(n+2)!}$$

$$\therefore \frac{1}{I_n} = \frac{(n+2)!}{n! \times 2}$$

14) a)





(i) $A(y) = \pi\left(R_1^2 - R_2^2\right)$

$= \pi\left((4-x)^2 - 1\right)$

$= \pi\left((4-y^2+1)^2 - 1\right)$

$= \pi\left((5-y^2)^2 - 1\right)$

(ii) $\Delta V = \pi\left((5-y^2)^2 - 1\right)\Delta y$.

$V = \lim\limits_{\Delta y \to 0} \sum\limits_{y=1}^{2} \pi\left((5-y^2)^2 - 1\right)\Delta y$.

$= \pi\int_1^2 (25 - 10y^2 + y^4 - 1)\,dy$

$= \pi\int_1^2 (24 - 10y^2 + y^4)\,dy$

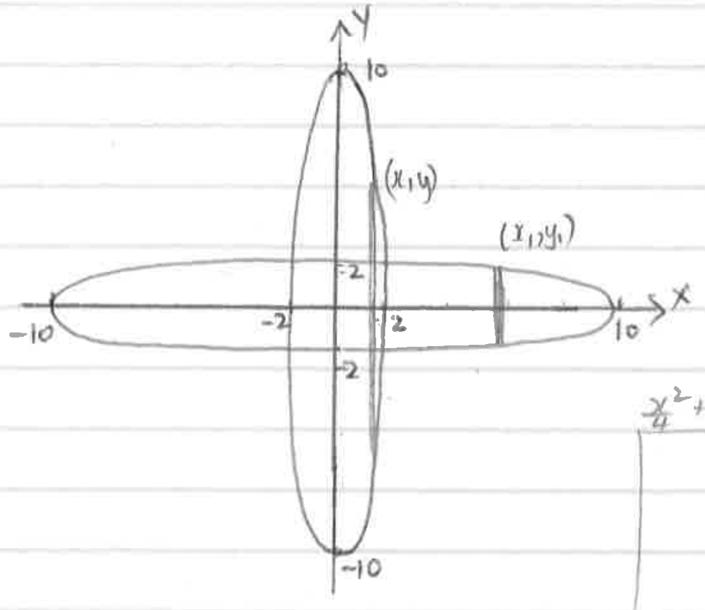
$= \pi\left[24y - \frac{10y^3}{3} + \frac{y^5}{5}\right]_1^2$

$= \pi\left[48 - \frac{80}{3} + \frac{32}{5} - \left(24 - \frac{10}{3} + \frac{1}{5}\right)\right]$

$= \pi\left[\frac{103}{15}\right]$

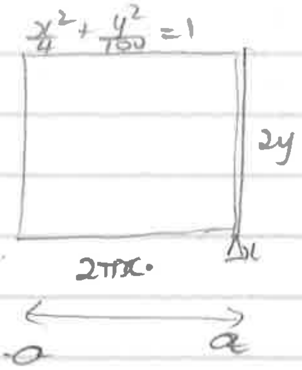
$= \frac{103\pi}{15}$ c.units

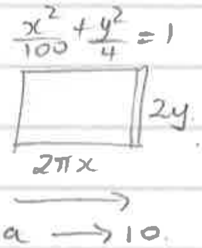
b) (i)



$$\frac{x^2}{100}+\frac{y^2}{4}=1$$

$$\frac{x^2}{4}+\frac{y^2}{100}=1$$





(ii)

$$A(x) = A_1 + A_2$$

$$=(2\pi x)(2y) \quad + 2\pi x_1(2y_1)$$

$$=2\pi x\, 2\sqrt{100-25x^2} \quad + 2\pi x_1\left(2\sqrt{\frac{100-x^2}{25}}\right)$$

$$=2\pi x.10\sqrt{4-x^2} \quad + \frac{4\pi x_1}{5}\sqrt{100-x_1^2}$$

$$\Delta V = 20\pi x\sqrt{4-x^2}\,\Delta x + \frac{4\pi x_1}{5}\sqrt{100-x_1^2}\,\Delta x_1$$

$$V = \lim_{\Delta x\to 0}\sum_{x=0}^{a}\Delta V_1 \quad + \lim_{\Delta x_1\to 0}\sum_{x=a}^{10}\Delta V_2.$$

$$V = \int_0^a 20\pi x\sqrt{4-x^2}\,dx \quad + \int_a^{10}\frac{4\pi x}{5}\sqrt{100-x^2}\,dx$$

$$=\frac{20\pi}{-2}\int_0^a -2x\sqrt{4-x^2}\,dx \quad + \frac{4\pi}{5\times 2}\int_a^{10} -2x\sqrt{100-x^2}\,dx$$

$$=-10\pi\left[\frac{2(4-x^2)^{3/2}}{3}\right]_0^a \quad - \frac{4\pi}{10}\left[\frac{2(100-x^2)^{3/2}}{3}\right]_a^{10}$$

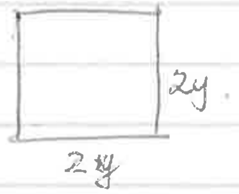
$$=-\frac{20\pi}{3}\left[(4-a^2)^{3/2}-4^{3/2}\right] - \frac{4\pi}{15}\left[0-(100-a^2)^{3/2}\right]$$

$$=\frac{20\pi}{3}\left[8-(4-a^2)^{3/2}\right] + \frac{4\pi}{15}(100-a^2)^{3/2} \quad \text{c.u.}$$

c) (i) $A(x) = 4y^2$

$= 4 \times \dfrac{1}{4+x^2}$

$= \dfrac{4}{4+x^2}$



$4y^2 + x^2y^2 = 1$

$(4+x^2)y^2 = 1$

$y^2 = \dfrac{1}{4+x^2}$

$\Delta V = \dfrac{4}{4+x^2}\Delta x.$

(ii) $V = \lim_{\Delta x \to 0} \sum_{x=-a}^{x=a} \Delta V$

$= \int_{-a}^{a} \dfrac{4}{4+x^2}\, dx.$

$= 8\int_0^a \dfrac{1}{4+x^2}\, dx$

$= \dfrac{8}{2}\left[\tan^{-1}\dfrac{x}{2}\right]_0^a.$

$= 4\left[\tan^{-1}\dfrac{a}{2}\right]$

(iii) As $a \to \infty$.

$\tan^{-1}\dfrac{a}{2} \to \dfrac{\pi}{2}$

$\therefore V \to 4 \times \dfrac{\pi}{2} = 2\pi$ c. units.

15, a) $P(x) = x^4 + 7x^3 + 9x^2 - 27x + C$

(i) $P'(x) = 4x^3 + 21x^2 + 18x - 27$

$P''(x) = 12x^2 + 42x + 18$.

Triple zero $\therefore P''(x) = 0$

$12x^2 + 42x + 18 = 0$.

$2x^2 + 7x + 3 = 0$

$\begin{matrix} 2x & 1 \\ x & 3 \end{matrix}$ $(2x+1)(x+3) = 0$

$\therefore x = \frac{-1}{2}, x = -3$.

$P'(-3) = 4(-27) + 21(9) + 18(-3) - 27$

$= 0$

zero at $x = -3$.

(ii) $P(-3) = 0$.

$\therefore 0 = 81 + 7(-27) + 9(9) - 27(-3) + C$

$0 = 54 + C$

$\therefore C = -54$

(iii) $\therefore P(x) = (x+3)^3 Q(x)$

$C = -54$

$C = -(27 \times 2)$

$P(2) = 0$

$\therefore (x-2)$ is the other factor

$\therefore P(x) = (x+3)^3(x-2)$

b) $x^3 - x^2 + 3 = 0$.

(i) For $\alpha^2, \beta^2, \gamma^2$, $\sqrt{x}$ satisfies above eq'n

$(\sqrt{x})^3 - (\sqrt{x})^2 + 3 = 0$

$x^{3/2} - x + 3 = 0$

$x^{3/2} = x - 3$

$x^3 = (x-3)^2$

$x^3 - x^2 + 6x - 9 = 0$

(ii) $\alpha^4 + \beta^4 + \gamma^4 = (\alpha^2 + \beta^2 + \gamma^2)^2 - 2\sum\alpha^2\beta^2$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= 1 - 2 \times 6$

$= -11$

c) $f(x) = x^4 + px^3 + qx^2 + rx + s$

$C = \alpha + \beta = \gamma + \delta$

$P = 3\alpha$

$Q = \gamma\delta$

(i) $\alpha + \beta + \gamma + \delta = -p$

$2C = -p$

$\therefore \boxed{p = -2C}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$

$P + Q + \alpha(\gamma+\delta) + \beta(\gamma+\delta) = q$

$P + Q + (\gamma+\delta)(\alpha+\beta) = q$

$\boxed{P + Q + C^2 = q}$

$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta a = -r$

$\gamma\delta(\alpha+\beta) + \alpha\beta(\gamma+\delta) = -r$

$\boxed{QC + PC = -r}$

$\alpha\beta\gamma\delta = s$

$\boxed{PQ = s}$

(ii) $p^3 + 8r = 4pq$

$LHS = p^3 + 8r$

$= (-2C)^3 + 8(-PC - QC)$

$= -8C^3 - 8PC - 8QC$

$= -8C(C^2 + P + Q)$

$= -8Cq$

$= -8\left(\frac{p}{-2}\right)q$

$= 4pq$

$= RHS$

(iii) $g(x) = x^4 - 18x^3 + 79x^2 + 18x - 440$

$-2C = -18$

$\therefore C = 9$

$P + Q + C^2 = 79$

$\therefore P + Q = -2$

$PQ = -440$

$P(-2-P) = -440$

$-P^2 - 2P + 440 = 0$

$P^2 + 2P - 440 = 0$

$(P+22)(P-20) = 0$

$P = 20, -22$

ie. $P = 20$, $Q = -22$

Now $\alpha + \beta = \gamma + \delta = 9$

$\alpha\beta\gamma\delta = -440$

$\therefore \alpha\beta = 20, \gamma\delta = -22$

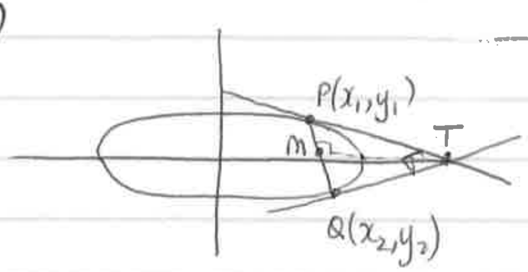
$\left.\begin{array}{l}\alpha+\beta = 9\\ \alpha\beta = 20\end{array}\right\rangle 5, 4$

$\left.\begin{array}{l}\gamma+\delta = 9\\ \gamma\delta = -22\end{array}\right\rangle 11, -2$

$\therefore$ Roots are

$4, 5, 11, -2$

16) a)



(i) $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$

(ii) T lies on $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

and T lies on $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$

Now $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{xx_2}{a^2}+\frac{yy_2}{b^2}$ (*)

∴ T satisfies above equn as well.

Now (*) becomes

$\frac{xx_1}{a^2}-\frac{xx_2}{a^2}+\frac{yy_1}{b^2}-\frac{yy_2}{b^2}=0$

ie. $\frac{x(x_1-x_2)}{a^2}+\frac{y(y_1-y_2)}{b^2}=0$ (**)

Coords of $M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Sub M into (**)

$\frac{(x_1+x_2)(x_1-x_2)}{2a^2}+\frac{(y_1+y_2)(y_1-y_2)}{2b^2}=$ LHS

LHS $=\frac{x_1^2-x_2^2}{2a^2}+\frac{y_1^2-y_2^2}{2b^2}$

$=\frac{1}{2}\left[\frac{x_1^2}{a^2}-\frac{x_2^2}{a^2}+\frac{y_1^2}{b^2}-\frac{y_2^2}{b^2}\right]$

$=\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-\frac{1}{2}\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)$

$=\frac{1}{2}\times 1-\frac{1}{2}\times 1$

$=0$ = RHS.

(iii) T & M satisfy

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{xx_2}{a^2}+\frac{yy_2}{b^2}$

∴ it is equn of MT.

Sub (0,0)

LHS = 0

RHS = 0 ∴ O(0,0) satisfies equn as well. Hence, O,T,M are collinear.

(iv) $m_{PQ}=\frac{y_2-y_1}{x_2-x_1}$

MT : $\frac{x(x_1-x_2)}{a^2}+\frac{y(y_1-y_2)}{b^2}=0$.

$\therefore m_{MT}=-\frac{(x_1-x_2)}{a^2}\div\frac{y_1-y_2}{b^2}$

$=\frac{(x_2-x_1)}{(y_1-y_2)}\cdot\frac{b^2}{a^2}$

$m_{PQ}\times m_{MT}=\frac{y_2-y_1}{x_2-x_1}\times\frac{(x_2-x_1)}{(y_1-y_2)}\frac{b^2}{a^2}$

$=-\frac{b^2}{a^2}$ = CONSTANT

(v) $m_{PT}=\frac{-x_1}{a^2}\div\frac{y_1}{b^2}=\frac{-x_1}{y_1}\cdot\frac{b^2}{a^2}$

$m_{QT}=\frac{-x_2}{a^2}\div\frac{y_2}{b^2}=\frac{-x_2}{y_2}\cdot\frac{b^2}{a^2}$

$m_{PT}\times m_{QT}=-1$ as $\angle PTQ=90°$.

$\therefore \frac{-x_1}{y_1}\cdot\frac{b^2}{a^2}\times\frac{-x_2}{y_2}\cdot\frac{b^2}{a^2}=-1$

$\frac{x_1x_2b^4}{y_1y_2a^4}=-1$

$x_1x_2b^4=-y_1y_2a^4$ $(\div a^4b^4)$

$\frac{x_1x_2}{a^4}+\frac{y_1y_2}{b^4}=0$.

b) Tangent at P

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} - 1 = 0$$

If tangent to circle then perpendicular distance equals radius of circle.

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$$

$$= \frac{\left|\frac{\sec\theta}{a}(ae) - \left(\frac{\tan\theta}{b}\right).0 - 1\right|}{\sqrt{\left(\frac{\sec\theta}{a}\right)^2 + \left(\frac{\tan\theta}{b}\right)^2}}$$

$$= \frac{|e\sec\theta - 1|}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}}$$

$$= \frac{e\sec\theta - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{a^2(e^2-1)}}}$$

$$= \frac{e\sec\theta - 1}{\sqrt{\frac{(e^2-1)\sec^2\theta + \tan^2\theta}{a^2(e^2-1)}}}$$

$$= \frac{(e\sec\theta - 1)\, a\sqrt{e^2-1}}{\sqrt{e^2\sec^2\theta - \sec^2 + \tan^2\theta}}$$

$$= \frac{(e\sec\theta - 1)\, a\sqrt{e^2-1}}{\sqrt{e^2\sec^2\theta - \sec^2\theta + \sec^2\theta - 1}}$$

$$= \frac{(e\sec\theta - 1)\, a\sqrt{e^2-1}}{\sqrt{e^2\sec^2\theta - 1}}$$

$$= \frac{(e\sec\theta - 1)\, a\sqrt{e^2-1}}{\sqrt{e\sec\theta - 1}\sqrt{e\sec\theta + 1}}$$

$$= \frac{\sqrt{e\sec\theta - 1}}{\sqrt{e\sec\theta + 1}} \cdot a\sqrt{e^2-1} = a\sqrt{e^2+1} \quad \text{(radius)}$$

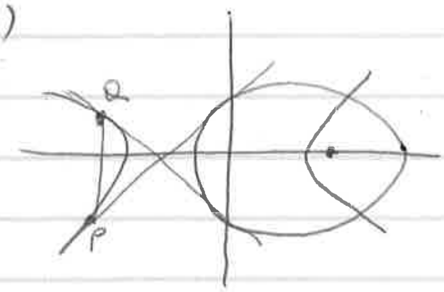
$$\therefore \frac{e^2+1}{e^2-1} = \frac{e\sec\theta - 1}{e\sec\theta + 1}$$

$$e^3\sec\theta + e^2 + e\sec\theta + 1 = e^3\sec\theta - e^2 - e\sec\theta + 1$$

$$2e^2 = -2e\sec\theta$$

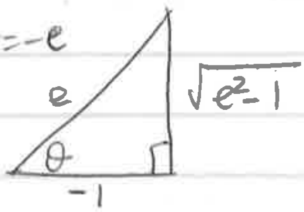
$$\therefore \boxed{\sec\theta = -e}$$

(ii)



$P(a\sec\theta, b\tan\theta)$

$\sec\theta = -e$



$\therefore P(a\sec\theta, b\tan\theta)$

$= P(a(-e), b(-\sqrt{e^2-1}))$

$= P(-ae, -b\sqrt{e^2-1})$

Sim, by symmetry.

$Q(ae, b\sqrt{e^2-1})$

Latus Rectum : $x = ae$

$\therefore$ P, Q are extremities of latus rectum.