NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2017 HSC Course Assessment Task 3 (Trial Examination)
Friday 23rd of June, 2017

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Dr Jomaa ◯ 12M4B – Miss Lee ◯ 12M4C – Mr Lin

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. Let $z = 5 - i$ and $\omega = 2 + 3i$. What is the value of $2z + \bar{\omega}$? **1**

(A) $12 + i$

(B) $12 + 2i$

(C) $12 - 4i$

(D) $12 - 5i$

2. If $-2 + 2i\sqrt{3}$ is expressed in modulus-argument form, the result is **1**

(A) $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(B) $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(C) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(D) $4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

3. It is known that $2 + 3i$ is a solution to $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$. Another solution is **1**

(A) $-2 - 3i$

(B) $-1 - 2i$

(C) $1 - 2i$

(D) $-2 + i$

4. Let α, β, and γ be the roots of the equation $x^3 + px^2 + q = 0$. The polynomial with roots 2α, 2β and 2γ is: **1**

(A) $x^3 - 2px^2 + 8q = 0$

(B) $x^3 + 2px^2 + 4q = 0$

(C) $x^3 - 2px^2 - 8q = 0$

(D) $x^3 + 2px^2 + 8q = 0$

5. Given the eccentricity of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is e, then the eccentricity of the ellipse $\dfrac{x^2}{a^2+b^2} + \dfrac{y^2}{b^2} = 1$ is **1**

(A) $-e$

(B) $\dfrac{1}{e}$

(C) e

(D) e^2

6. $\displaystyle\int \frac{1}{1+\operatorname{cosec} x}\,dx =$ **1**

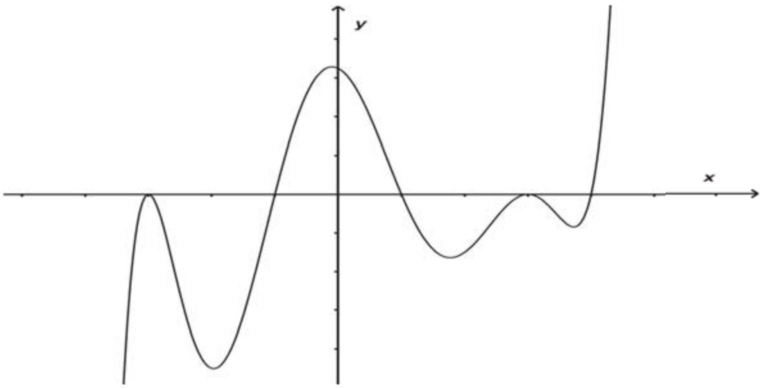
(A) $\dfrac{\left(1+\tan\dfrac{x}{2}\right)^2}{1+(\tan\dfrac{x}{2})^2} + x + c$

(B) $\dfrac{\left(1-\tan\dfrac{x}{2}\right)^2}{1+(\tan\dfrac{x}{2})^2} - x + c$

(C) $\sec x - \tan x + x + c$

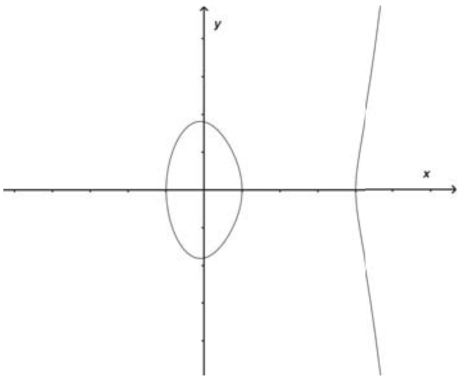
(D) $\tan x - \sec x - x + c$

7. Consider the graph of $y = f(x)$ drawn below. **1**

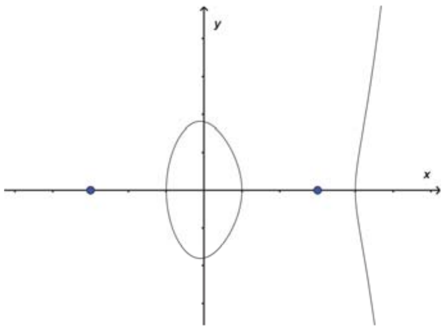


Which of the following diagrams show the graph of $y^2 = f(x)$?

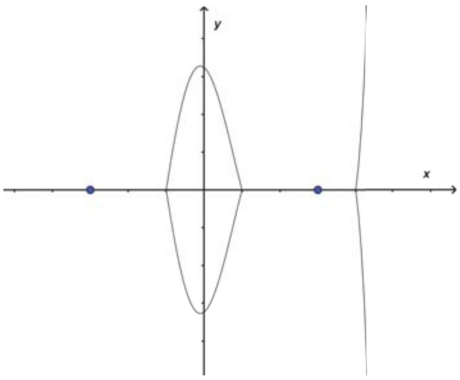
(A)



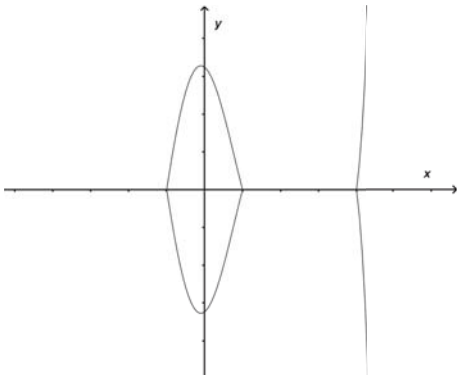
(C)



(B)



(D)



8. If $\sqrt{yx^2 + xy^2} = 3$, then at the point $(1, -1)$, the value of $\dfrac{dy}{dx}$ is **1**

(A) 1

(B) -1

(C) $\dfrac{1}{3}$

(D) $\dfrac{-1}{3}$

9. The cross section perpendicular to the $x-$ axis between two curves $y = \sqrt{x}$ and $y = 2\sqrt{x}$ is a circle. If the two curves are drawn between $x = 0$ and $x = 4$, the volume of the horn is given by **1**

(A) $\displaystyle\int_0^4 \sqrt{x}\, dx$

(B) $\displaystyle\int_0^4 \pi\sqrt{x}\, dx$

(C) $\displaystyle\int_0^4 \frac{\pi}{2}x\, dx$

(D) $\displaystyle\int_0^4 \frac{\pi x}{4}\, dx$

10. The value of **1**

$$\lim_{h\to 0} \frac{1}{h}\left(\int_{\frac{\pi}{2}-h}^{\frac{\pi}{2}+h} \frac{\sin x}{x}\, dx\right)$$

is

(A) 0

(B) 1

(C) $\dfrac{2}{\pi}$

(D) $\dfrac{4}{\pi}$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Let $z = 4 + i$ and $w = \bar{z}$. Find $\dfrac{z}{w}$ in the form $x + iy$. **1**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$. **2**

(c) Given that

$$\frac{25}{(x-1)^2(x^2+4)} = \frac{ax+b}{(x-1)^2} + \frac{cx+d}{x^2+4}.$$

i. Find a, b, c and d. **2**

ii. Hence, find $\displaystyle\int \frac{25}{(x-1)^2(x^2+4)}dx$. **3**

(d) The equation $z^5 = 1$ has roots $1, \omega, \omega^2, \omega^3, \omega^4$, where $\omega = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$.

i. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. **1**

ii. Show that $\left(\omega + \dfrac{1}{\omega}\right)^2 + \left(\omega + \dfrac{1}{\omega}\right) - 1 = 0$. **1**

iii. Hence, show that $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$. **2**

(e) The region enclosed by the curves $y = x + 1$ and $y = (x-1)^2$ is rotated about the $y-$axis. Find the volume of the solid formed. **3**

End of Question 11

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) For the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, find:

i. The eccentricity. **1**

ii. The coordinates of the foci S and S' and the equations of its directrices. **2**

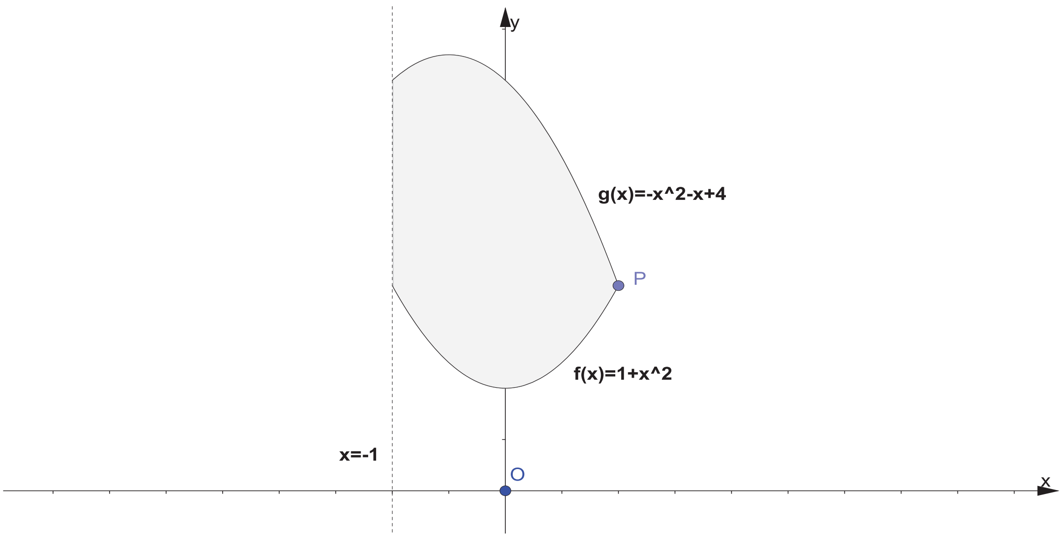
iii. Sketch the ellipse showing all the above features. **1**

(b) Given the polynomial $P(x) = 2x^3 + 3x^2 - x + 1$ has roots α, β and γ:

i. Find the polynomial whose roots are α^2, β^2 and γ^2. **2**

ii. Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. **3**

(c) The shaded region bounded by $g(x) = -x^2 - x + 4$, $f(x) = 1 + x^2$ and $x = -1$ is rotated about the line $x = -1$. The point P is the intersection of $f(x)$ and $g(x)$ in the first quadrant.



i. Find the $x-$coordinate of P. **1**

ii. Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. **3**

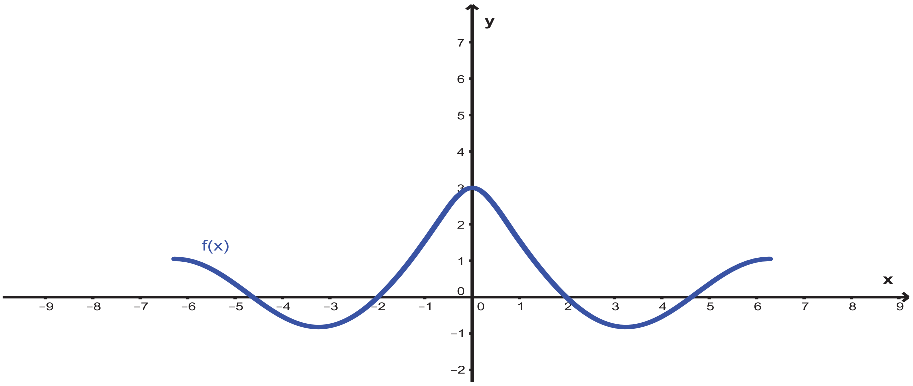
iii. Evaluate the integral in part (ii). **2**

End of Question 12

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) Use integration by parts to evaluate $\int x \ln(x^3 + x)dx$. **3**

(b) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of graphs of the following:

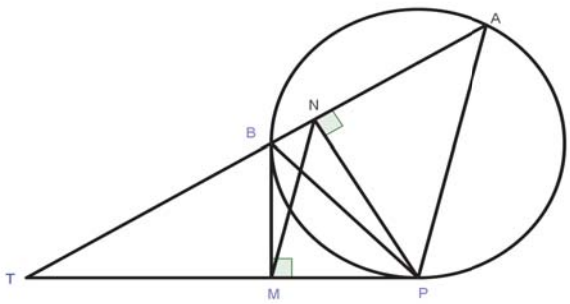
i. $y = \sqrt{f(x)}$ **2**

ii. $|y| = f(x)$ **2**

iii. $y = f(x)^2$ **2**

iv. $y = e^{-f(x)}$ **2**

(c) The points A, B and P lie on a circle. The chord AB produced and the tangent at P intersect at the point T, as shown in the diagram. The point N is the foot of the perpendicular to AB through P, and the point M is the foot of the perpendicular to PT through B.



Copy or trace this diagram into your writing booklet.

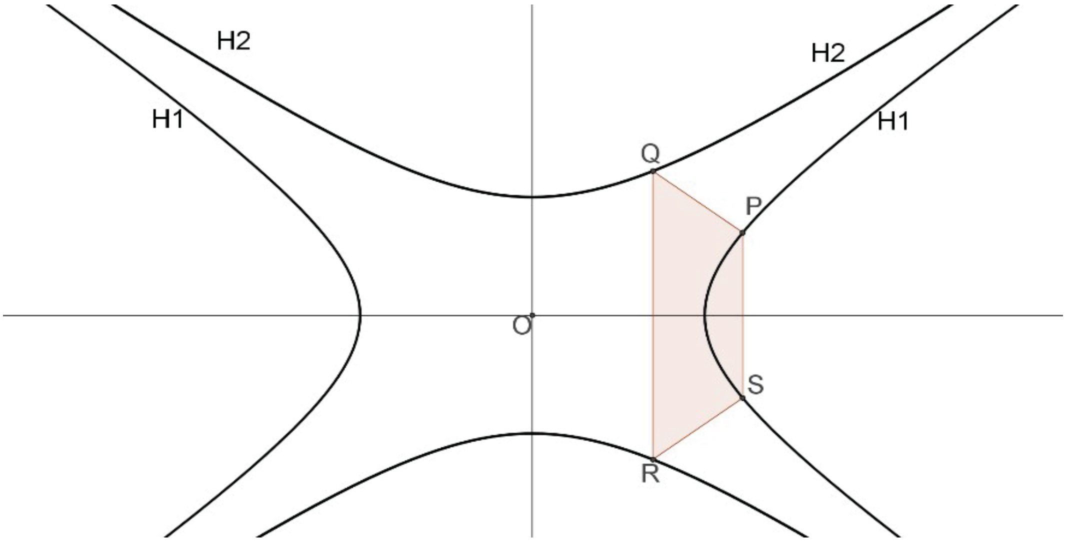
i. Explain why $BNPM$ is a cyclic quadrilateral. **1**

ii. Prove that MN is parallel to PA. **3**

End of Question 13

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) The hyperbola $\mathcal{H}_1 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\mathcal{H}_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ as shown in the diagram. The line $x = a\sec\theta$ cuts $\mathcal{H}_1$ at P and S. Similarly the line $x = a\tan\theta$ cuts $\mathcal{H}_2$ at Q and R.



i. Show that the $y-$ coordinates of P and S are $\pm b\tan\theta$ respectively and the $y-$ coordinates of Q and R are $\pm b\sec\theta$ respectively. **2**

ii. Prove that the area of trapezium $PQRS$ is independent of θ. **2**

iii. Show that the equation of the line PQ is $bx + ay = ab\left(\tan\theta + \sec\theta\right)$. **2**

iv. Prove that the area of triangle OPQ equals to half the area of the trapezium $PQRS$. **3**

(b) Given that $I_n = \displaystyle\int_0^1 (1 - x^2)^n \, dx$.

i. Evaluate I_1 and I_2. **2**

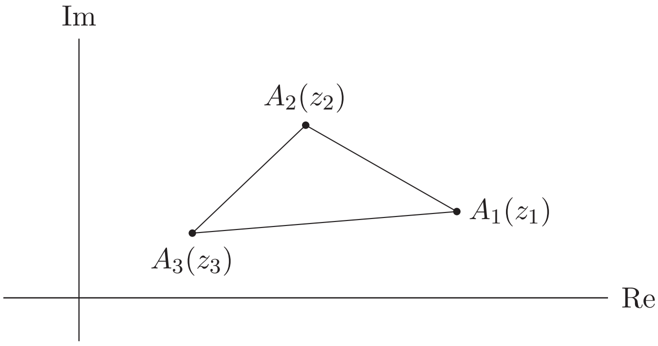
ii. Show that $I_{n+1} = \dfrac{2(n+1)}{2n+3} I_n$. **2**

iii. Hence or otherwise prove that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$. **2**

End of Question 14

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) $A_1A_2A_3$ is an equilateral triangle, the vertices occurring in the positive direction of rotation.



i. Prove by geometric means or otherwise that **2**

$$\overrightarrow{A_3A_1} = (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})\overrightarrow{A_2A_3} = \omega\overrightarrow{A_2A_3},$$

where ω is a complex cube root of unity.

ii. z_1, z_2 and z_3 are the complex numbers corresponding to A_1, A_2 and A_3 respectively. The triangle $A_1A_2A_3$ is inscribed in a circle of centre z_0 and radius r. Show that $z_0 = \frac{1}{3}[z_1 + z_2 + z_3]$ and $r = \frac{1}{\sqrt{3}}|z_1 - z_2|$. **3**

iii. Use (i) or otherwise, prove that $z_1 + \omega z_2 + \omega^2 z_3 = 0$. **2**

iv. Hence or otherwise, prove that **3**

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

(b) Given that $x_n = \frac{1}{2}\left[\left(1+i\sqrt{2}\right)^n + \left(1-i\sqrt{2}\right)^n\right]$, $n \geq 0$.

Let $y_0 = 6$, $y_1 = 2$ and $3y_n = 2y_{n-1} - y_{n-2}$, $n \geq 2$.

i. Prove by mathematical induction that $y_n = \frac{2}{3^{n-1}}x_n$, $n \geq 0$. **3**

ii. Hence or otherwise, show that **2**

$$y_n = 3\left[\left(\frac{1}{1+i\sqrt{2}}\right)^n + \left(\frac{1}{1-i\sqrt{2}}\right)^n\right],\ n \geq 0.$$

End of Question 15

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) i. Given that $a+b \geq 2\sqrt{ab}$. Prove that $a^2+b^2+c^2 \geq ab+ac+bc$. **1**

ii. Given that $a+b+c \geq 3\sqrt[3]{abc}$. Hence or otherwise prove that **3**

$$\frac{a^3}{b-c}+\frac{b^3}{c-a}+\frac{c^3}{a-b} \geq \frac{3}{2}\left(ab+ac+bc\right).$$

(b) i. Prove that for all positive values of x, $x > \ln(1+x)$. **3**

ii. Given that $x_n = (1+\frac{1}{3})(1+\frac{1}{3^2})\ldots(1+\frac{1}{3^n})$, $n \geq 1$. Show that $x_{n+1} > x_n$. **1**

iii. Prove that **3**

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{x_{k+1}-x_k}{x_k} = \frac{1}{6}.$$

iv. Show that $\ln x_n < \sum_{k=1}^{n}\frac{1}{3^k}$. **2**

v. Hence or otherwise, show that $x_n < \sqrt{e}$ for all positive integer n. **2**

End of paper.

MC:

1. $z = 5 - i, \omega = 2 + 3i$

$$2z + \overline{\omega} = 2(5 - i) + 2 - 3i = 12 - 5i$$

The answer is D.

2. $-2 + 2i\sqrt{3} = 4\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

The answer is D.

3. $2 + 3i$ is a root of $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$

$2 - 3i$ is also a root (complex conjugate theorem).

Assume $a + ib$ is another root, so $a - ib$.

Sum of roots are $=2 + 3i + 2 - 3i + a + ib + a - ib = 4 + 2a$

But sum of roots =6. Hence a=1. And the answer is C.

4. Replace x by $\frac{x}{2}$. $\left(\frac{x}{2}\right)^3 + p\left(\frac{x}{2}\right)^2 + q = 0$

$$\frac{x^3}{8} + p\frac{x^2}{4} + q = 0$$

$$x^3 + 2px^2 + 8q = 0$$

The answer is D.

5. $b^2 = a^2(e^2 - 1) \therefore e^2 = \frac{a^2+b^2}{a^2}$

Let E the eccentricity of the ellipse $\therefore b^2 = (a^2 + b^2)(1 - E^2)$

$$E^2 = \frac{a^2}{a^2 + b^2} = \frac{1}{e^2} \;\therefore E = \frac{1}{e}$$

The answer is B.

6. $\int \frac{1}{1+\csc x}dx = \int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}dx = \int \frac{\sin x-(\sin x)^2}{(\cos x)^2}dx =$

$$\int \frac{\sin x}{(\cos x)^2}dx - \int (\tan x)^2 dx = -\int \frac{du}{u^2} - \int ((\sec x)^2 - 1)dx$$

$$\frac{1}{u} - \tan x + x + c = \frac{1}{\cos x} - \tan x + x + c = \sec x - \tan x + x + c$$

(for $\int \frac{\sin x}{(\cos x)^2}dx$, use $u = \cos x$)

The answer is C.

7. The answer is C.

8. $\sqrt{yx^2 + xy^2} = 3 \;\therefore\; yx^2 + xy^2 = 9$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2xy + y^2}{x^2 + 2xy}$$

At (1,-1), $\frac{dy}{dx} = -\frac{2\times1\times-1+(-1)^2}{1^2+2\times1\times-1} = -\frac{-1}{-1} = -1$

The answer is B.

9. The diameter of the circle is $2\sqrt{x} - \sqrt{x} = \sqrt{x}$

So the area of the circle is $\pi\left(\frac{d}{2}\right)^2 = \pi\frac{x}{4}$

Volume=$\int_0^4 \pi\frac{x}{4}dx$. The answer is D.

10. $\lim_{h\to 0}\frac{1}{h}\left(\int_{\frac{\pi}{2}-h}^{\frac{\pi}{2}+h}\frac{\sin x}{x}dx\right) = \lim_{h\to 0}\frac{1}{h}\left(\int_{\frac{\pi}{2}-h}^{\frac{\pi}{2}}\frac{\sin x}{x}dx\right) + \lim_{h\to 0}\frac{1}{h}\left(\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+h}\frac{\sin x}{x}dx\right)$

$$= \lim_{h\to 0}\frac{1}{h}\left[F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{2} - h\right)\right] + \lim_{h\to 0}\frac{1}{h}\left[F\left(\frac{\pi}{2} + h\right) - F\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4}{\pi}$$

Where $\frac{dF}{dx} = \frac{\sin x}{x}$.

The answer is D.

Question 11.

a) $z = 4 + i, \omega = \bar{z} = 4 - i$

$$\frac{z}{\omega} = \frac{4+i}{4-i} = \frac{4+i}{4-i} \times \frac{4+i}{4+i} = \frac{16+8i-1}{16+1} = \frac{15}{17} + \frac{8}{17}i$$

b) $\int \frac{1}{\sqrt{2x-x^2}}dx = \int \frac{dx}{\sqrt{1-1+2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + c = \sin^{-1}(x-1) + c$

$$(where\ u = x - 1).$$

c) i) $\frac{25}{(x-1)^2(x^2+4)} = \frac{ax+b}{(x-1)^2} + \frac{cx+d}{x^2+4}$

$$\therefore 25 = (ax+b)(x^2+4) + (cx+d)(x-1)^2$$

$$25 = ax^3 + 4ax + bx^2 + 4b + (cx+d)(x^2 - 2x + 1)$$

$$25 = ax^3 + 4ax + bx^2 + 4b + cx^3 - 2cx^2 + cx + dx^2 - 2dx + d$$

$$25 = (a+c)x^3 + (b - 2c + d)x^2 + (4a + c - 2d)x + 4b + d$$

$$\therefore a + c = 0\ (1), b - 2c + d = 0\ (2), 4a + c - 2d = 0\ (3) and\ 4b + d = 25\ (4)$$

$$(1) \to c = -a$$

$(1)\&\ (3) \to 3a - 2d = 0, or\ d = \frac{3}{2}a$ (5)

$(1)\&(2) \to b + d = -2a, or\ b = -\frac{7}{2}a$ (6)

$$(4), (5) and\ (6) \to -14a + \frac{3}{2}a = 25 \to a = -2$$

So $b = 7, c = 2\ and\ d = -3$

ii) $\int \frac{25}{(x-1)^2(x^2+4)}dx = \int \frac{-2x+7}{(x-1)^2}dx + \int \frac{2x-3}{x^2+4}dx =$

$$-2\int \frac{x-1}{(x-1)^2}dx + 5\int \frac{1}{(x-1)^2}dx + \int \frac{2x}{x^2+4}dx - 3\int \frac{1}{x^2+4}dx$$

$$= -2\ln|x-1| - \frac{5}{x-1} + \ln(x^2+4) - \frac{3}{2}\tan^{-1}\frac{x}{2} + c$$

d) i) $z^5 = 1 \to z^5 - 1 = 0 \to (z-1)(1 + z + z^2 + z^3 + z^4) = 0 \to 1 + z + z^2 + z^3 + z^4 = 0$

since ω is a root of $z^5 - 1 = 0. \to 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

ii) $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. Divide by ω^2, we obtain:

$$\frac{1}{\omega^2} + \frac{1}{\omega} + 1 + \omega + \omega^2 = 0$$

$$\left(\frac{1}{\omega^2} + \omega^2\right) + \left(\frac{1}{\omega} + \omega\right) + 1 = 0$$

$$\left(\frac{1}{\omega^2} + \omega^2 + 2\right) + \left(\frac{1}{\omega} + \omega\right) + 1 - 2 = 0$$

$\left(\frac{1}{\omega} + \omega\right)^2 + \left(\frac{1}{\omega} + \omega\right) - 1 = 0$ (***)

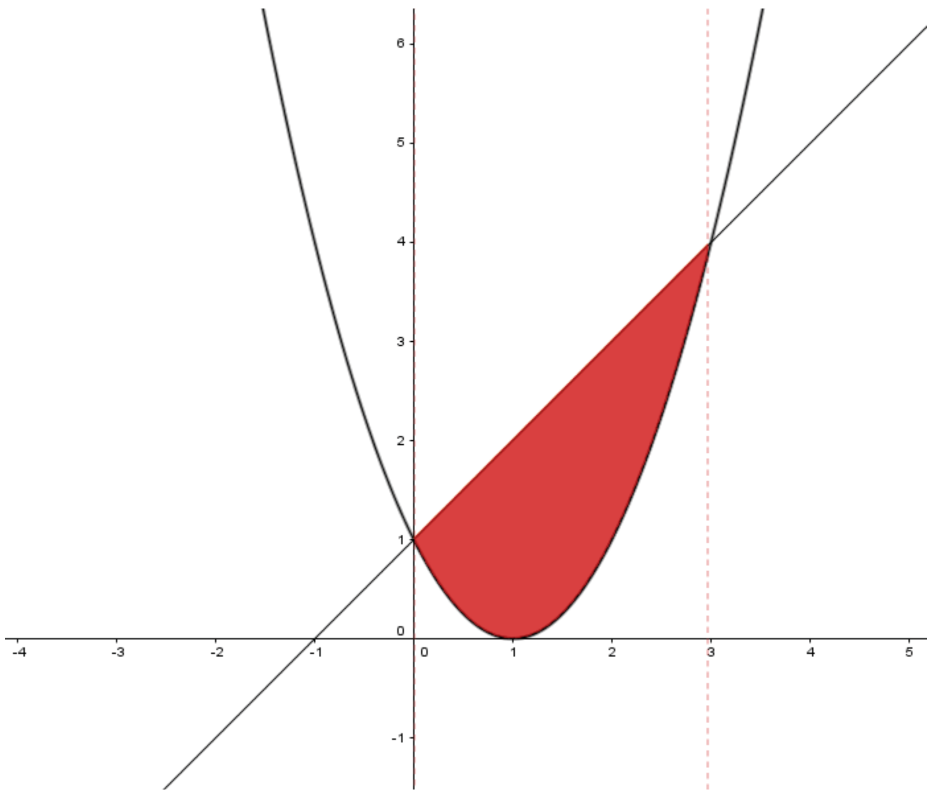
iii) But $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, and\ \frac{1}{\omega} = \bar{\omega}, so\ \frac{1}{\omega} + \omega = 2\cos\frac{2\pi}{5}$.

Let $X = \frac{1}{\omega} + \omega, so\ (***) \rightarrow X^2 + X - 1 = 0$ and $X = \frac{-1\pm\sqrt{(1)^2-4(1)(-1)}}{2}$

Since $\frac{2\pi}{5}$ is in the first quadrant, the cosine will be positive. $2\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{2}$

$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$.

e)



Using the cylindricall Shell Method.

$$x + 1 = (x-1)^2 = x^2 - 2x + 1 \ \therefore x^2 - 3x = 0 \ \therefore x = 0 \ or \ x = 3.$$

$$\delta V = 2\pi r h \delta x = 2\pi x(x + 1 - (x-1)^2)\delta x = 2\pi x(3x - x^2)\delta x$$

$$V = 2\pi \lim_{\delta x \to 0} \sum_{x=0}^{x=3} (3x^2 - x^3)\delta x$$

$$V = 2\pi \int_0^3 (3x^2 - x^3)dx = 2\pi \left[x^3 - \frac{x^4}{4}\right]_0^3 = 2\pi\left(27 - \frac{81}{4} - 0\right) = \frac{27}{2}\pi\ U^3.$$

Using the Washer method:

$$V = \pi \int_0^1 [((1+\sqrt{y})^2 - (1-\sqrt{y})^2)]dx + \pi \int_1^4 [((1+\sqrt{y})^2 - (y-1)^2)]dx$$

For the first integral needs to solve for x: $y = x^2 - 2x + 1, x = 1 \pm \sqrt{y}$.

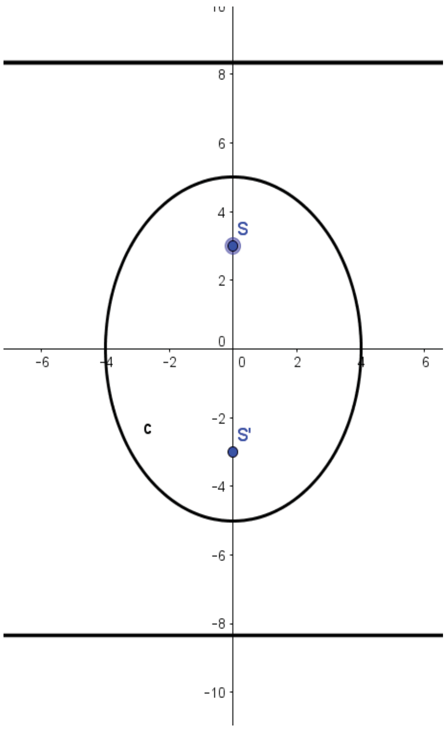
$V = \frac{65\pi}{6} + \frac{8\pi}{6} = \frac{27}{2}\pi\ U^3$.

Question 12

a) i) $16 = 25(1 - e^2) \therefore e^2 = \frac{9}{25}, e = \frac{3}{5}$.

ii) S(0,3), S'(0,-3), directrices $y = \pm\frac{b}{e} = \pm\frac{25}{3}$

iii)



b) $P(x) = 2x^3 + 3x^2 - x + 1$

i) Replace x by $\sqrt{x}$ in $P(x) = 0$.

$$2(\sqrt{x})^3 + 3(\sqrt{x})^2 - \sqrt{x} + 1 = 0 \ \therefore 2x\sqrt{x} + 3x - \sqrt{x} + 1 = 0$$
$$\sqrt{x}(2x-1) + 3x + 1 = 0 \ \therefore 3x + 1 = \sqrt{x}(1-2x)$$
$$(3x+1)^2 = (\sqrt{x}(1-2x))^2 \ \therefore 9x^2 + 6x + 1 = x(1 - 4x + 4x^2)$$
$$4x^3 - 13x^2 - 5x - 1 = 0.$$

ii) α, β, γ are roots of $2x^3 + 3x^2 - x + 1 = 0$

$$\alpha + \beta + \gamma = -\frac{3}{2}$$

$\alpha^2, \beta^2, \gamma^2$ are roots of $4x^3 - 13x^2 - 5x - 1 = 0$.

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{13}{4}$$

α, β, γ are roots of $2x^3 + 3x^2 - x + 1 = 0$

$$2\alpha^3 = -3\alpha^2 + \alpha - 1$$
$$2\beta^3 = -3\beta^2 + \beta - 1$$
$$2\gamma^3 = -3\gamma^2 + \gamma - 1$$

$$\therefore 2(\alpha^3 + \beta^3 + \gamma^3) = -3(\alpha^2 + \beta^2 + \gamma^2) + \alpha + \beta + \gamma - 3 = -3\frac{13}{4} + \frac{-3}{2} - 3 = \frac{-57}{4}$$

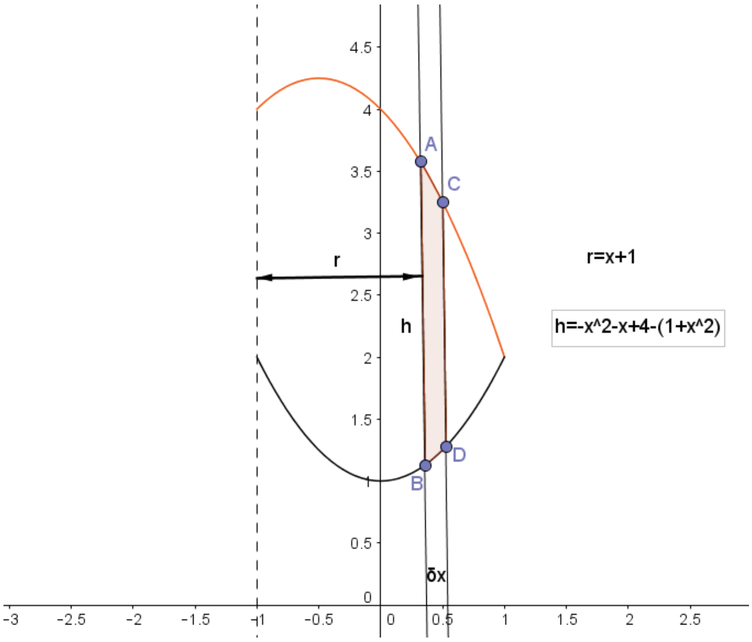
$$\alpha^3 + \beta^3 + \gamma^3 = \frac{-57}{8}$$

c)

i) $-x^2 - x - 4 = 1 + x^2 \quad \therefore 2x^2 + x - 3 = 0$

$$(2x + 3)(x - 1) = 0 \quad \therefore x = \frac{-3}{2} \quad and\ x = 1.$$

$P\ is\ in\ the\ first\ quadrant\ \therefore x = 1.$



ii) $\delta V = 2\pi r h \delta x =$

$$2\pi(x + 1)(-x^2 - x + 4 - 1 - x^2)\delta x = 2\pi(x + 1)(3 - x - x^2)\delta x$$

$$V = 2\pi \lim_{\delta x \to 0} \sum_{x=-1}^{x=1} (x + 1)(3 - x - 2x^2)\delta x$$

$$V = 2\pi \int_{-1}^{1} (x + 1)(3 - x - 2x^2)dx$$

iii) $V = 2\pi \int_{-1}^{1} (x + 1)(3 - x - 2x^2)dx = 2\pi \int_{-1}^{1} (-2x^3 - 3x^2 + 2x + 3)dx$

$$= 2\pi \left[-\frac{x^4}{4} - x^3 + x^2 + 3x \right]_{-1}^{1}$$

$$2\pi \left[\frac{-1}{2} - 1 + 1 + 3 - (\frac{-1}{2} + 1 + 1 - 3) \right] = 2\pi[4] = 8\pi\ U^3.$$

Question 13

a) $\int x\ln(x^3+x)dx$ Let $u=\ln(x^3+x)$ *and* $dv = xdx$

$\frac{du}{dx}=\frac{3x^2+1}{x(x^2+1)}=\frac{3(x^2+1)}{x(x^2+1)}-\frac{2}{x(x^2+1)}$ and $v=\frac{x^2}{2}$

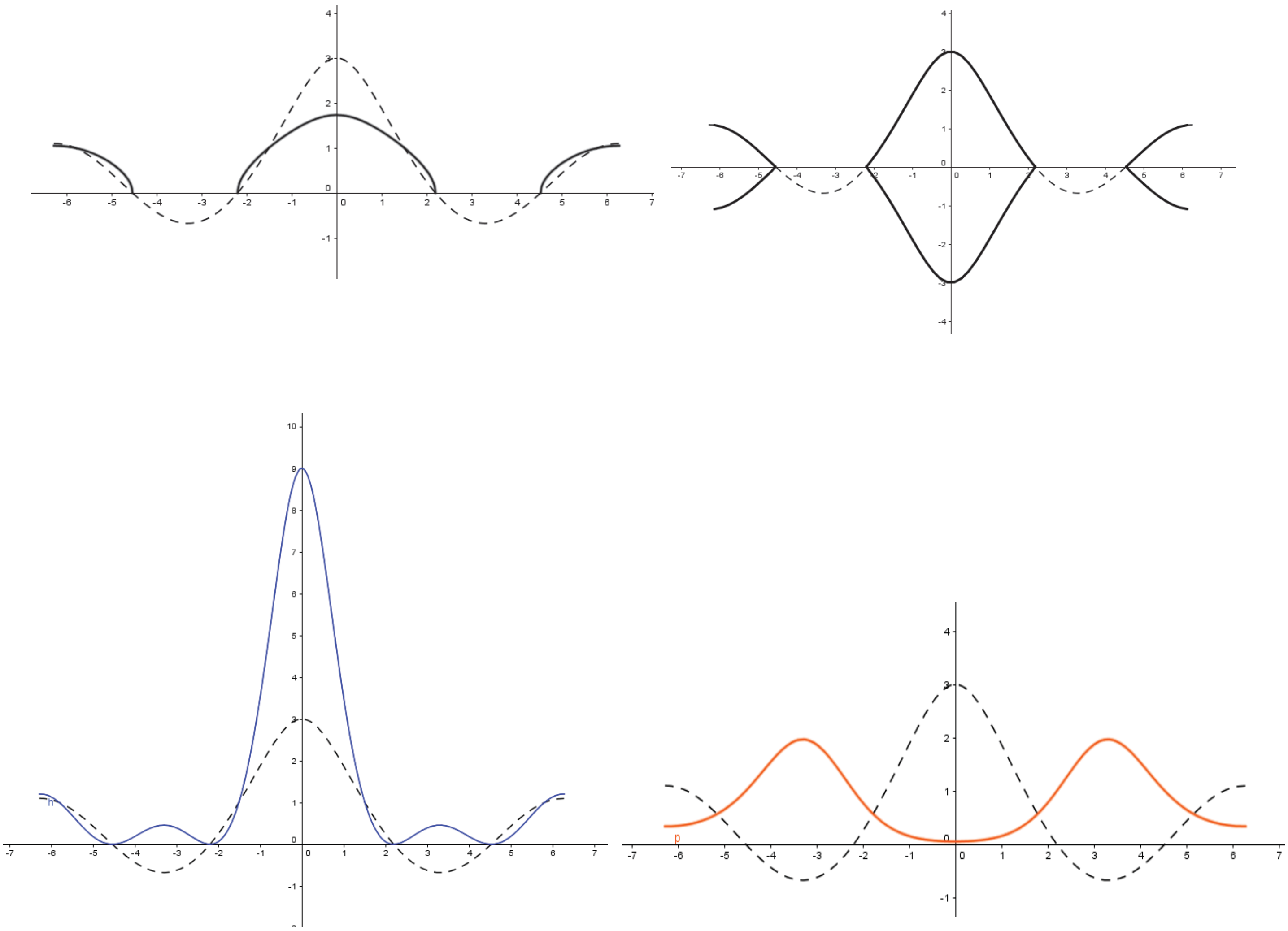
$$\int x\ln(x^3+x)dx=\frac{x^2}{2}\ln(x^3+x)-\frac{1}{2}\int x^2\frac{3(x^2+1)}{x(x^2+1)}dx+\frac{1}{2}\int x^2\frac{2}{x(x^2+1)}dx$$

$$=\frac{x^2}{2}\ln(x^3+x)-\frac{3}{2}\int xdx+\frac{1}{2}\int\frac{2x}{x^2+1}dx$$

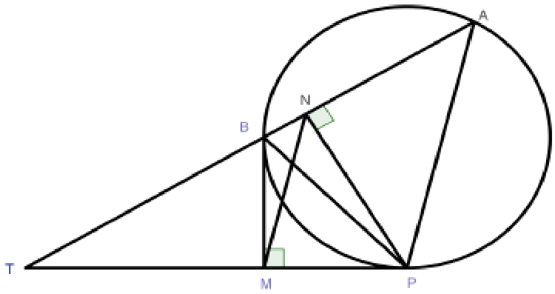
$$=\frac{x^2}{2}\ln(x^3+x)-\frac{3}{2}\frac{x^2}{2}+\frac{1}{2}\ln(x^2+1)+c$$

$$=\frac{x^2}{2}\ln(x^3+x)-\frac{3}{4}x^2+\frac{1}{2}\ln(x^2+1)+c$$

b)



c)



(i) In the quadrilateral BNPM, $\angle BMP + \angle BNP = 180°$
(Opposite angles in cyclic quadrilateral are supplementary.)

(ii) $\angle TPB = \angle TAP$ (angle between the tangent is equal to angle in the alternate segment = $\theta\ say$). $\angle NPA = 90 - \theta$.
$\angle NBP = 90 - \theta$. But $\angle MBP = \angle MNP$ (angles in the same segment).

$$\therefore \angle MNP = \angle NPA.$$

But $\angle MNP$ and $\angle NPA$ are alternate angle on lines MN and PA. $\therefore MN \parallel PA$.

Question 14.

a) i) $x = a\sec\theta \ \therefore \frac{a^2(\sec\theta)^2}{a^2} - \frac{y^2}{b^2} = 1 \ \therefore \frac{y^2}{b^2} = (\sec\theta)^2 - 1 = (\tan\theta)^2$

$$y^2 = (b\tan\theta)^2 \therefore y = \pm b\tan\theta.$$

$$x = a\tan\theta \ \therefore \ \frac{a^2(\tan\theta)^2}{a^2} - \frac{y^2}{b^2} = -1 \ \therefore \ \frac{y^2}{b^2} = (\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$y^2 = (b\sec\theta)^2 \therefore y = \pm b\sec\theta$$

ii) Let M be the foot of the Perpendicular from P to QR.

Area of trapezium PQRS=$\frac{1}{2}(PM)(PS+QR)$

$$PM = a\sec\theta - a\tan\theta = a(\sec\theta - \tan\theta)$$

$$PS = 2y(P) = 2b\tan\theta$$

$$QR = 2y(Q) = 2b\sec\theta$$

Area of PQRS=$\frac{1}{2}a(\sec\theta - \tan\theta) \times 2b(\sec\theta + \tan\theta) = ab((\sec\theta)^2 - (\tan\theta)^2) = ab.$

iii) Gradient of PQ=$\frac{b\sec\theta - b\tan\theta}{a\tan\theta - a\sec\theta} = -\frac{b}{a}$

Equation of PQ : $y - b\tan\theta = -\frac{b}{a}(x - a\sec\theta)$

$\therefore bx + ay = ab(\sec\theta + \tan\theta)$.

iv) d=Perpendicular distance from O to PQ is $\frac{|b\times 0 + a\times 0 - ab(\sec\theta + \tan\theta)|}{\sqrt{a^2+b^2}}$

PQ=$\sqrt{(b\sec\theta - b\tan\theta)^2 + (a\tan\theta - a\sec\theta)^2}$

$$= \sqrt{b^2(\sec\theta)^2 - 2b^2\sec\theta\tan\theta + b^2(\tan\theta)^2 + a^2(\tan\theta)^2 - 2a^2\sec\theta\tan\theta + a^2(\sec\theta)^2}$$

$$= \sqrt{(a^2+b^2)((\sec\theta)^2+(\tan\theta)^2) - 2(a^2+b^2)\sec\theta\tan\theta}$$
$$= \sqrt{(a^2+b^2)(\sec\theta-\tan\theta)^2} = (\sec\theta-\tan\theta)\sqrt{a^2+b^2}$$

Area of OPQ $=\frac{1}{2}\times d\times PQ = \frac{1}{2}\frac{|-ab(\sec\theta+\tan\theta)|}{\sqrt{a^2+b^2}}\times(\sec\theta-\tan\theta)\sqrt{a^2+b^2} = \frac{1}{2}ab =$ $\frac{1}{2}\ Area\ of\ PQRS.$

b)

i) $I_1 = \int_0^1 (1-x^2)dx = \left[x-\frac{x^3}{3}\right]_0^1 = 1-\frac{1}{3}-0 = \frac{2}{3}$

$$I_2 = \int_0^1 (1-x^2)^2dx = \int_0^1 (1-2x^2+x^4)dx = \left[x-2\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1 = 1-\frac{2}{3}+\frac{1}{5}-0 = \frac{8}{15}$$

ii) $I_{n+1} = \int_0^1 (1-x^2)^{n+1}\,dx = \int_0^1 (1-x^2)(1-x^2)^n dx =$
$$\int_0^1 (1-x^2)^n\,dx - \int_0^1 x^2(1-x^2)^n\,dx = I_n - \int_0^1 x^2(1-x^2)^n\,dx$$

Now for $\int_0^1 x^2(1-x^2)^n\,dx$, let $u = x\ and\ dv = x(1-x^2)^n dx$, then

$$du = dx\ and\ v = -\frac{1}{2}\times\frac{(1-x^2)^{n+1}}{n+1}$$

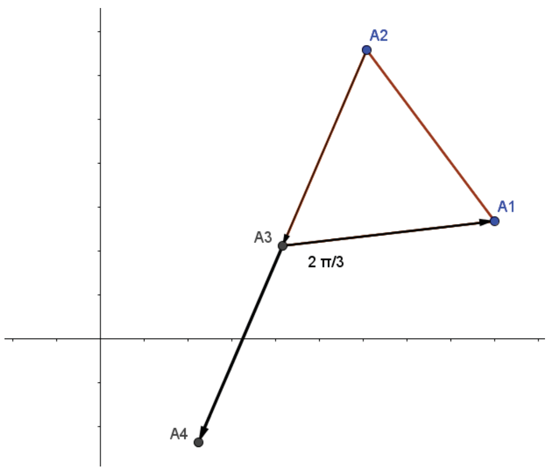
So $\int_0^1 x^2(1-x^2)^n\,dx = \left[-\frac{1}{2}\times x\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1 + \frac{1}{2(n+1)}\int_0^1(1-x^2)^{n+1}\,dx = \frac{1}{2(n+1)}I_{n+1}$

$$\therefore I_{n+1} = I_n - \frac{1}{2(n+1)}I_{n+1}, \qquad \therefore I_{n+1} = \frac{2(n+1)}{2n+3}I_n.$$

iii) $I_n = \frac{2n}{2n+1}I_{n-1},\ I_{n-1} = \frac{2(n-1)}{2n-1}I_{n-2},\ \ldots,\ I_2 = \frac{2\times2}{2\times2+1}I_1$
$$I_n = \frac{2n}{2n+1}\times\frac{2(n-1)}{2n-1}\times\ldots\times\frac{2\times2}{2\times2+1}I_1$$

$$I_n = \frac{2n}{2n+1}\times\frac{2n}{2n}\times\frac{2(n-1)}{2n-1}\times\frac{2(n-1)}{2(n-1)}\times\ldots\times\frac{2(2)}{2(2)}\times\frac{2\times2}{2\times2+1}I_1$$
$$= \frac{(2\times2\times\ldots\times2)^2\times(n\times(n-1)\times\ldots\times2\times1)^2}{(2n+1)\times(2n)\times(2n-1)\ldots5\times4\times3\times2\times1} = \frac{2^{2n}\times n!^2}{(2n+1)!}$$

Question 15 a)



i) $\overrightarrow{A_3A_4} = \overrightarrow{A_2A_3}$
$\overrightarrow{A_3A_1} = (\cos\frac{2\pi}{3} + i\ \sin\frac{2\pi}{3})\overrightarrow{A_3A_4}$
$= (\cos\frac{2\pi}{3} + i\ \sin\frac{2\pi}{3})\overrightarrow{A_2A_3}$

$= \omega\overrightarrow{A_2A_3}$

ii) The triangle $A_1A_2A_3$ is inscribed in a circle.
Let O be the centre and r its radius.
$$OA_1 = OA_2 = OA_3 = r.$$

Let z_0 the complex number corresponding to O. Since $z_1,\ z_2,\ z_3$ are the complex numbers corresponding to A_1, A_2, A_3 respectively.

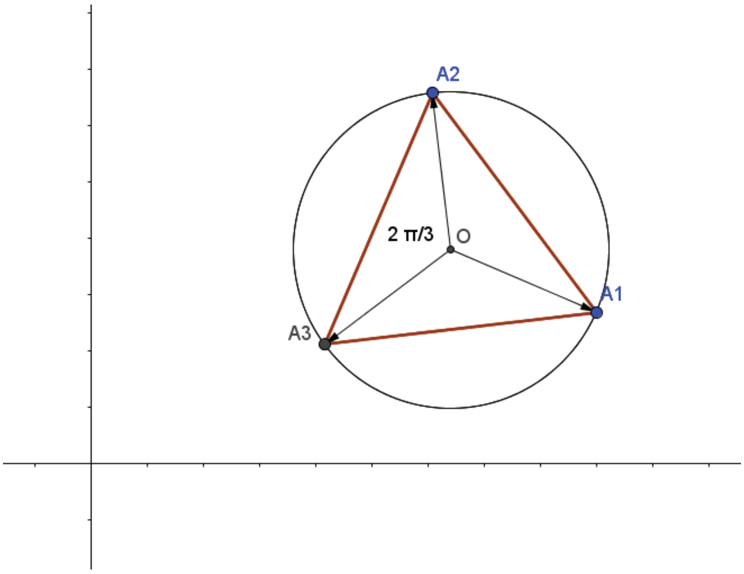
Now

$$\overrightarrow{OA_3} = \omega\overrightarrow{OA_2}$$
$$\overrightarrow{OA_1} = \omega\overrightarrow{OA_3}$$
$$\overrightarrow{OA_2} = \omega\overrightarrow{OA_1}$$

$\therefore \quad z_3 - z_0 = \omega(z_2 - z_0)$ (1)

$z_1 - z_0 = \omega(z_3 - z_0)$ (2)

$z_2 - z_0 = \omega(z_1 - z_0)$ (3)



Add the three equations $\therefore$

$$z_1 + z_2 + z_3 - 3z_0 = \omega(z_1 + z_2 + z_3) - 3\omega z_0$$

$$(1-\omega)(z_1 + z_2 + z_3) = 3(1-\omega)z_0$$

$$z_0 = \frac{1}{3}(z_1 + z_2 + z_3)$$

In triangle OA_1A_2, applying the cosine rule:

$A_2{A_1}^2 = {OA_1}^2 + {OA_2}^2 - 2 \times OA_1 \times OA_2 \cos\frac{2\pi}{3}$

$$|z_1 - z_2|^2 = r^2 + r^2 - 2 \times r \times r \times \cos\frac{2\pi}{3} = 3r^2$$

$r = \frac{1}{\sqrt{3}}|z_1 - z_2|.$

iii) ω is the complex cube root of unity $\therefore\ \omega^3 = 1,\ \omega^3 - 1 = 0, (\omega - 1)(\omega^2 + \omega + 1) = 0$

$\omega^2 + \omega + 1 = 0$ (*)

Using (i) $z_1 - z_3 = \omega(z_3 - z_2)$

$z_1 + \omega z_2 - (1+\omega)z_3 = 0$, but from (*) $\omega + 1 = -\omega^2$

$\therefore z_1 + \omega z_2 + \omega^2 z_3 = 0.$

iv) Using (iii) $z_1 = -(\omega z_2 + \omega^2 z_3)$, also $z_2 = -(\omega z_3 + \omega^2 z_1)$, $z_3 = -(\omega z_1 + \omega^2 z_2)$.

$${z_1}^2 + {z_2}^2 + {z_3}^2 = [-(\omega z_2 + \omega^2 z_3)]^2 + [-(\omega z_3 + \omega^2 z_1)]^2 + [-(\omega z_1 + \omega^2 z_2)]^2$$

$$= \omega^2{z_2}^2 + 2\omega\omega^2 z_2 z_3 + \omega^4{z_3}^2 + \omega^2{z_3}^2 + 2\omega\omega^2 z_1 z_3 + \omega^4{z_1}^2 + \omega^2{z_1}^2 + 2\omega\omega^2 z_1 z_2 + \omega^4{z_2}^2$$

$$= 2\omega^3(z_1z_2 + z_2z_3 + z_1z_3) + (\omega^2 + \omega^4)({z_1}^2 + {z_2}^2 + {z_3}^2)$$

But $\omega^3 = 1,\ \omega^4 = \omega\ and\ \omega^2 + \omega^4 = -1.$

$${z_1}^2 + {z_2}^2 + {z_3}^2 = 2(z_1z_2 + z_2z_3 + z_1z_3) - ({z_1}^2 + {z_2}^2 + {z_3}^2)$$

$$2({z_1}^2 + {z_2}^2 + {z_3}^2) = 2(z_1z_2 + z_2z_3 + z_1z_3)$$

${z_1}^2 + {z_2}^2 + {z_3}^2 = z_1z_2 + z_2z_3 + z_1z_3.$

b) i) Prove by mathematical induction that $y_n = \frac{2}{3^{n-1}} x_n \quad n \geq 0$.

$y_0 = \frac{2}{3^{0-1}} x_0 = 6x_0 = 6 \times \frac{1}{2}[1+1] = 6$, true for $n = 0$.

$y_1 = \frac{2}{3^{1-1}} x_1 = 2x_1 = 2 \times \frac{1}{2}\left[1 + i\sqrt{2} + 1 - i\sqrt{2}\right] = 2$, true for $n = 1$.

Assume it is true for $n = k$, i.e. $y_k = \frac{2}{3^{k-1}} x_k \quad k \geq 0\ (**)$ Induction Hypothesis

And prove it true for $n = k + 1$.

i.e. $y_{k+1} = \frac{2}{3^k} x_{k+1}$

Using the recurrence relation of $y, \therefore 3y_{k+1} = 2y_k - y_{k-1} = 2 \times \frac{2}{3^{k-1}} x_k - \frac{2}{3^{k-2}} x_{k-1}$

$$= \frac{1}{3^{k-1}}\left[2\left(1+i\sqrt{2}\right)^k + 2(1-i\sqrt{2})^k - 3\left(1+i\sqrt{2}\right)^{k-1} - 3\left(1-i\sqrt{2}\right)^{k-1}\right]$$

$$= \frac{1}{3^{k-1}}\left[\left(1+i\sqrt{2}\right)^{k-1}\left\{2\left(1+i\sqrt{2}\right) - 3\right\} + \left(1-i\sqrt{2}\right)^{k-1}\left\{2\left(1-i\sqrt{2}\right) - 3\right\}\right]$$

$$= \frac{1}{3^{k-1}}\left[\left(1+i\sqrt{2}\right)^{k-1}\left(-1+2i\sqrt{2}\right) + \left(1-i\sqrt{2}\right)^{k-1}\left(-1-2i\sqrt{2}\right)\right]$$

$$= \frac{1}{3^{k-1}}\left[\left(1+i\sqrt{2}\right)^{k-1}\left(1+i\sqrt{2}\right)^2 + \left(1-i\sqrt{2}\right)^{k-1}\left(1-2i\sqrt{2}\right)^2\right]$$

$$= \frac{1}{3^{k-1}}\left[\left(1+i\sqrt{2}\right)^{k+1} + \left(1-i\sqrt{2}\right)^{k+1}\right]$$

$$= \frac{2}{3^{k-1}} \times \frac{1}{2}\left[\left(1+i\sqrt{2}\right)^{k+1} + \left(1-i\sqrt{2}\right)^{k+1}\right]$$

$$3y_{k+1} = \frac{2}{3^{k-1}} x_{k+1}$$

$$y_{k+1} = \frac{2}{3^k} x_{k+1}$$

Hence by mathematical induction it is true for all $n \geq 0$.

ii) $y_n = \frac{2}{3^{n-1}} x_n = \frac{2}{3^{n-1}} \times \frac{1}{2}\left[\left(1+i\sqrt{2}\right)^n + \left(1-i\sqrt{2}\right)^n\right]$

$$= \frac{1}{3^{n-1}}\left[\left(1+i\sqrt{2}\right)^n + \left(1-i\sqrt{2}\right)^n\right]$$

$$= 3\left[\left(\frac{1+i\sqrt{2}}{3}\right)^n + \left(\frac{1-i\sqrt{2}}{3}\right)^n\right]$$

$$= 3\left[\left(\frac{1+i\sqrt{2}}{3} \times \frac{1-i\sqrt{2}}{1-i\sqrt{2}}\right)^n + \left(\frac{1-i\sqrt{2}}{3} \times \frac{1+i\sqrt{2}}{1+i\sqrt{2}}\right)^n\right]$$

$$= 3\left[\left(\frac{1}{1-i\sqrt{2}}\right)^n + \left(\frac{1}{1+i\sqrt{2}}\right)^n\right]$$

Since $(1+i\sqrt{2}) \times \left(1-i\sqrt{2}\right) = 3$.

Question 16

a) i) $a+b \geq 2\sqrt{ab}$

$$a^2+b^2 \geq 2ab$$
$$a^2+c^2 \geq 2ac$$
$$b^2+c^2 \geq 2bc$$
$$a^2+b^2+a^2+c^2+b^2+c^2 \geq 2ab+2ac+2bc$$
$$2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$$
$$a^2+b^2+c^2 \geq ab+ac+bc$$

ii) $a+b+c \geq 3\sqrt[3]{abc}$

$$\frac{a^3}{b-c}+\frac{b^3}{c-a}+(b-c)(c-a) \geq 3ab$$

$$\frac{a^3}{b-c}+\frac{c^3}{a-b}+(b-c)(a-b) \geq 3ac$$

$$\frac{b^3}{c-a}+\frac{c^3}{a-b}+(c-a)(a-b) \geq 3bc$$

$$2[\frac{a^3}{b-c}+\frac{b^3}{c-a}+\frac{c^3}{a-b}]+(b-c)(c-a)+(b-c)(a-b)+(c-a)(a-b)$$
$$\geq 3(ab+ac+bc)$$

Now $(b-c)(c-a)+(b-c)(a-b)+(c-a)(a-b) = bc-ab-c^2+ac+ab-b^2-ac+bc+ac-bc-a^2+ab = ab+ac+bc-(a^2+b^2+c^2)$

$$2[\frac{a^3}{b-c}+\frac{b^3}{c-a}+\frac{c^3}{a-b}]+ab+ac+bc-(a^2+b^2+c^2) \geq 3(ab+ac+bc)$$

$$2\left[\frac{a^3}{b-c}+\frac{b^3}{c-a}+\frac{c^3}{a-b}\right] \geq -(ab+ac+bc)+(a^2+b^2+c^2)+3(ab+ac+bc)$$
$$\geq 3(ab+ac+bc)$$
$$\frac{a^3}{b-c}+\frac{b^3}{c-a}+\frac{c^3}{a-b} \geq \frac{3}{2}(ab+ac+bc)$$

b) i) Let $f(x) = x-\ln(1+x)$

$$f'(x) = 1-\frac{1}{1+x} = \frac{x}{1+x} \geq 0, for\ x \geq 0$$

$f(x)$ is increasing function.

$$f'(x) = 0 \ \therefore x = 0$$
$$f''(x) = \frac{1}{(1+x)^2} > 0 \ \therefore \big(0, f(0)\big)\ is\ a\ minimum.$$
$$f(0) = 0\ which\ is\ minimum\ and\ f(x) is\ increasing\ for\ x > 0$$
$$f(x) > 0\ for\ x > 0.$$

ii) $x_n = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots(1+\frac{1}{3^n})$

$$x_{n+1} = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots(1+\frac{1}{3^n})(1+\frac{1}{3^{n+1}})$$

$$\frac{x_{n+1}}{x_n}=\frac{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots(1+\frac{1}{3^n})(1+\frac{1}{3^{n+1}})}{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots(1+\frac{1}{3^n})}=1+\frac{1}{3^{n+1}}>1$$

$$x_{n+1}>x_n.$$

iii) $\frac{x_{n+1}}{x_n}=1+\frac{1}{3^{n+1}}$, so $\frac{x_{k+1}-x_k}{x_k}=1+\frac{1}{3^{k+1}}-1=\frac{1}{3^{k+1}}$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{x_{k+1}-x_k}{x_k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3^{k+1}}=\frac{\frac{1}{3^2}}{1-\frac{1}{3}}=\frac{1}{6}$$

iv) $x_n=\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots(1+\frac{1}{3^n})$

$$\ln x_n=\ln\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\dots\left(1+\frac{1}{3^n}\right)=\ln\left(1+\frac{1}{3}\right)+\ln\left(1+\frac{1}{3^2}\right)+\dots+\ln\left(1+\frac{1}{3^n}\right)$$

Using b)(ii)

$$\ln x_n<\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^n}=\sum_{k=1}^{n}\frac{1}{3^k}$$

v) $\ln x_n<\sum_{k=1}^{n}\frac{1}{3^k}<\sum_{k=1}^{\infty}\frac{1}{3^k}=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}$

$$e^{\ln x_n}<e^{\frac{1}{2}}$$

$x_n<\sqrt{e}$.

The End.