Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

VINCIT QUI SE VINCIT

2016 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- For Section I, shade the correct box on the sheet provided
- For Section II, write in the booklet provided
- Each new question is to be started on a **new page.**
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- ○ Mr Ireland
- ○ Mr Lin
- ○ Dr Jomaa

Student Number

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	15	16	Total	Total
Mark	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$	$\overline{100}$

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10 located at the end of this section.

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1 If $z = 2 - 5i$ find z^{-1} expressed with a real denominator

(A) $\frac{1}{24}(2 + 5i)$

(B) $\frac{1}{29}(2 + 5i)$

(C) $\frac{1}{29}(2 - 5i)$

(D) $\frac{1}{24}(-2 + 5i)$

2 If the line $y = mx + b$ is a tangent to the hyperbola $xy = c^2$, which of the following is true?

(A) $b^2 = -4mc^2$

(B) $b^2 = 4mc^2$

(C) $b = 4mc$

(D) $c^2 = 4mb$

3 For a certain function $= f(x)$, the function $f(|x|)$ is represented by:

(A) A reflection of $y = f(x)$ in the y axis

(B) A reflection of $y = f(x)$ in the x axis

(C) A reflection of $y = f(x)$ in the x axis for $y \geq 0$

(D) A reflection of $y = f(x)$ in the y axis for $x \geq 0$

4 A hyperbola has equation $x^2 - 4y^2 = 4$. The distance between its directrices is:

(A) $\sqrt{5}$

(B) $\dfrac{4\sqrt{5}}{5}$

(C) $2\sqrt{5}$

(D) $\dfrac{8\sqrt{5}}{5}$

5 If $e^x + e^y = 1$, $\frac{dy}{dx} =$

(A) $-e^{x-y}$

(B) e^{y-x}

(C) e^{x-y}

(D) $-e^{y-x}$

6 The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and

$P(2i) = P(2+i) = 0$ What is $a + b + c + d$?

(A) 0

(B) 1

(C) 4

(D) 9

7 The solutions to the equation $x^4 - 10x^2 + 5 = 0$ are

$$x = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$$

What is the value of: $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} + \tan^2\frac{3\pi}{5} + \tan^2\frac{4\pi}{5}$?

(A) 20

(B) 5

(C) -20

(D) 10

8 Which expression is equal to $\int \frac{1}{1-\sin x}\,dx$

(A) $\tan x - \sec x + c$

(B) $\tan x + \sec x + c$

(C) $\log_e(1 - \sin x) + c$

(D) $\frac{\log_e(1 - \sin x)}{-\cos x} + c$

9 Let ω be the complex root of unity such that $\omega^n = 1$, $\omega \neq 1$

Find the value of $\sum_{k=0}^{n}(w^k + \frac{1}{w^k})$

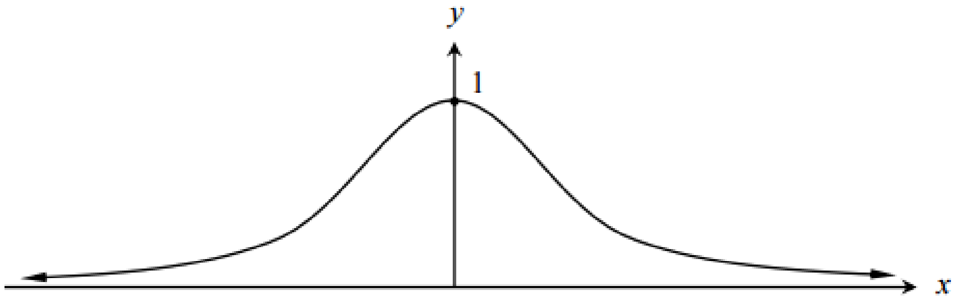
(A) 0

(B) 1

(C) 2

(D) 3

10 Which of the following equations best describe this curve?



(A) $y = e^{x^2}$

(B) $y = \dfrac{1}{(x+1)^2}$

(C) $y = \dfrac{1}{x^2+1}$

(D) $y = e^{-x^2-x}$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question on a NEW page. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page in your writing booklet.

(a) Find $\displaystyle\int \frac{(\ln x)^2}{x}\,dx.$ **2**

(b) Find $\displaystyle\int \sin^3 x \cos^2 x\,dx$ **2**

(c) Find $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$ **2**

(d) By using the substitution of $x = 2\tan\theta$, evaluate $\displaystyle\int_0^2 \frac{dx}{(4+x^2)^{\frac{3}{2}}}$ **2**

(e) (i) Express $\dfrac{7x}{(x^2+3)(x+2)}$ in the form of $\dfrac{Ax+B}{x^2+3}+\dfrac{C}{x+2}$ **2**

(ii) Hence evaluate $\displaystyle\int_0^3 \frac{7x}{(x^2+3)(x+2)}\,dx$ **2**

(f) The area bounded by the curve $y = 2\cos x$ for $0 \le x \le 2\pi$ and the line $y = 2$ is rotated about the y axis. Find the volume of the solid formed using the method of cylindrical shells. **3**

Question 12 (15 marks) Start a NEW page in your writing booklet.

(a) For the hyperbola $\frac{y^2}{25} - \frac{x^2}{16} = 1$, find

(i) the eccentricity **1**

(ii) the coordinates of the foci S and S' and the equations of its directrices **2**

(iii) Sketch the hyperbola showing all the above features. **1**

(b) Explain why the equation $\frac{x^2}{\lambda - 23} + \frac{y^2}{5 - \lambda} = 1$ cannot represent the equation of an ellipse. **1**

(c) Let $f(x) = x^2(x - \frac{3}{2})$ be a function on the domain $-1 \leq x \leq 2$.

(i) Draw a neat sketch of $y = f(x)$, labelling all intersections with the coordinate axes and turning points. You are not required to test the nature of the turning points. **2**

(ii) Sketch $y = \frac{1}{f(x)}$. **2**

(iii) Sketch $y^2 = f(x)$ **2**

(d) Find the coordinates of the stationary points for the curve $x^2 - 4xy = y^2 - 20$ **4**

Question 13 (15 marks) Start a NEW page in your writing booklet.

(a) Shade the region defined by the intersection of $|\arg z| \leq \frac{\pi}{3}$, $z + \bar{z} \leq 4$ and $|z| \geq 2$ **3**

(b) If $z = \sqrt{3} + i$

(i) Find the exact value of |z| and $\arg z$ **2**

(ii) By using De Moivre's theorem write $\frac{1}{z^5}$ in the form of $x + iy$ **2**

(c) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$.

Factorise $P(x)$ over the field of:

(i) real numbers, **2**

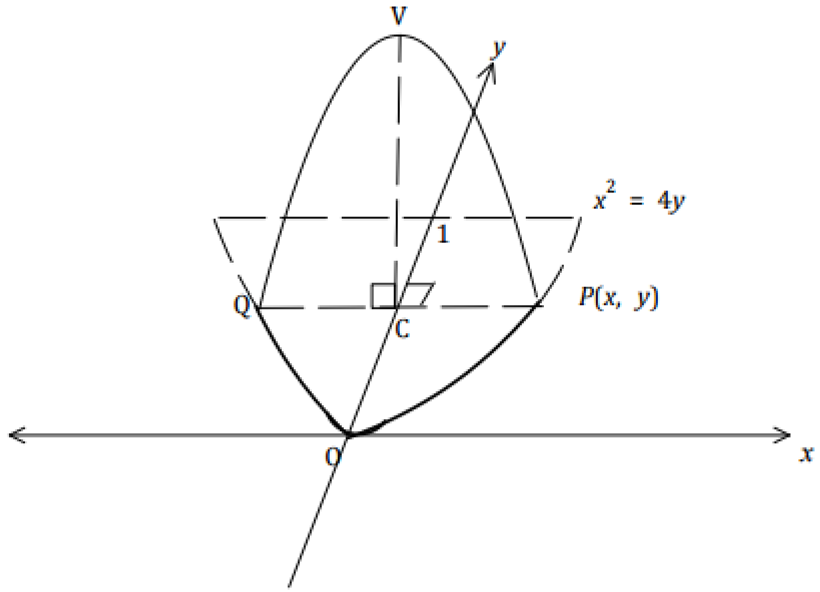
(ii) complex numbers. **1**

(d) Find the complex square roots of $7 + 6i\sqrt{2}$, giving your answer in the form of $a + ib$, where a and b are real. **2**

(e) R is a positive number and z_1, z_2 are complex numbers. Show that the points on the Argand diagram which represent respectively the numbers z_1, z_2, $\frac{z_1 - iRz_2}{1 - iR}$ form the vertices of a right angled triangle. **3**

Question 14 (15 marks) Start a NEW page in your writing booklet.

(a)



The base of the solid is the region bounded by the parabola $x^2 = 4y$ and the line $y = 1$.

Cross sections perpendicular to this base and the y axis are parabolic segments with their vertices V directly above the y axis. The diagram shows a typical segment PVQ. All the segments have the property that the vertical height VC is three times the base length PQ.

Let $P(x, y)$ where $x \geq 0$ be a point on the parabola $x^2 = 4y$

(i) Show that the area of the segment PVQ is $8x^2$. **2**

(ii) Hence, find the volume of the solid. **2**

Question 14 continues on Page 11

Question 14 (continued)

(b) let $z = \cos\theta + i\sin\theta$ be any complex number of modulus 1.

(i) Show that $\dfrac{z^2 - 1}{z} = 2i\sin\theta$ **2**

(ii) Hence, prove that **2**

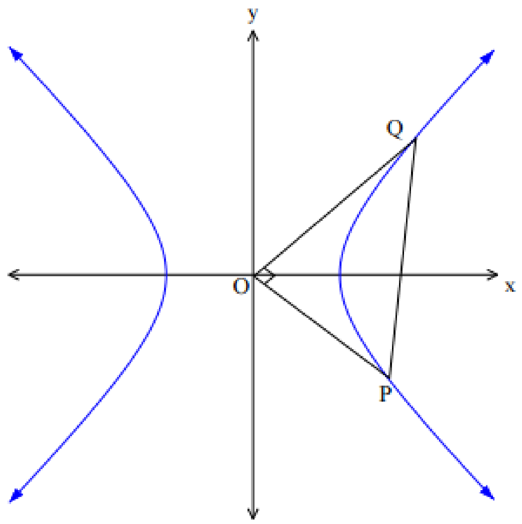
$$z + z^3 + z^5 + z^7 + z^9 = \frac{\sin 10\theta + i(1 - \cos 10\theta)}{2\sin\theta}$$

(iii) Hence write down a simplified expression for **1**

$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + cos9\theta$

(c) The diagram shows the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.

The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\alpha, b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



(i) Show that $\sin\theta\sin\alpha = -\dfrac{a^2}{b^2}$. **3**

(ii) Hence show that the gradient of the curve at $P(a\sec\theta, b\tan\theta)$ is **3**

$$\frac{dy}{dx} = -\frac{b^3}{a^3}\sin\alpha$$

Question 15 (15 marks) Start a NEW page in your writing booklet.

(a) Let $I_n = \int_0^1 x^n \ln(1+x)dx \, , n = 0, 1, 2 \ldots$

(i) Show that $\int \ln(1+x)dx = (1+x)\ln(1+x) - x + c$ **1**

(ii) Show that $(n+1)I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1} \, , n = 1, 2, \ldots$ **2**

(iii) Evaluate $3I_2$ and $4I_3$ **2**

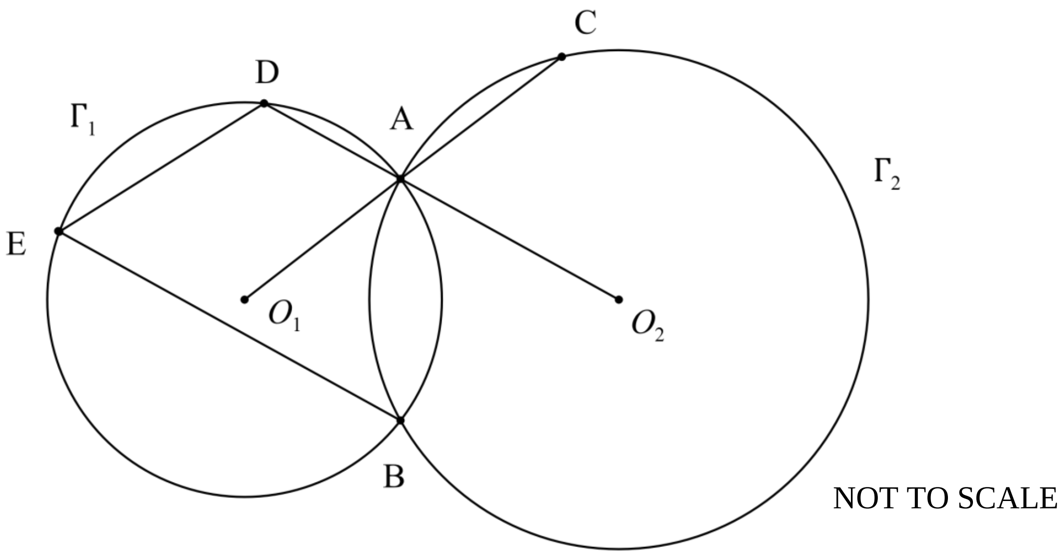
(iv) Show that $(n+1)I_n = \begin{cases} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n+1}, & \text{for } n \text{ is odd} \\ 2\ln 2 - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n+1}\right), & \text{for } n \text{ is even} \end{cases}$ **2**

(b) Given $S_n = \sum_{k=1}^{n} k^2$. Prove by mathematical induction that:

$$nS_n - \sum_{r=1}^{n-1} S_r = \sum_{r=1}^{n} r^3 \text{ for n} > 1$$

Question 15 continued on Page 13

(c)



In the figure, the two circles Γ_1 and Γ_2 have centres O_1 and O_2 respectively. They intersect at the points A and B. The extensions of O_1A meets Γ_2 at C and the extension of O_2A meets Γ_1 at D.

Given that $BE \parallel O_2A$ and $ED \parallel O_1A$. Let $\angle BED = \alpha$

(i) Copy or trace the diagram into your answer booklet.

(ii) Prove that O_1DCO_2 is a cyclic quadrilateral **2**

(iii) Hence, prove that $DC \perp CO_2$ **2**

Question 16 (15 marks) Start a NEW page in your writing booklet.

(a) (i) Given that $\dfrac{a+b}{2} \geq \sqrt{ab}$ **1**

Prove that $\dfrac{\mathrm{a+b+c+d}}{4} \geq \sqrt[4]{abcd}$

(ii) Using part (i) and the fact that $\dfrac{a+b+c}{3} = \dfrac{1}{4}\left(\mathrm{a+b+c}+\dfrac{a+b+c}{3}\right)$, **2**
prove that $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

(b) (i) Prove that for all positive values of x, $e^x > 1 + x$ **3**

(ii) If $x_1, x_2, \dots, x_n$ are positive numbers with $S_n = x_1 + x_2 + \dots + x_n$ and **2**

$R_n = (1 + x_1)(1 + x_2) \dots (1 + x_n)$, show that $e^{S_n} > R_n > 1 + S_n$

(iii) Let $b_k = \dfrac{k^2+k+1}{k^2+k}$ and $P_n = b_1 \times b_2 \times \dots \times b_n$ **3**

Show that $P_n < e$ for any positive whole numbers n

(c) Given a polynomial $P(x) = x^3 - ax^2 + bx - a$ where a and b are positive real numbers. Let a be the smallest positive real number such that all the roots of the polynomial are positive and real.

(i) Explain why all three roots of the polynomial are positive real numbers. **1**

(ii) Using the result from part (a) (ii), find the value of a and b **3**

End of Examination

Multiple Choice

1. B   2. A   3. D   4. D   5. A   6. D   7. A

8. B   9. C   10. C

Question 11

a) $I = \int (\ln x)^2 \cdot \frac{1}{x}\, dx$

By using the reverse chain rule

$I = \frac{(\ln x)^3}{3} + C$

b) $I = \int \sin^3 x \cos^2 x \, dx$

$= \int \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$

$= \int \sin x \cos^2 x - \sin x \cos^4 x \, dx$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$

c) $\int \frac{dx}{\sqrt{3 - (2x + x^2)}} = \int \frac{dx}{\sqrt{4 - (x+1)^2}}$

$= \sin^{-1}\left(\frac{x+1}{2}\right) + C.$

d) let $x = 2\tan\theta$

when $x = 0$ $\quad \theta = 0$

when $x = 2$ $\quad \theta = \frac{\pi}{4}$

$dx = 2\sec^2\theta \, d\theta$

$\therefore \quad I = \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{(4 + 4\tan^2\theta)^{\frac{3}{2}}} \, d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos\theta \, d\theta$

$= \frac{1}{4}\Big[\sin\theta\Big]_0^{\frac{\pi}{4}}$

$= \frac{\sqrt{2}}{8}$

e) $\frac{7x}{(x^2+3)(x+2)} \equiv \frac{Ax+B}{x^2+3} + \frac{C}{x+2}$

$(Ax+B)(x+2) + C(x^2+3) \equiv 7x$

when $x = -2$

$C = -2$

$A + C = 0 \Rightarrow A = 2$

when $x = 0$

$2B + 3C = 0$

$B = 3$

$\frac{7x}{(x^2+3)(x+2)} \equiv \frac{2x+3}{x^2+3} - \frac{2}{x+2}$

(ii) $\displaystyle\int_0^3 \frac{7x}{(x^2+3)(x+2)}\,dx = \int_0^3 \frac{2x+3}{x^2+3} - \frac{2}{x+2}\,dx$

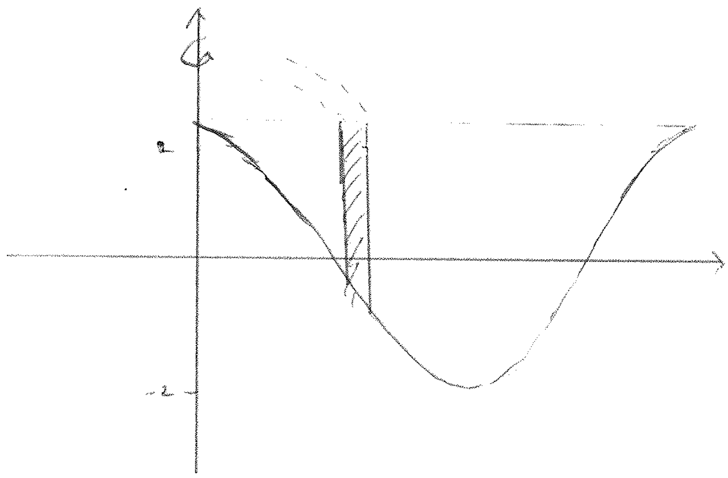
$\displaystyle = \int_0^3 \frac{2x}{x^2+3} + \frac{3}{x^2+3} - \frac{2}{x+2}\,dx$

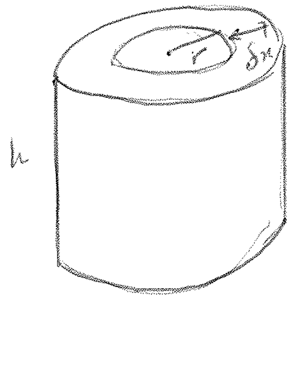
$\displaystyle = \left[\log(x^2+3) + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\log|x+2|\right]_0^3$

$\displaystyle = \log 12 + \frac{\pi}{\sqrt{3}} - \log 25 - \log 3 + \log 4$

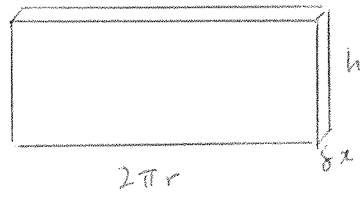
$\displaystyle = \log\left(\frac{16}{25}\right) + \frac{\pi}{\sqrt{3}}.$

f)





This volume is equivalent to



$\delta V = 2\pi r h\,\delta x$

$= 2\pi x \cdot (2 - 2\cos x)\,\delta x$

$= 4\pi x(1-\cos x)\,\delta x$

$\displaystyle \therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{2\pi} 4\pi x(1-\cos x)$

$\displaystyle = 4\pi \int_0^{2\pi} x(1-\cos x)\,dx$

$\displaystyle = 4\pi\left[\big[x(x-\sin x)\big]_0^{2\pi} - \int_0^{2\pi} x - \sin x\,dx\right]$

$\displaystyle = 4\pi\left[4\pi^2 - \left[\frac{x^2}{2} + \cos x\right]_0^{2\pi}\right]$

$= 4\pi\left[4\pi^2 - 2\pi^2\right]$

$= 8\pi^3 \ u^3$

Question 12

a)(i) $a^2 = b^2(e^2 - 1)$

$16 = 25(e^2 - 1)$

$\frac{16}{25} = e^2 - 1$

$e^2 = \frac{41}{25}$

$e = \frac{\sqrt{41}}{5}$

(ii) Foci

$(0, \pm 5 \times \frac{\sqrt{41}}{5})$

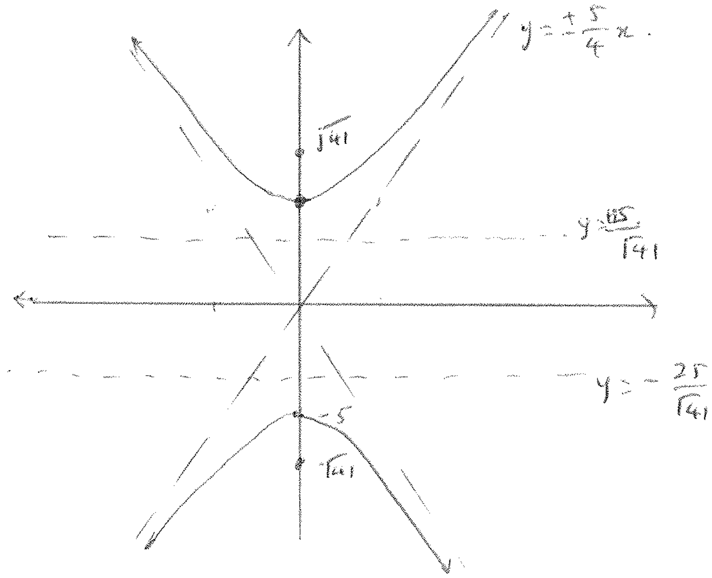
$= (0, \pm\sqrt{41})$

Directrices

$y = \pm \frac{5}{\frac{\sqrt{41}}{5}}$

$= \pm \frac{25}{\sqrt{41}}$

(iii) asymptotes are $y = \pm\frac{5}{4}x$. vertices are $(0,5)$, $(0,-5)$



b) For an ellipse $\lambda - 23 > 0$ $\quad 5 - \lambda > 0$

$\lambda > 23$ $\quad 5 > \lambda$.

There are no values of λ that will satisfy this condition

$\therefore \frac{x^2}{\lambda - 23} + \frac{y^2}{5 - \lambda} = 1$ cannot represent the equation of an ellipse.

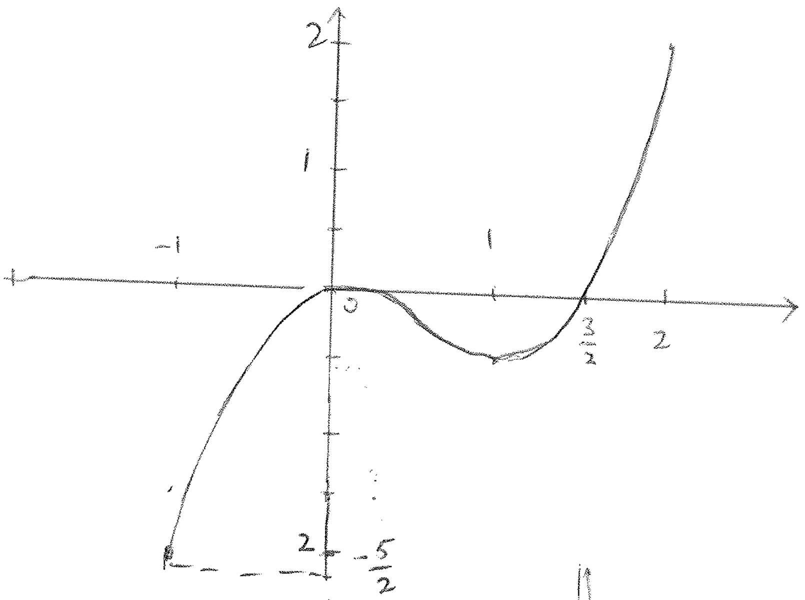
c) (i) $f(x) = x^2(x - \frac{3}{2}) \quad -1 \leq x \leq 2$

$= x^3 - \frac{3x^2}{2}$

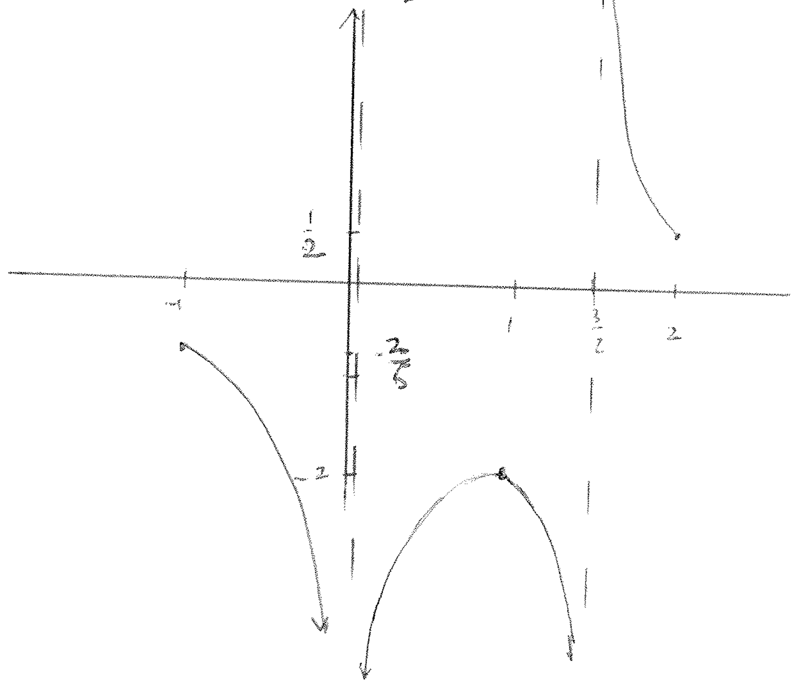
$f'(x) = 3x^2 - 3x$

$= 3x(x-1)$

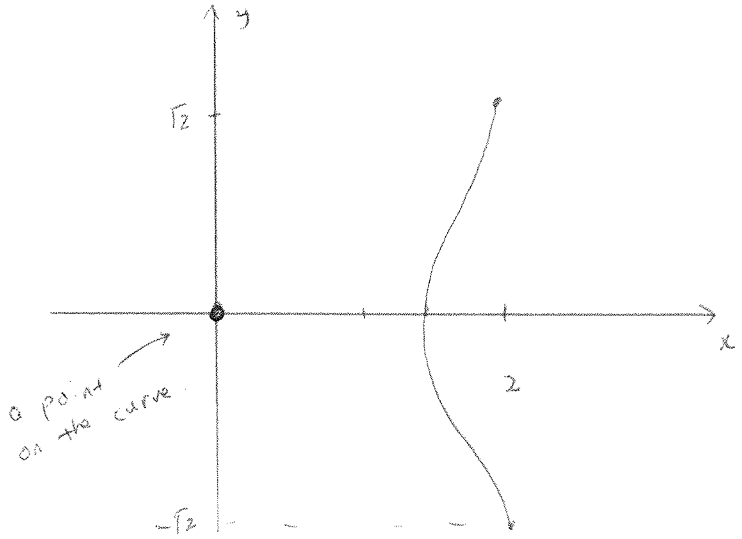
$\therefore$ stationary points are at $(0,0)$ and $(1, -\frac{1}{2})$



(ii)



iii)



d) $2x - \left(4x\frac{dy}{dx} + 4y\right) = 2y\frac{dy}{dx}$

$2x - 4x\frac{dy}{dx} - 4y = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x-4y}{2y+4x}$

$= \frac{x-2y}{y+2x}$

stationary points when $x - 2y = 0$

i.e. $x = 2y$

$\therefore \quad 4y^2 - 8y^2 = y^2 - 20$

$5y^2 = 20$

$y^2 = 4$

$y = \pm 2$

$\therefore$ when $y = 2,\ x = 4$

$y = -2,\ x = -4$

$\therefore$ stationary points are

$(4, 2)$, $(-4, -2)$

Question 13

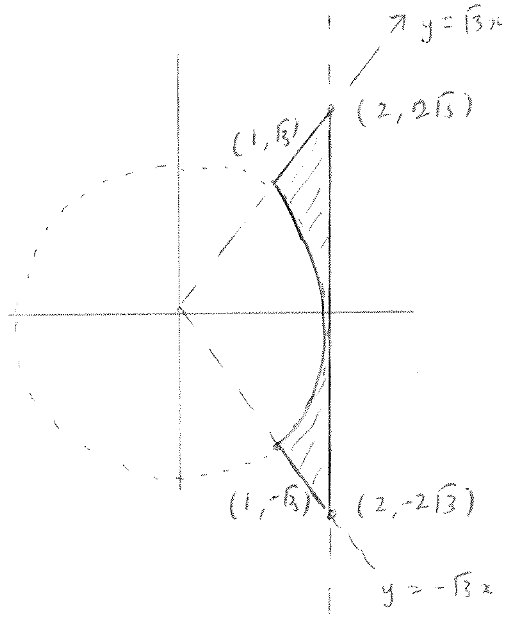
a) $z + \frac{1}{z} \leq 4$

$\therefore \quad 2x \leq 4$

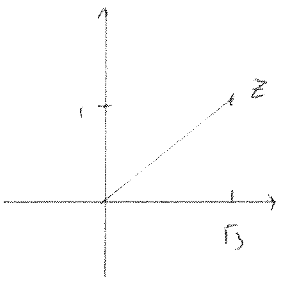
$x \leq 2$

$|\arg z| \leq \frac{\pi}{3}$

$\therefore \quad -\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$



b)



(i) $|z| = \sqrt{3+1}$

$= 2$

$\arg z = \frac{\pi}{6}$

(ii) $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\therefore \quad z^{-5} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{-5}$

By De Moirve's Thm

$z^{-5} = \frac{1}{32}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

$= \frac{1}{32}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$

$= -\frac{\sqrt{3}}{64} - \frac{1}{64}i$

(c) i) $P(x) = x^4(x+1) + 13x^2(x+1) - 48(x+1)$

$= (x+1)(x^4 + 13x^2 - 48)$

$= (x+1)(x^2-3)(x^2+16)$

$= (x+1)(x-\sqrt{3})(x+\sqrt{3})(x^2+16)$

ii) $P(x) = (x+1)(x-\sqrt{3})(x+\sqrt{3})(x-4i)(x+4i)$

d) $z^2 = 7+6\sqrt{2}i$

let $z = a+ib$

$\therefore (a+ib)^2 = a^2-b^2+2abi$

$\therefore a^2-b^2 = 7$

$2ab = 6\sqrt{2}$

$ab = 3\sqrt{2}$

$a^2+b^2 = 11$

$\therefore 2a^2 = 18$

$a^2 = 9$

$a = \pm 3 \quad \therefore b = \pm\sqrt{2}$

$\therefore z = \pm(3+\sqrt{2}i)$

e) $\arg\left(z_1 - \left(\frac{z_1 - Riz_2}{1-Ri}\right)\right)$ and $\arg\left(z_2 - \left(\frac{z_1 - Riz_2}{1-Ri}\right)\right)$ where $z_3 = \frac{z_1 - Riz_2}{1-Ri}$

$= \arg\left(\frac{Ri(z_2-z_1)}{1-Ri}\right)$ $\qquad = \arg\left(\frac{z_2-z_1}{1-Ri}\right)$

Now $\arg(z_1-z_3) - \arg(z_2-z_3)$

$= \arg\left(\frac{Ri(z_2-z_1)}{1-Ri}\right) - \arg\left(\frac{z_2-z_1}{1-Ri}\right)$

$= \arg\left(\frac{Ri(z_2-z_1)}{z_2-z_1}\right)$

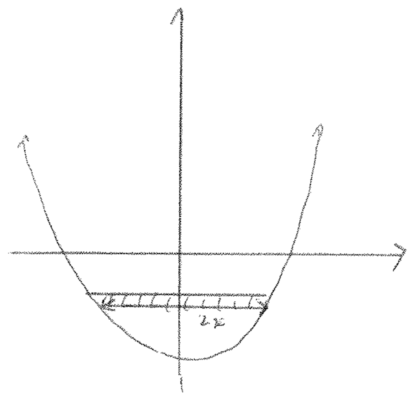
$= \arg(Ri)$

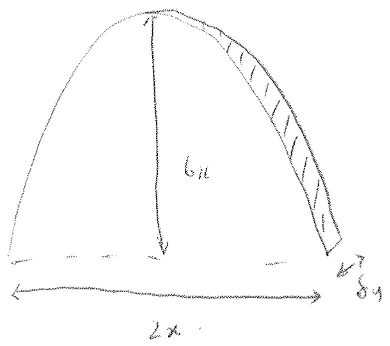
$= \frac{\pi}{2}$ as $R>0$.

$\therefore z_1, z_2, z_3$ form a right angle triangle.

Question 14

(i)





using simpson's rule

$$A(x) = \frac{x}{3}(0 + 4 \times 6x + 0)$$
$$= \frac{24x^2}{3}$$
$$= 8x^2$$

(ii)

$$\delta V = 8x^2 \delta y$$
$$= 8(4y)\delta y$$

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 32y \, \delta y$$
$$= 32\int_0^1 y \, dy$$
$$= 32\left[\frac{y^2}{2}\right]_0^1$$
$$= 16 \text{ units}^3$$

14 b) (i) $\frac{z^2-1}{z} = z - \frac{1}{z}$ let $z = \cos\theta + i\sin\theta$ as modulus is 1

$= (\cos\theta + i\sin\theta) - (\cos\theta + i\sin\theta)^{-1}$

$= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))$ By De Moivre's Thm

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

(ii) $z + z^3 + z^5 + z^7 + z^9$ is a geometric progression

where the common ratio $r = z^2$, first term is $a = z$

and $n = 5$

$\therefore \quad S_5 = \frac{z((z^2)^5 - 1)}{z^2 - 1}$

$= \frac{z}{z^2-1}(z^{10} - 1)$

$= \frac{1}{2i\sin\theta} \cdot (\cos 10\theta + i\sin 10\theta - 1)$ From part (i)

$= \frac{i\cos 10\theta - \sin 10\theta - i}{-2\sin\theta}$

$= \frac{\sin 10\theta + i(1 - \cos 10\theta)}{2\sin\theta}$

(iii) Hence $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta$ is the real part of the expression found in part (ii)

$\therefore \quad \cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{\sin 10\theta}{2\sin\theta}$.

c) (i) Gradient of chord OQ

$$m_{OQ} = \frac{b\tan\alpha}{a\sec\alpha} = \frac{b\sin\alpha}{\cos\alpha} \times \frac{\cos\alpha}{a}$$

$$= \frac{b\sin\alpha}{a}$$

Similarly gradient of chord OP

$$m_{OP} = \frac{b\sin\theta}{a}$$

Since $OQ \perp OP$

$$m_{OQ} \times m_{OP} = -1$$

$$\therefore \quad \frac{b\sin\alpha}{a} \times \frac{b\sin\theta}{a} = -1$$

$$\frac{b^2\sin\alpha\sin\theta}{a^2} = -1$$

$$\sin\alpha\sin\theta = \frac{-a^2}{b^2}$$

Differentiating implicitly

(ii) $$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Substituting the point P.

$$= \frac{b^2 a\sec\theta}{a^2 b\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

$$= \frac{b}{a\sin\theta}$$

Since $\sin\theta = -\dfrac{a^2}{b^2\sin\alpha}.$

$$\therefore \quad \frac{1}{\sin\theta} = \frac{-b^2\sin\alpha}{a^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{b}{a} \times \frac{-b^2\sin\alpha}{a^2}$$

$$= -\frac{b^3}{a^3}\sin\alpha.$$

Question 15

a) $I_n = \int_0^1 x^n \ln(1+x)\,dx \quad , n = 0, 1, 2.$

i) $I_0 = \int_0^1 1 \cdot \ln(1+x)\,dx$

$= x\ln(1+x) - \int \frac{x}{1+x}\,dx$

$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\,dx$

$= x\ln(1+x) - x + \ln(1+x) + C.$

$= (x+1)\ln(1+x) - x + C.$

ii) $I_n = \Big[x^n(1+x)\ln(1+x) - x\Big]_0^1 - n\int x^{n-1}\Big[(1+x)\ln(1+x) - x\Big]\,dx$

$= 2\ln 2 - 1 - n\int_0^1 (x^{n-1} + x^n)\ln(1+x) - x^n\,dx$

$= (2\ln 2 - 1) - n\int_0^1 x^{n-1}\ln(1+x) + x^n\ln(1+x) - x^n\,dx$

$= 2\ln 2 - 1 - n(I_{n-1} + I_n) + n\int x^n\,dx$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + n\left[\frac{x^{n+1}}{n+1}\right]_0^1$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}$

$I_n + nI_n = 2\ln 2 - 1 - 1 + \frac{n}{n+1} - nI_{n-1}$

$\therefore \quad I_n(n+1) = 2\ln 2 + \frac{-n-1+n}{n+1} - nI_{n-1}$

$I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$

iii)

$$I_0 = \int_0^1 \ln(1+x)\,dx$$

$$= \Big[(1+x)\ln(1+x) - x\Big]_0^1$$

$$= (2\ln 2 - 1)$$

$$2I_1 = 2\ln 2 - \frac{1}{2} - I_0$$

$$= 2\ln 2 - \frac{1}{2} - (2\ln 2 - 1)$$

$$= 1 - \frac{1}{2}.$$

$$3I_2 = 2\ln 2 - \frac{1}{3} - 2I_1$$

$$= 2\ln 2 - \frac{1}{3} - \left(1 - \frac{1}{2}\right)$$

$$= 2\ln 2 - 1 + \frac{1}{2} - \frac{1}{3}.$$

$$4I_3 = 2\ln 2 - \frac{1}{4} - 3I_2$$

$$= 2\ln 2 - \frac{1}{4} - \left(2\ln 2 - 1 + \frac{1}{2} - \frac{1}{3}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}.$$

iv)

$$(n+1)I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$$

$$= 2\ln 2 - \frac{1}{n+1} - \left[2\ln 2 - \frac{1}{n} - (n-1)I_{n-2}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n} + (n-1)I_{n-2}.$$

when n is odd the last term will be $2I_1$ i.e. $n=3$.

i.e. $2I_1 = 1 - \frac{1}{2}$

$\therefore$ no log term.

when n is even the last term will be I_0

where $I_0 = (2\ln 2 - 1)$

$$(n+1) I_n = \begin{cases} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{n+1}, & \text{for } n \text{ is odd} \\ 2\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n+1}\right) & \text{for } n \text{ is even} \end{cases}$$

b) Given $S_n = \sum_{k=1}^{n} k^2$

RTP $n S_n - \sum_{r=1}^{n-1} S_r = \sum_{r=1}^{n} r^3$ for $n > 1$

Base case when $n = 2$.

$$LHS = 2 S_2 - \sum_{r=1}^{1} S_r \qquad RHS = \sum_{r=1}^{2} r^3$$

$$= 2(1^2 + 2^2) - 1^2 \qquad = 1^3 + 2^3$$

$$= 9 \qquad = 1 + 8$$

$$= 9$$

$\therefore$ LHS = RHS $\therefore$ true for $n = 2$.

Assume true for $n = k$ i.e. $k \in \mathbb{Z}$

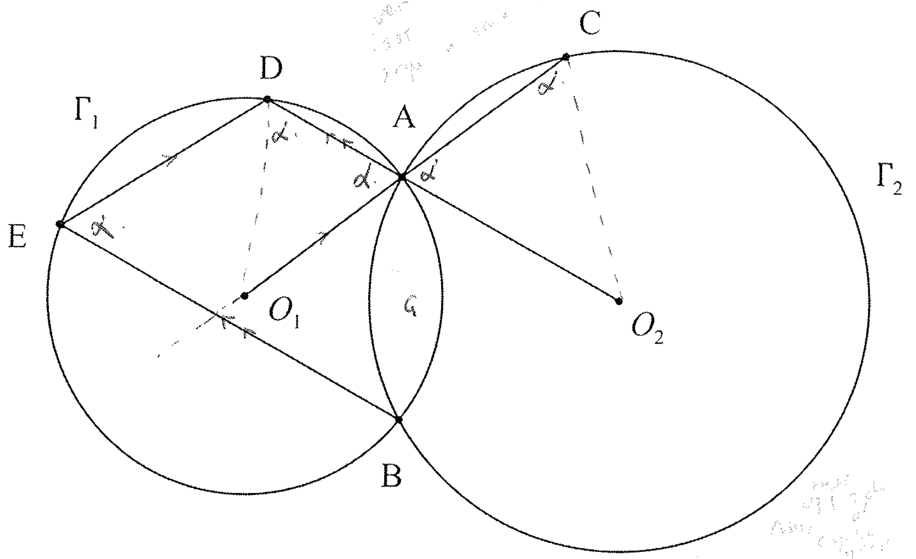
$$k S_k - \sum_{r=1}^{k-1} S_k = \sum_{r=1}^{k} r^3$$

Required to prove true for $n = k+1$

i.e. $(k+1) S_{k+1} - \sum_{r=1}^{k} S_r = \sum_{r=1}^{k+1} r^3$

$$LHS = (k+1) S_{k+1} - \sum_{r=1}^{k} S_r$$

$$= (k+1)\left[S_k + (k+1)^2\right] - \sum_{r=1}^{k-1} S_r - S_k.$$



(i) Extending O_1A to line EB, let the point of intersection be F

$\therefore$ ADEF is a parallelogram (2 pairs of parallel sides)

$\therefore$ $\angle DAO_1 = \alpha$ (opposite angles of a parallelogram are equal)

$\angle CAO_2 = \alpha$ (vertically opposite angles are equal).

$\because$ $\triangle AO_1D$ and $\triangle AO_2C$ are isosceles (as $O_1D = O_1A$ and $O_2A = O_2C$)

$\angle O_1DA = \alpha$ and $\angle ACO_2 = \alpha$ (base angles of isosceles triangle are equal)

$\therefore$ $\angle O_1DA = \angle ACO_2$

$\therefore$ O_1DCO_2 is a cyclic quadrilateral (equal angles subtended by equal chords in the same segment)

Join O_1O_2 and AB. Intersect at G.

ii) $\because$ $\triangle O_1AO_2 \equiv \triangle O_1BO_2$ (SSS)

$\triangle O_1AG \equiv O_1BG$ (SAS)

$\therefore$ $O_1O_2 \perp AB$ ($\angle O_1GA = \angle BGO_1$ and AGB is a straight angle)

$\angle DAG = 180 - \alpha$ (opposite angles of cyclic quadrilateral ADEB)

$\therefore$ $\angle O_1AG = 180 - 2\alpha$

$\therefore$ $\angle AO_1G = 2\alpha - 90°$ (angle sum of triangle)

$\because$ $\angle DO_1A = 180 - 2\alpha$ (angle sum of triangle)

then $\angle DO_1G = 90° \Rightarrow \angle DCO_2 = 90°$ (opposite angles of cyc. quad are supplementary)

Question 16

a) i) $\dfrac{a+b}{2} + \dfrac{c+d}{2} \geq \sqrt{ab} + \sqrt{cd}$

applying the AM-GM inequality again

$$\geq 2\sqrt{\sqrt{ab}\cdot\sqrt{cd}}$$

$$= 2\,(abcd)^{\frac{1}{4}}$$

$\therefore \dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}.$

(ii) Given $\dfrac{a+b+c}{3} = \dfrac{1}{4}\left(a+b+c+\dfrac{a+b+c}{3}\right)$

here we can use port (i)

$$\dfrac{a+b+c}{3} \geq \sqrt[4]{abc\left(\dfrac{a+b+c}{3}\right)}$$

$$\dfrac{(a+b+c)}{3} \geq (abc)^{\frac{1}{4}}\cdot\left(\dfrac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\dfrac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

$\therefore$ taking powers of $\frac{4}{3}$ on both sides

$$\left(\dfrac{a+b+c}{3}\right) \geq (abc)^{\frac{1}{3}}$$

c (i) Any valid explanation will suffice. (Conjugate root theorem)

ii) Let the roots be p, q, r.

$\therefore \quad pqr = a \qquad p+q+r = a$

applying the AM - GM.

$$\frac{p+q+r}{3} \geq \sqrt[3]{pqr}$$

$$\frac{a}{3} \geq \sqrt[3]{a}$$

$$\therefore \quad \frac{a^2}{27} \geq 1$$

$$a \geq 3\sqrt{3}$$

$\therefore$ smallest positive value of a is $3\sqrt{3}$.

$\therefore \quad b = 9 \qquad$ (From product of roots 2 at a time)