North Sydney Boys High School

MATHEMATICS (EXTENSION 2)

2014 HSC Course Assessment Task 3 (Trial Examination)
Wednesday June 18, 2014

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

○ 12M4A – Ms Ziaziaris ○ 12M4B – Mr Lam ○ 12M4C – Mr Ireland

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. Which of the following represents the region defined by the upper part of the semicircle in the Argand diagram with centre $(1, 1)$ and radius 1, cut off by the line $y = x$? **1**

(A) $|z + 1 + i| \leq \sqrt{2}$ and $\arg(z) \leq \frac{\pi}{4}$

(B) $|z - 1 - i| \leq 1$ and $\arg(z) \geq \frac{\pi}{4}$

(C) $|z + 1 + i| \leq 1$ and $\arg(z) \geq \frac{\pi}{4}$

(D) $|z - 1 - i| \leq \sqrt{2}$ and $\arg(z) \leq \frac{\pi}{4}$

2. Which of the following shapes represents the locus of the point P representing the complex number z, moving in the Argand diagram such that **1**

$$|z - 4i| = \text{Arg}\left(\sqrt{3} + i\right) + |z + 4i|$$

(A) Parabola

(B) Ellipse

(C) Hyperbola

(D) Circle

3. Let $z = \sqrt{48} - 4i$. What is the value of $\text{Arg}\left(z^7\right)$? **1**

(A) $-\dfrac{2\pi}{3}$

(B) $\dfrac{2\pi}{3}$

(C) $\dfrac{5\pi}{6}$

(D) $\dfrac{-5\pi}{6}$

4. What is the eccentricity of the hyperbola $3x^2 - 4y^2 = 1$? **1**

(A) $\dfrac{\sqrt{7}}{2}$

(B) $\dfrac{\sqrt{7}}{\sqrt{3}}$

(C) $\dfrac{1}{2}$

(D) $\dfrac{1}{\sqrt{3}}$

5. Which of the following integrals does NOT involve integration by parts? **1**

(A) $\displaystyle\int 7xe^{-x^2}\,dx$

(B) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

(C) $\displaystyle\int x^2 \sin x\,dx$

(D) $\displaystyle\int e^x \cos x\,dx$

6. Which of the following expressions will lead to the location of the vertical tangent(s) to the graph of $x^4 + y^4 = 4xy$? **1**

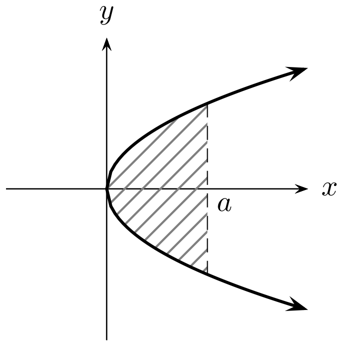
(A) $x^3 + y = 0$

(B) $x^3 - y = 0$

(C) $x - y^3 = 0$

(D) $x + y^3 = 0$

7. A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex $(0,0)$ and the line $x = a$ about the y axis. **1**



Which of the following integrals gives the volume of this area by *slicing*?

(A) $2\pi\sqrt{a}\int_0^a z^{\frac{3}{2}}\,dz$

(B) $4\pi\sqrt{a}\int_0^a z^{\frac{3}{2}}\,dz$

(C) $\pi\int_0^{2a}\left(a^2 - \frac{z^4}{16a^2}\right)\,dz$

(D) $2\pi\int_0^{2a}\left(a^2 - \frac{z^4}{16a^2}\right)\,dz$

8. Without evaluating the integrals, which of the following is greater than zero? **1**

(A) $\int_{-1}^{1} \tan^{-1}(\sin x)\,dx$

(B) $\int_{-1}^{1} \frac{2x}{\sin^2 x}\,dx$

(C) $\int_{-1}^{1} \left((e^x)^3 + x^7\right)\,dx$

(D) $\int_{-1}^{1} \frac{x^5}{\cos^3 x}\,dx$

9. An ellipse has foci $(0, -3)$ and $(0, 5)$. **1**

Which of the following could be the equation of the ellipse?

(A) $\dfrac{x^2}{8} + \dfrac{(y-1)^2}{12} = 1$

(B) $\dfrac{x^2}{12} + \dfrac{(y-1)^2}{8} = 1$

(C) $\dfrac{x^2}{9} + \dfrac{(y-1)^2}{25} = 1$

(D) $\dfrac{x^2}{8} + \dfrac{(y+1)^2}{12} = 1$

10. The polynomial equation $P(x) = 0$ has real coefficients and has roots that include $x = 4i - 3$, $x = -3$ and $x = -4i + 3$. What is the smallest possible degree of $P(x)$? **1**

(A) 2

(B) 3

(C) 4

(D) 5

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) i. Find the partial fraction decomposition of $\dfrac{1}{x^2-1}$. **2**

ii. Hence or otherwise, evaluate $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$. **3**

(b) i. Given $I_n = \displaystyle\int_0^1 x^n 2^x\,dx$ $(n \in \mathbb{Z}^+)$, show that **3**

$$I_n = \frac{2}{\ln 2} - \frac{n}{\ln 2} I_{n-1}$$

ii. Hence evaluate $\displaystyle\int_0^1 x^3 2^x\,dx$. **3**

(c) Evaluate $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$, giving your answer in simplest form. **4**

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) i. Find the three roots of $z^3 - 1 = 0$ in modulus-argument form. **3**

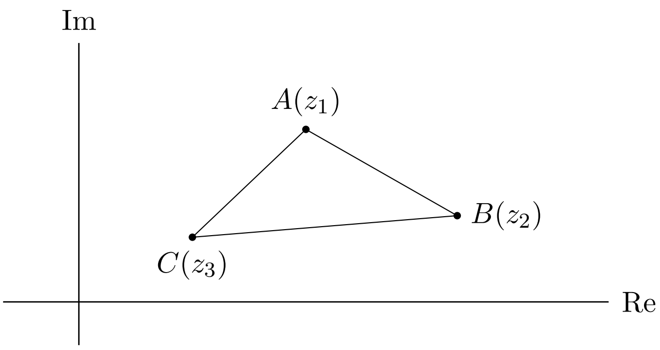
ii. Write each of the complex roots in the form $x + iy$. **2**

iii. If one of the complex roots is ω, find the area of the triangle formed by 1, ω and ω^2. **1**

iv. Show that $1 + \omega + \omega^2 = 0$. **2**

v. Evaluate $(1+\omega)\left(1+\omega^2\right)\left(1+\omega^5\right)\left(1+\omega^8\right)\left(1+\omega^{11}\right)$ **3**

(b) A, B and C are the points that represent the complex number numbers z_1, z_2 and z_3 on the Argand diagram. **4**



Prove that if

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{z_1 - z_3}{z_1 - z_2}$$

then $\triangle ABC$ is equilateral.

Hint: To commence, extend lengths AC, AB and BC to real axis, and use angles.

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

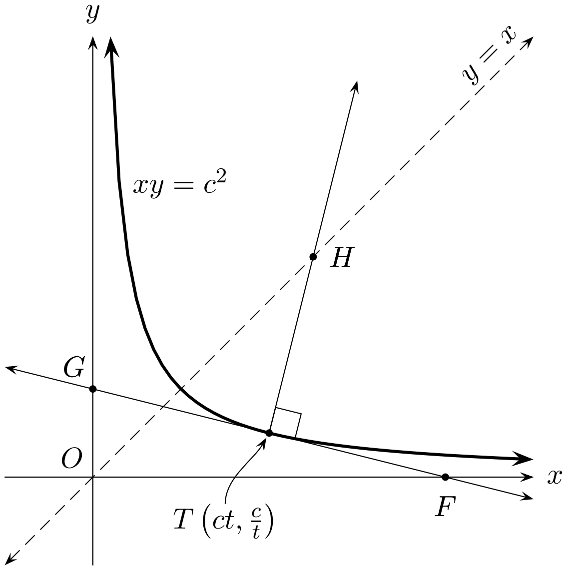
i. Show that the equation of the normal to P is: **3**

$$ax\sin\theta - by\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta$$

ii. G is the point where this normal meets the x axis. N is the foot of the perpendicular from P to the x axis, O is the origin and e is the eccentricity. **2**

Show that $\dfrac{OG}{ON}=e^2$.

(b) The tangent to the hyperbola $xy=c^2$ at the point $T\left(ct, \dfrac{c}{t}\right)$ meets the x and y axes at F and G respectively, and the normal at T meets the line $y=x$ at H.



i. Show that the tangent at T is **3**

$$x+t^2y=2ct$$

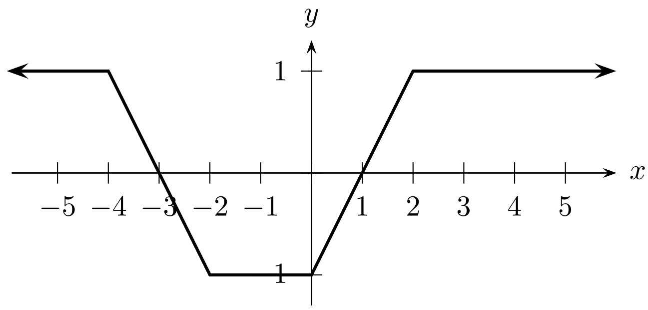
ii. Show that the normal at T is **2**

$$t^3x-ty=c\left(t^4-1\right)$$

iii. Prove that $FH\perp HG$. **5**

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) The following diagram shows the sketch of the function $y = f(x)$.



On separate diagrams of $\frac{1}{3}$ page each, carefully sketch:

i. $y = f(|x|)$. **2**

ii. $y = \dfrac{1}{f(x)}$. **2**

iii. $y = \cos^{-1}(f(x))$. **2**

(b) Given $x \in \mathbb{R}$, and $\lceil x \rceil$ be a real number that is the smallest integer that is greater than, or equal to x.

i. Evaluate $\lceil 2.5 \rceil$ and $(2.5 + \lceil 2.5 \rceil)^2$. **2**

ii. Sketch a graph of $y = \lceil x \rceil + (x + \lceil x \rceil)^2$. **3**

(c) By using the substitution $x = 2\tan\theta$, evaluate the definite integral **4**

$$\int \frac{dx}{(4 + x^2)^{\frac{3}{2}}}$$

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) A solid is formed by rotating the region bounded $y = \sqrt{x}$, the x axis and the line $x = 4$, about the line $x = 4$.

i. By drawing a diagram and taking slices perpendicular to the axis of rotation, show that the element of volume δV is **2**

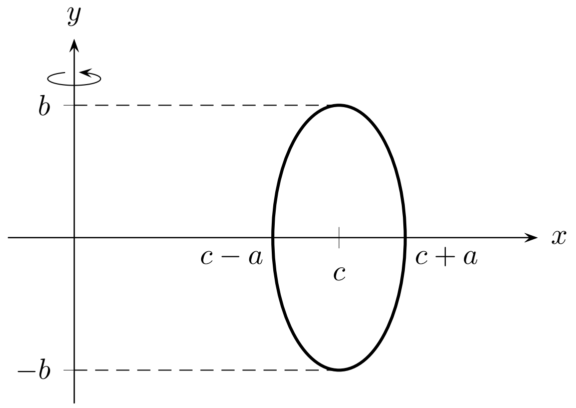
$$\delta V = \pi \left(16 - 8y^2 + y^4\right) \delta y$$

ii. Hence or otherwise, find the volume generated. **2**

(b) i. Sketch the curve $x = \sqrt{b^2 - y^2}$, and hence explain why **2**

$$\int_0^b \sqrt{b^2 - y^2}\, dy = \frac{\pi b^2}{4}$$

ii. The ellipse $\dfrac{(x-c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a$ and $c > a$ is shown in the diagram. The region bounded by the ellipse is rotated about the y axis to form a ring. **4**



By taking slices perpendicular to the y axis, show that the ring has volume $2abc\pi^2$.

(c) i. Show that $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **2**

ii. Hence or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}}\, dx$. **3**

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) The equation $x^3 - 4x^2 + 5x + 2 = 0$ has roots α, β and γ.

i. Show that $\alpha^2 + \beta^2 + \gamma^2 = 6$. **1**

ii. Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

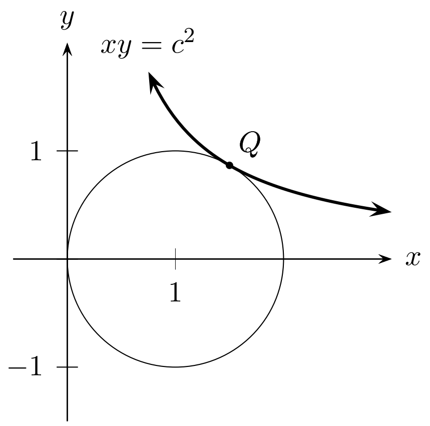
(b) A polynomial $P(x)$ is divided by $x^2 - a^2$ (where $a \neq 0$) and the remainder is $px + q$.

i. Show that **3**

$$p = \frac{1}{2a}[P(a) - P(-a)] \quad \text{and} \quad q = \frac{1}{2}[P(a) + P(-a)]$$

ii. Find the remainder when $P(x) = x^n - a^n$ for $n \in \mathbb{Z}^+$, is divided by $x^2 - a^2$. **2**

(c) The hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point Q.



i. By considering the diagram provided or otherwise, deduce that the equation $x^2(x-1)^2 + c^4 = x^2$ has a repeated real root $\beta > 0$, as well as two non-real complex roots. **2**

ii. Find the values of β and c^2. **3**

Hint: Consider a property of β being a repeated root of $P(x) = 0$.

iii. Find the equation of the common tangent to the hyperbola and the circle at Q. **2**

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 12M4A – Ms Ziaziaris ○ 12M4B – Mr Lam ○ 12M4C – Mr Ireland

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
7 – Ⓐ Ⓑ Ⓒ Ⓓ
8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (B) **2.** (C) **3.** (C) **4.** (A) **5.** (A) **6.** (C) **7.** (D)
8. (C) **9.** (C) **10.** (D)

Question 11 (Lam)

(a) i. (2 marks)

$$\underbrace{\frac{1}{(x-1)(x+1)}}_{\times(x-1)(x+1)} \equiv \underbrace{\frac{A}{x-1}}_{\times(x-1)(x+1)} + \underbrace{\frac{B}{x+1}}_{\times(x-1)(x+1)}$$
$$1 \equiv A(x+1) + B(x-1)$$

- When $x = 1$,

$$1 = A(1+1) + B(1-1)$$
$$\therefore A = \frac{1}{2}$$

- When $x = -1$,

$$1 = A(-1+1) + B(-1-1)$$
$$\therefore B = -\frac{1}{2}$$

$$\therefore \frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

ii. (3 marks)

$$\begin{aligned}\int \frac{x^2+1}{x^2-1}\,dx &= \int \frac{x^2-1+2}{x^2-1}\,dx\\ &= \int \left(1 + \frac{2}{x^2-1}\right)dx\\ &= \int \left(1 + 2\left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right)\right)dx\\ &= \int \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right)dx\\ &= x + \ln(x-1) - \ln(x+1) + C\\ &= x + \ln\left(\frac{x-1}{x+1}\right) + C\end{aligned}$$

(b) i. (3 marks)

$$I_n = \int_0^1 \underbrace{x^n}_{=u}\underbrace{2^x}_{=dv}\,dx$$

$$u = x^n \qquad v = \frac{1}{\ln 2}2^x$$
$$du = nx^{n-1} \quad dv = 2^x$$

$$\begin{aligned}\therefore I_n &= \left[\frac{x^n2^x}{\ln 2}\right]_0^1 - \int_0^1 nx^{n-1}\times\frac{1}{\ln 2}2^x\,dx\\ &= \left(\frac{2}{\ln 2} - 0\right) - \frac{n}{\ln 2}\int_0^1 x^{n-1}2^x\,dx\\ &= \frac{2}{\ln 2} - \frac{n}{\ln 2}I_{n-1}\end{aligned}$$

ii. (3 marks)

$$\begin{aligned}I_0 &= \int_0^1 x^0 2^x\,dx = \int_0^1 2^x\\ &= \frac{1}{\ln 2}\left[2^x\right]_0^1\\ &= \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2}\\ I_1 &= \frac{2}{\ln 2} - \frac{1}{\ln 2}I_0\\ &= \frac{2}{\ln 2} - \frac{1}{\ln 2}\left(\frac{1}{\ln 2}\right)\\ &= \frac{2}{\ln 2} - \frac{1}{\ln^2 2}\\ I_2 &= \frac{2}{\ln 2} - \frac{2}{\ln 2}I_1\\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2}\left(\frac{2}{\ln 2} - \frac{1}{\ln^2 2}\right)\\ &= \frac{2}{\ln 2} - \frac{4}{\ln^2 2} + \frac{2}{\ln^3 2}\\ I_3 &= \frac{2}{\ln 2} - \frac{3}{\ln 2}I_2\\ &= \frac{2}{\ln 2} - \frac{3}{\ln 2}\left(\frac{2}{\ln 2} - \frac{4}{\ln^2 2} + \frac{2}{\ln^3 2}\right)\\ &= \frac{2}{\ln 2} - \frac{6}{\ln^2 2} + \frac{12}{\ln^3 2} - \frac{6}{\ln^4 2}\end{aligned}$$

(c) (4 marks)

$$\begin{aligned}
&\int \frac{x}{x^2+2x+10}\,dx \\
&= \frac{1}{2}\int \frac{2x+2 \quad -2}{x^2+2x+10}\,dx \\
&= \frac{1}{2}\int \frac{2x+2}{x^2+2x+10}\,dx - \int \frac{1}{(x^2+2x+1)+9}\,dx \\
&= \frac{1}{2}\ln\left(x^2+2x+10\right) - \int \frac{1}{(x+1)^2+9}\,dx \\
&= \frac{1}{2}\ln\left(x^2+2x+10\right) - \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C
\end{aligned}$$

Question 12 (Ireland)

(a) i. (3 marks)

$$\begin{aligned}
z^3 - 1 &= 0 \\
z^3 = 1 &= \cos(2k\pi) + i\sin(2k\pi)
\end{aligned}$$

Applying De Moivre's Theorem,

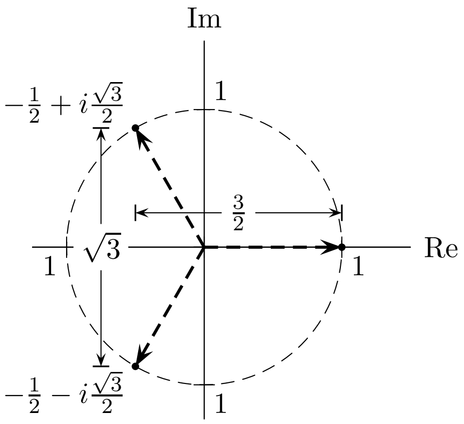
$$z = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$

- $k=0,\ z=1$.
- $k=1,\ z=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$.
- $k=-1$,
  $z=\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)$.

ii. (2 marks)

$$-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

iii. (1 mark)



$$A = \frac{1}{2}bh = \frac{1}{2}\times\frac{3}{2}\times\sqrt{3} = \frac{3\sqrt{3}}{4}$$

iv. (2 marks)

$$\begin{aligned}
\omega^3 - 1 &= 0 \\
(\omega-1)\left(\omega^2+\omega+1\right) &= 0
\end{aligned}$$

Given $\mathrm{Im}(\omega) \neq 0$,

$$\therefore \omega^2+\omega+1=0$$

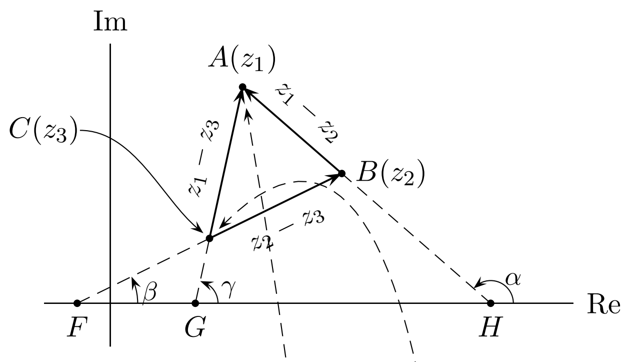
v. (3 marks)

$$\begin{aligned}
\omega^2+\omega+1 &= 0 \\
\therefore \omega^2 &= -(\omega+1)
\end{aligned}$$

Now examine expression:

$$\begin{aligned}
&(1+\omega)(1+\omega^2)(1+\omega^5)(1+\omega^8)(1+\omega^{11}) \\
&= -\omega^2\left(\cancel{1}-(\cancel{1}+\omega)\right)\left(1+\cancel{\omega^3}\omega^2\right) \\
&\qquad \times\left(1+\cancel{\omega^6}\omega^2\right)\left(1+\cancel{\omega^9}\omega^2\right) \\
&= \cancel{\omega^3}\left(1+\omega^2\right)^3 \\
&= \left(\cancel{1}-(\cancel{1}+\omega)\right)^3 \\
&= -1
\end{aligned}$$

(b) (4 marks)



where

- $\alpha = \arg(z_1 - z_2)$
- $\beta = \arg(z_2 - z_3)$
- $\gamma = \arg(z_1 - z_3)$

Given

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{z_1 - z_3}{z_1 - z_2} \qquad (\flat)$$

Then by taking arguments,

$$\arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \arg\left(\frac{z_1 - z_3}{z_1 - z_2}\right)$$

$$\begin{aligned}\therefore \arg(z_2 - z_3) - \arg(z_1 - z_3) \\ = \arg(z_1 - z_3) - \arg(z_1 - z_2)\end{aligned}$$

From diagram,

$$\beta - \gamma = \gamma - \alpha \qquad (\diamondsuit)$$

- Now in $\triangle FCG$,
  - $\angle ACB = \angle FCG$ (vertically opposite)
  - Hence $\beta + \angle FCG = \gamma$, and

$$\angle FCG = \angle ACB = \gamma - \beta$$

- In $\triangle AGH$
  - $\gamma + \angle CAB = \alpha$ (exterior $\angle$ of $\triangle$)
  - Hence $\angle CAB = \alpha - \gamma$. But from $(\diamondsuit)$,

$$\begin{aligned}\alpha - \gamma &= \gamma - \beta \\ \therefore \angle CAB &= \gamma - \beta = \angle ACB\end{aligned}$$

- Hence $\triangle ABC$ is now isosceles with $AB = BC$, or $|z_1 - z_2| = |z_2 - z_3|$.

- Apply modulus to $(\flat)$,

$$\frac{|z_2 - z_3|}{|z_1 - z_3|} = \frac{|z_1 - z_3|}{|z_1 - z_2|}$$
$$|z_1 - z_2|\,|z_2 - z_3| = |z_1 - z_3|^2$$

But $|z_1 - z_2| = |z_2 - z_3|$,

$$\therefore |z_1 - z_2|^2 = |z_1 - z_3|^3$$
$$\therefore |z_1 - z_2| = |z_1 - z_3|$$

i.e. $AC = AB$. Hence $AB = AC = BC$, and $\triangle ABC$ is equilateral.

Question 13 (Ziaziaris)

(a) i. (3 marks)

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

Differentiating to obtain gradient of tangent:

$$\frac{dx}{d\theta} = -a\sin\theta$$
$$\frac{dy}{d\theta} = b\cos\theta$$
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$$

Hence gradient of normal is

$$m_\perp = \frac{a\sin\theta}{b\cos\theta}$$

Apply point-gradient formula,

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$
$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$
$$ax\sin\theta - by\cos\theta = a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$$
$$\therefore ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta \qquad (\ddagger)$$

ii. (2 marks)

- G occurs when normal meets x axis: substitute $y = 0$ into $(\ddagger)$:

$$ax\cancel{\sin\theta} - 0 = \left(a^2 - b^2\right)\cancel{\sin\theta}\cos\theta$$
$$x = \frac{\left(a^2 - b^2\right)\cos\theta}{a}$$

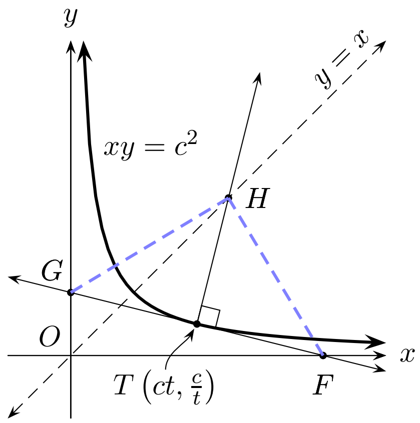
- N occurs at $x = a\cos\theta$, $y = 0$ (on x axis):
- Hence $OG = \frac{(a^2-b^2)\cos\theta}{a}$, $ON = a\cos\theta$:

$$\begin{aligned}\frac{OG}{ON} &= \frac{\frac{(a^2-b^2)\cancel{\cos\theta}}{a}}{a\cancel{\cos\theta}}\\ &= \frac{a^2-b^2}{a^2}\\ &= 1-\frac{b^2}{a^2}\end{aligned}$$

As $b^2 = a^2\left(1-e^2\right)$, then $\frac{b^2}{a^2} = 1-e^2$:

$$\begin{aligned}\therefore \frac{OG}{ON} &= 1-\left(1-e^2\right)\\ &= e^2\end{aligned}$$

(b) i. (3 marks)



$$\begin{aligned}xy &= c^2\\ y &= c^2x^{-1}\\ \frac{dy}{dx} &= -c^2x^{-2}\Big|_{x=ct}\\ &= -\cancel{c^2}\times\frac{1}{\cancel{c^2}t^2}\\ &= -\frac{1}{t^2}\end{aligned}$$

Use pt-gradient formula to find equation of tangent:

$$\underbrace{y-\frac{c}{t}}_{\times(-t^2)} = \underset{\times(-t^2)}{-\frac{1}{t^2}}(x-ct) \qquad (\spadesuit)$$

$$\begin{aligned}-t^2y+ct &= x-ct\\ x+t^2y &= 2ct\end{aligned}$$

ii. (2 marks)

$$m_\perp = t^2$$

Use pt-gradient formula to find equation of normal:

$$\begin{aligned}y-\frac{c}{t} &= t^2(x-ct)\\ \underbrace{y-\frac{c}{t}}_{\times t} &= \underbrace{t^2x-ct^3}_{\times t} \qquad (\clubsuit)\\ ty-c &= t^3x-ct^4\\ t^3x-ty &= ct^4-c\\ \therefore t^3x-ty &= c\left(t^4-1\right)\end{aligned}$$

iii. (5 marks)

- Point H: Normal meets $y = x$. Replace y with x in ($\clubsuit$):

$$\begin{aligned}t^3x-tx &= c\left(t^4-1\right)\\ x(t)\cancel{\left(t^2-1\right)} &= c\cancel{\left(t^2-1\right)}\left(t^2+1\right)\\ \therefore x = y &= \frac{c\left(t^2+1\right)}{t} \qquad (\heartsuit)\end{aligned}$$

- Point F: when tangent meets x axis, $y = 0$. Use ($\spadesuit$):

$$\begin{aligned}0-\frac{c}{t} &= -\frac{1}{t^2}(x-ct)\\ ct &= x-ct\\ x &= 2ct\\ \therefore F&(2ct,0)\end{aligned}$$

- Point G: when tangent meets y axis, $x = 0$. Use ($\spadesuit$):

$$\begin{aligned}y-\frac{c}{t} &= -\frac{1}{t^2}(0-ct)\\ y &= \frac{c}{t}+\frac{c}{t} = \frac{2c}{t}\\ \therefore G&\left(0,\frac{2c}{t}\right)\end{aligned}$$

- Gradient of FH:

$$m_{FH} = \frac{y_h - y_f}{x_h - x_f} = \frac{\frac{c(t^2+1)}{t} - 0}{\frac{c(t^2+1)}{t} - 2ct}$$
$$= \frac{\frac{c(t^2+1)}{\not t}}{\frac{ct^2+c-2ct^2}{\not t}} = \frac{\not c\left(t^2+1\right)}{\not c\left(1-t^2\right)}$$
$$= \frac{1+t^2}{1-t^2}$$

- Gradient of GH:

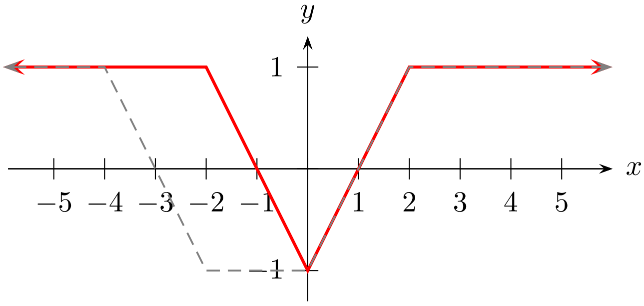
$$m_{GH} = \frac{y_h - y_g}{x_h - x_g} = \frac{\frac{c(t^2+1)}{t} - \frac{2c}{t}}{\frac{c(t^2+1)}{t} - 0}$$
$$= \frac{\frac{ct^2+c-2c}{\not t}}{\frac{ct^2+c}{\not t}} = \frac{\not c\left(t^2-1\right)}{\not c\left(1+t^2\right)}$$
$$= \frac{t^2-1}{1+t^2}$$

- Multiply gradients,

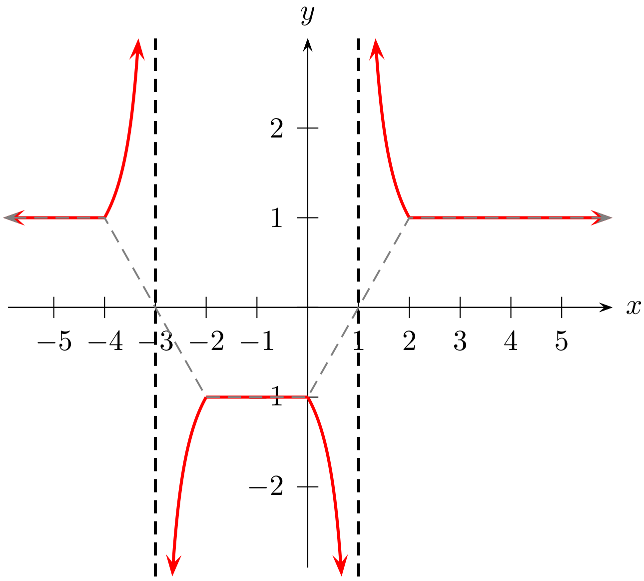
$$m_{GH} \times m_{GH} = \frac{t^2-1}{\cancel{t^2+1}} \times \frac{\cancel{t^2+1}}{1-t^2}$$
$$= -1$$
$$\therefore FH \perp GH$$

Question 14(Lam)

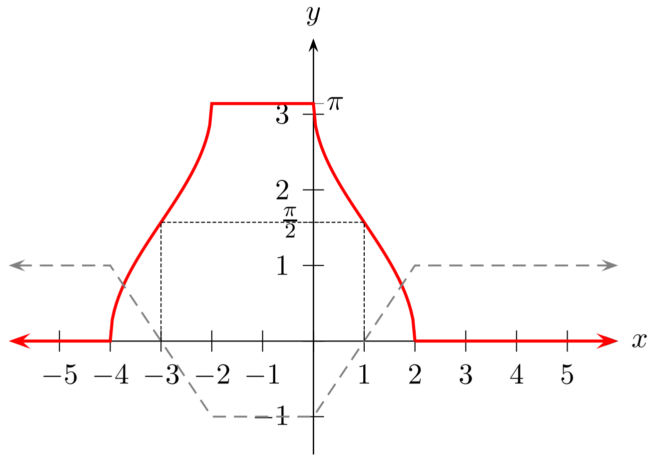
(a) i. (2 marks) – $y = f(|x|)$. (Old curve in gray dashes)



ii. (2 marks) – $y = \dfrac{1}{f(x)}$



iii. (2 marks)



(b) i. (2 marks)

$$\begin{aligned}\lceil 2.5 \rceil &= 3\\ (2.5 + \lceil 2.5 \rceil)^2 &= (2.5+3)^2\\ &= 5.5^2\\ &= \frac{121}{4}\end{aligned}$$

ii. (3 marks)
By cases,

- From $x = -4$ to $x = -3$, $\lceil x \rceil = -3$

$$\therefore y = -3 + (x-3)^2$$

- From $x = -3$ to $x = -2$, $\lceil x \rceil = -2$

$$\therefore y = -2 + (x-2)^2$$

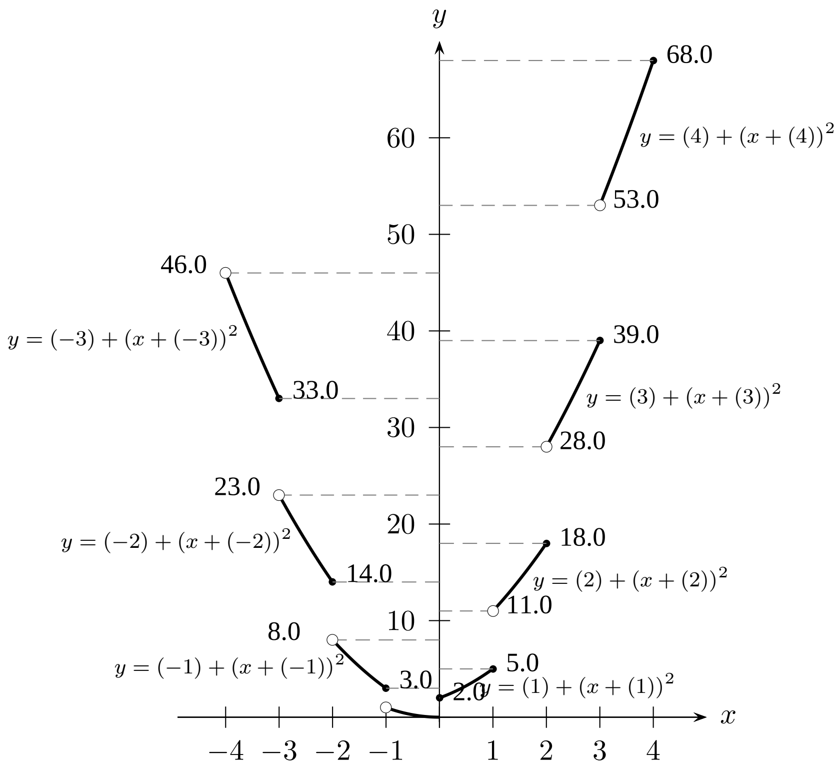
- From $x = -2$ to $x = -1$, $\lceil x \rceil = -1$

$$\therefore y = -1 + (x-1)^2$$

- From $x = -1$ to $x = 0$, $\lceil x \rceil = 0$

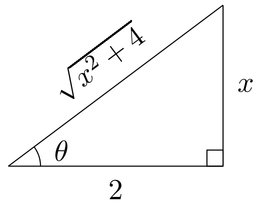
$$\therefore y = 0 + (x-0)^2$$

etc.



(c) (4 marks)

Letting $x = 2\tan\theta$,



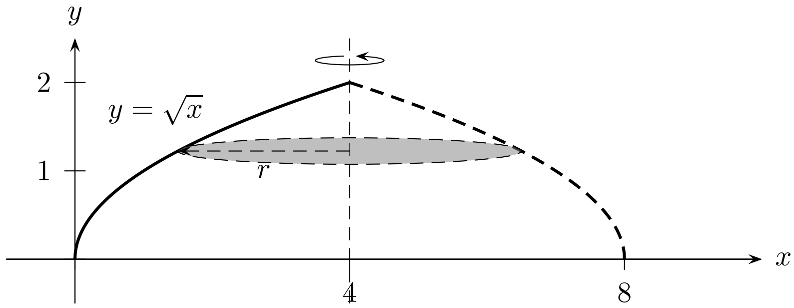
$$\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$$

$$\frac{dx}{d\theta} = 2\sec^2\theta\, d\theta$$

$$\begin{aligned}
\int \frac{dx}{(4+x^2)^{\frac{3}{2}}} &= \int \frac{2\sec^2\theta\, d\theta}{(4+4\tan^2\theta)^{\frac{3}{2}}} \\
&= \int \frac{2\sec^2\theta\, d\theta}{(4\left(1+\tan^2\theta\right))^{\frac{3}{2}}} \\
&= \int \frac{2\sec^2\theta\, d\theta}{8\sec^3\theta} \\
&= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta \\
&= \frac{1}{4}\int \cos\theta\, d\theta \\
&= \frac{1}{4}\sin\theta + C \\
&= \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C
\end{aligned}$$

Question 15(Ziaziaris)

(a) i. (2 marks)



Area of disc:

$$\begin{aligned} r &= (4 - x) \\ A &= \pi r^2 \\ &= \pi (4 - x)^2 \end{aligned}$$

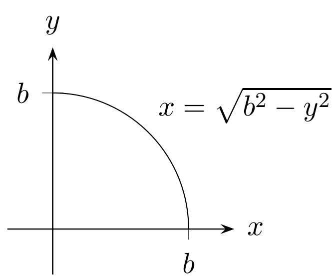
Variable sized discs run from $y = 0$ to $y = 2$, i.e. use δy for thickness

$$\begin{aligned} \therefore \delta V &= A \times \delta y \\ &= \pi \left(4 - y^2\right)^2 \, \delta y \\ &= \pi \left(16 - 8y^2 + y^4\right) \, \delta y \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} V &= \pi \lim_{\delta y \to 0} \sum_{y=0}^{y=2} \left(16 - 8y^2 + y^4\right) \, \delta y \\ &= \pi \int_0^2 \left(16 - 8y^2 + y^4\right) \, dy \\ &= \pi \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 \\ &= \pi \left(16(2) - \frac{8}{3}(8) + \frac{1}{5}(32)\right) \\ &= \frac{256}{15}\pi \end{aligned}$$

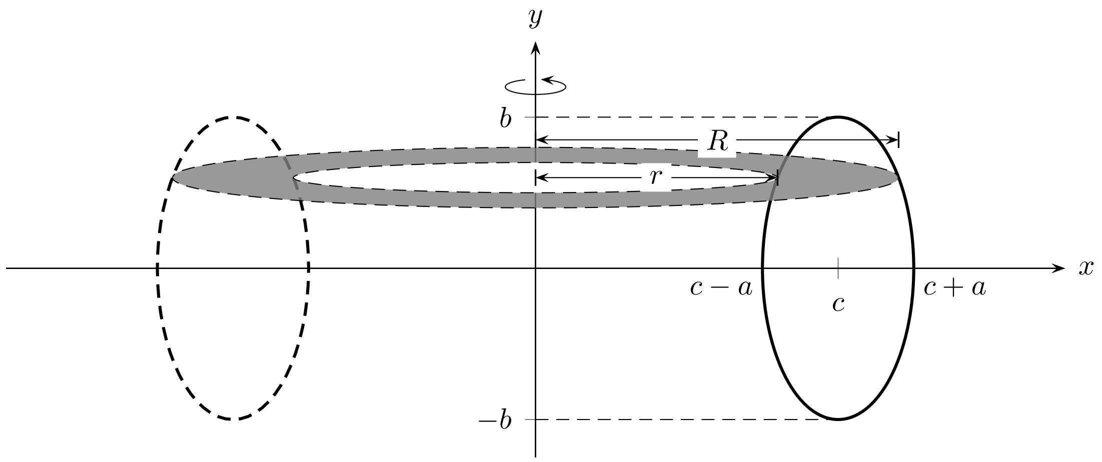
(b) i. (2 marks)



- Curve $x = \sqrt{b^2 - y^2}$ is the top half of the sideways semicircle with radius b.
- Hence $\int_0^1 \sqrt{b^2 - y^2}\, dy$ is the area of the quarter circle:

$$\begin{aligned} A &= \int_0^b \sqrt{b^2 - y^2}\, dy \\ &= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi b^2 \end{aligned}$$

ii. (4 marks)



Ellipse equation:

$$\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=1$$
$$(x-c)^2=a^2\left(1-\frac{y^2}{b^2}\right)$$
$$x-c=\pm a\sqrt{1-\frac{y^2}{b^2}}$$
$$x=c\pm a\sqrt{1-\frac{y^2}{b^2}}$$

- Inner radius r:

$$r=c-a\sqrt{1-\frac{y^2}{b^2}}$$

- Outer radius R:

$$R=c+a\sqrt{1-\frac{y^2}{b^2}}$$

Area of annulus:

$$\begin{aligned}A&=\pi\left(R^2-r^2\right)\\&=\pi\left(R-r\right)\left(R+r\right)\\&=\pi\left(2a\sqrt{1-\frac{y^2}{b^2}}\right)(2c)\\&=4\pi ac\sqrt{\frac{1}{b^2}\left(b^2-y^2\right)}\\&=\frac{4\pi ac}{b}\sqrt{b^2-y^2}\end{aligned}$$

Volume element & volume generated:

$$\delta V=A\times\delta y=\frac{4\pi ac}{b}\sqrt{b^2-y^2}\,\delta y$$
$$\begin{aligned}V&=\frac{4\pi ac}{b}\lim_{\delta y\to 0}\sum_{y=0}^{y=b}2\times\sqrt{b^2-y^2}\,\delta y\\&=\frac{4\pi ac}{b}\times 2\times\int_0^b\sqrt{b^2-y^2}\,dy\\&=\frac{\cancel{4}\pi ac}{\cancel{b}}\times 2\times\frac{1}{\cancel{4}}\pi b^{\cancel{2}}\\&=2\pi^2 abc\end{aligned}$$

(c) i. (2 marks)

$$\int_0^a f(x)\,dx$$

Let $u = a - x$, then

$$\begin{aligned}\frac{du}{dx} &= -1\\ du &= -dx\\ x = 0 &\qquad u = a\\ x = a &\qquad u = 0\\ \int_0^a f(x)\,dx &= \int_{u=a}^{u=0} f(a-u)\,(-du)\\ &= \int_0^a f(a-u)\,du\\ &= \int_0^a f(a-x)\,dx\end{aligned}$$

ii. (3 marks)

$$I = \int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos}}\,dx \left(\equiv \int_0^a f(x)\,dx\right) \qquad (\bigstar)$$

By using the result from above,

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{e^{\sin\left(\frac{\pi}{2}-x\right)}}{e^{\sin\left(\frac{\pi}{2}-x\right)} + e^{\cos\left(\frac{\pi}{2}-x\right)}}\,dx \quad = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x} + e^{\sin x}}\,dx \qquad (\blacktriangledown)$$

Adding $(\blacktriangledown)$ and $(\bigstar)$,

$$\begin{aligned}2I &= \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x} + e^{\sin x}}\,dx + \int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\cos x} + e^{\sin x}}\,dx\\ &= \int_0^{\frac{\pi}{2}} \frac{e^{\sin x} + e^{\cos x}}{e^{\sin x} + e^{\cos x}}\,dx\\ &= \int_0^{\frac{\pi}{2}} 1\,dx\\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\therefore 2I &= \frac{\pi}{2}\\ I &= \frac{\pi}{4}\end{aligned}$$

Question 16(Ireland)

(a) i. (1 mark)

$$x^3 - 4x^2 + 5x + 2 = 0$$

$$\begin{aligned} & (\alpha+\beta+\gamma)^2 \\ &= (\alpha+\beta+\gamma)(\alpha+\beta+\gamma) \\ &= \alpha^2 + \alpha\beta + \alpha\gamma \\ &\quad + \beta^2 + \alpha\beta + \beta\gamma \\ &\quad + \gamma^2 + \alpha\gamma + \beta\gamma \end{aligned}$$

$$\begin{aligned} \therefore\ & \alpha^2+\beta^2+\gamma^2 \\ &= (\alpha+\beta+\gamma)^2 \\ &\qquad - 2(\alpha\beta+\alpha\gamma+\beta\gamma) \\ &= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) \\ &= (4^2) - 2(5) = 6 \end{aligned}$$

ii. (2 marks)
If α, β and γ are roots, then they satisfy cubic equation

$$\begin{aligned} \alpha^3 - 4\alpha^2 + 5\alpha + 2 &= 0 \\ \beta^3 - 4\beta^2 + 5\beta + 2 &= 0 \\ \gamma^3 - 4\gamma^2 + 5\gamma + 2 &= 0 \end{aligned}$$

Adding equations, and subtracting to other side,

$$\begin{aligned} \alpha^3+\beta^3+\gamma^3 &= 4\left(\alpha^2+\beta^2+\gamma^2\right) \\ &\quad - 5(\alpha+\beta+\gamma) \\ &\quad - 6 \\ &= 4(6) - 5(4) - 6 \\ &= -2 \end{aligned}$$

(b) i. (3 marks)

$$P(x) = \left(x^2 - a^2\right)Q(x) + (px + q)$$

Evaluating at $x = a$,

$$P(a) = 0 + pa + q = ap + q \quad (\blacksquare)$$

Evaluating at $x = -a$,

$$P(-a) = 0 - pa + q = -ap + q \quad (\blacktriangle)$$

Adding ($\blacksquare$) and ($\blacktriangle$),

$$\begin{aligned} P(a) + P(-a) &= 2q \\ \therefore q &= \frac{1}{2}\left[P(a) + P(-a)\right] \end{aligned}$$

Subtracting ($\blacksquare$) and ($\blacktriangle$),

$$\begin{aligned} P(a) - P(-a) &= 2ap \\ \therefore p &= \frac{1}{2a}\left[P(a) - P(-a)\right] \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} P(x) &= x^n - a^n \\ &= \left(x^2 - a^2\right)Q(x) + (px + q) \end{aligned}$$

Finding p:

$$\begin{aligned} P(a) &= a^n - a^n = 0 \\ P(-a) &= (-a)^n - a^n \\ &= (-1)^n a^n - a^n \end{aligned}$$

$$\begin{aligned} p &= \frac{1}{2a}\left[P(a) - P(-a)\right] \\ &= \frac{1}{2a}\left[0 - ((-1)^n a^n - a^n)\right] \\ &= \frac{1}{2a}\left[a^n - (-1)^n a^n\right] \end{aligned}$$

When n is even, $(-1)^n = 1$,

$$p = 0$$

When n is odd, $(-1)^n = -1$,

$$\begin{aligned} p &= \frac{1}{2a}\left[a^n - (-)a^n\right] \\ &= \frac{1}{2a}\left(+2a^n\right) \\ &= a^{n-1} \end{aligned}$$

Finding q,

$$\begin{aligned} q &= \frac{1}{2}\left[P(a) + P(-a)\right] \\ &= \frac{1}{2}\left[0 + (-1)^n a^n - a^n\right] \\ &= \frac{1}{2}\left[(-1)^n a^n - a^n\right] \end{aligned}$$

When n is even, $(-1)^n = 1$,

$$q = \frac{1}{2} \times 0 = 0$$

When n is odd, $(-1)^n = -1$,

$$q = \frac{1}{2}\left[-2a^n\right] = -a^n$$

Hence when n is even, the remainder is zero, whilst when n is odd, the remainder is

$R(x) = a^{n-1}x - a^n = a^{n-1}(x-a)$

(c) i. (2 marks)

$$\begin{cases} y = \frac{c^2}{x} & (1) \\ (x-1)^2 + y^2 = 1 & (2) \end{cases}$$

Solve simultaneously by substituting (1) to (2):

$$\underset{\times x^2}{(x-1)^2} + \underset{\times x^2}{\left(\frac{c^4}{x^2}\right)} = \underset{\times x^2}{1}$$

$$x^2(x-1)^2 + c^4 = x^2$$

Since the curves *touch* at Q, the x coordinate of (β) of Q is a repeated real root of the equation. As there are no further intersections, the equation has no other real roots. Hence the remaining two roots are non-real complex conjugate roots, as the equation has real coefficients.

ii. (3 marks)

- If $x = \beta$ is a double root of $P(x) = 0$, then $x = \beta$ is also a root of $P'(x) = 0$:

$$\begin{aligned} P(x) &= x^2(x-1)^2 - x^2 + c^4 = 0 \\ &= x^2\left(x^2 - 2x + 1\right) - x^2 + c^4 \\ &= x^4 - 2x^3 + c^4 \\ P'(x) &= 4x^3 - 6x^2 \\ &= 2x^2\left(2x - 3\right) \end{aligned}$$

As $P(\beta) = 0$, then $P'(\beta) = 0$ and $\beta \neq 0$

$$\therefore 2x - 3 = 0$$

$$x = \beta = \frac{3}{2}$$

Finding c^2: substitute $x = \frac{3}{2}$ into quartic

$$\begin{aligned} \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 + c^4 &= 0 \\ c^4 &= \frac{27}{16} \\ \therefore c^2 &= \frac{3\sqrt{3}}{4} \end{aligned}$$

iii. (2 marks)

Equation of common tangent: find $\frac{dy}{dx}$.

$$\begin{aligned} y &= \frac{c^2}{x} = c^2x^{-1} \\ \frac{dy}{dx} &= -c^2x^{-2}\Big|_{\substack{x=\frac{3}{2} \\ c^2=\frac{3\sqrt{3}}{4}}} \\ &= -\frac{3\sqrt{3}}{4} \times \frac{4}{9} \\ &= -\frac{\sqrt{3}}{3} = -\frac{1}{\sqrt{3}} \end{aligned}$$

At the point of contact, $x = \frac{3}{2}$:

$$\begin{aligned} y^2 &= 1 - (x-1)^2 \\ y &= \sqrt{1 - \left(\frac{3}{2} - 1\right)^2} = \frac{\sqrt{3}}{2} \end{aligned}$$

Apply point-gradient formula,

$$\begin{aligned} y - \frac{\sqrt{3}}{2} &= -\frac{1}{\sqrt{3}}\left(x - \frac{3}{2}\right) \\ y &= -\frac{1}{\sqrt{3}}x + \frac{\overset{\sqrt{3}}{\cancel{3}}}{2\sqrt{3}} + \frac{\sqrt{3}}{2} \\ y &= -\frac{1}{\sqrt{3}}x + \sqrt{3} \end{aligned}$$