STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

NORTH SYDNEY GIRLS' HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

4 UNIT ADDITIONAL PAPER

Time Allowed: 3 hours plus 5 minutes reading time

This paper contains 8 questions.
ALL questions may be attempted.

Note: The questions are NOT necessarily arranged in order of difficulty.
ALL necessary working should be shown.
Candidates are advised to read the whole paper carefully at the start of the examination.
All questions are of approximately equal value.
Start each question on a new page, clearly marked Question 1, Question 2, etc.
Standard Integrals are provided on the last page.
Approved silent calculators may be used.

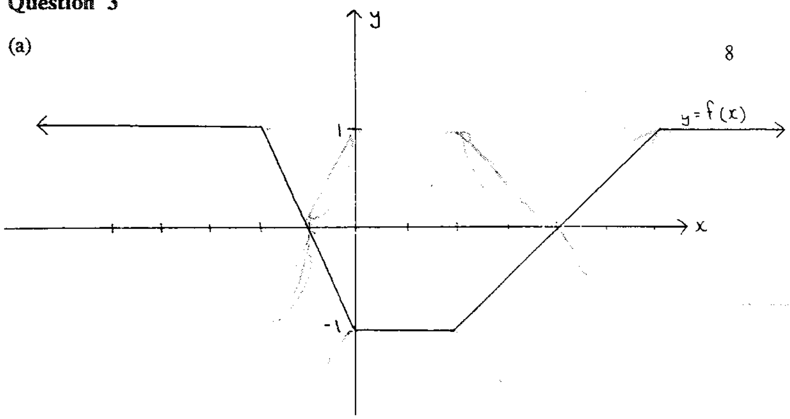
This is a trial paper ONLY. The content and format of this paper do not necessarily reflect the content and format of the final Higher School Certificate examination paper

31 July 1997/pd/Trial HSC Maths 4U

Question 3

Marks

(a) 8



The diagram above is a sketch of the function $y = f(x)$. On separate diagrams sketch

(i) $y = |f(x)|$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = [f(x)]^2$

(iv) $y^2 = f(x)$

(b) let $f(x) = \log_e(1+x) - \log_e(1-x), \ -1 < x < 1$ 7

(i) Show that $f'(x) > 0$ for $-1 < x < 1$

(ii) Draw a neat sketch of $y = f(x), \ -1 < x < 1$

(iii) Find an expression for the inverse function $y = f^{-1}(x)$

QUESTION 1.

Marks

(a) Find $\int \frac{Sin^{-1}x}{\sqrt{1-x^2}}\,dx$ 2

(b) Find $\int Sin^3 x \ dx$ 2

(c) Find $\int (\log_e x)^2 dx$ 3

(d) Using the substitution $t = \tan\frac{\theta}{2}$ or otherwise calculate $\int_0^{\frac{\pi}{3}} \frac{d\theta}{2 + 2Cos\theta}$ 4

(e) Evaluate $\int_0^2 \frac{dx}{(x+1)(x^2+4)}$ 4

Question 2.

(a) Given that a and b are real numbers express in the form $x+iy$ where x and y are real 4

(i) $(a+bi)\overline{(2-i)}$

(ii) $\dfrac{a+bi}{4+3i}$

(b) (i) Find all pairs of integers x and y such that 5

$$(x+iy)^2 = -3-4i$$

(ii) Hence or otherwise solve the quadratic equation

$$z^2 - 3z + (3+i) = 0$$

(c) Express $z = \sqrt{2} - i\sqrt{2}$ in modulus-argument form. Hence write z^{22} in the form $a + ib$ where a and b are real. 3

(d) On separate diagrams draw a neat sketch of the locus specified by 3

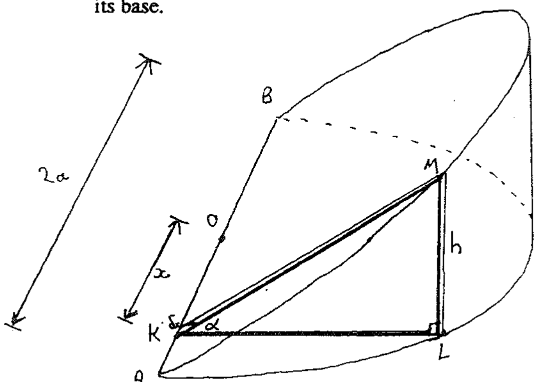
(i) $\arg\ (z-2i) = \dfrac{\pi}{6}$

(ii) $\text{Re}(z^2) = 4$

Question 6

Marks

(a) The solid S is a wedge that has been obtained by slicing a right circular cylinder radius a at an angle α through diameter AB at its base. 9



Consider a thin slice of thickness δx which is perpendicular to the base and the line AB and positioned at a distance x from the centre.

Let $KO = x$, $OB = OA = a$, $\angle MKL = \alpha$ and $ML = h$

(i) Show that the triangular slice has height

$$h = \sqrt{a^2 - x^2}\ \tan\alpha$$

(ii) Show that the volume of the triangular slice is given by

$$\delta v = \frac{\tan\alpha}{2}\left(a^2 - x^2\right)\delta x$$

(iii) Find the volume of S

(d) (i) Simplify $\dfrac{r\binom{n}{r}}{\binom{n}{r-1}}$ 6

(ii) Hence or otherwise show that

$$\frac{\binom{n}{1}}{\binom{n}{0}} + \frac{2\binom{n}{2}}{\binom{n}{1}} + \frac{3\binom{n}{3}}{\binom{n}{2}} + \ldots\ldots\ldots\ldots\ldots + \frac{n\binom{n}{n}}{\binom{n}{n-1}} = \frac{1}{2}n(n+1)$$

Question 4

Marks

(a) The area between the curve $y = 4x - x^2$ and the $x-axis$ is rotated about the $y-axis$. Use the method of cylindrical shells to evaluate this ~~area~~ volume. 5

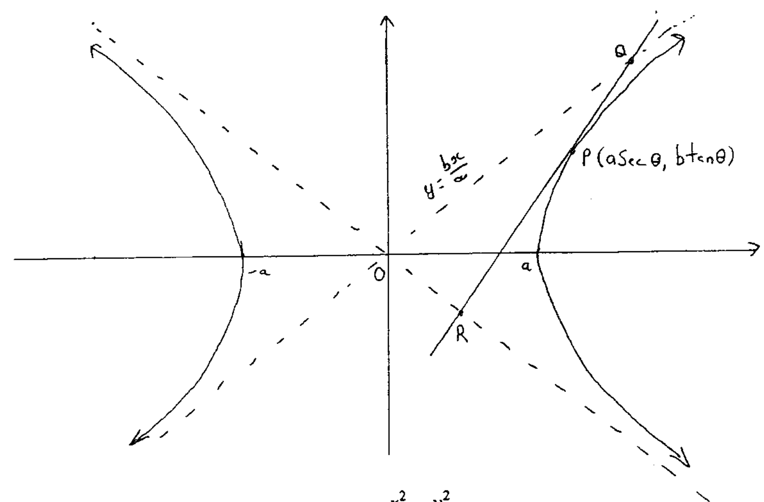
(b) Prove by induction $\dfrac{3^n + 5^n}{2} \geq 4^n$ for $n \geq 1$ 5

(c) Find all x such that 5

$$Cos\ x + Sin\ x = 1 + Sin\ 2x \text{ and } 0 \leq x \leq 2\pi$$

Question 5

(a) 11



P is the point $(aSec\theta,\ b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i) Derive the equation at the tangent at P.

(ii) If the tangent meets the asymptote $y = \dfrac{b}{a}x$ at Q, prove that Q is the point $\left[a(Sec\theta + \tan\theta),\ b(Sec\theta + \tan\theta)\right]$

(iii) If the tangent at P meets the other asymptote at R, prove that P is the midpoint at QR

(iv) If P is equidistant from Q, R and O (the origin) prove that the hyperbola is rectangular.

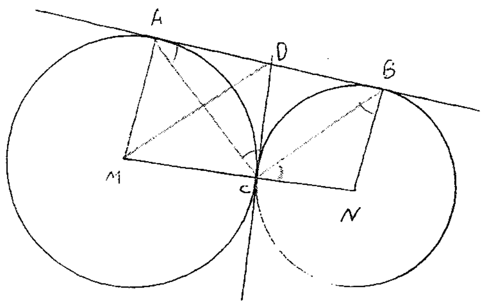
(b) Find the equation at the tangent to $y = x^3$ at the point (t, t^3) 4

Hence deduce the values of k for which the equation $x^3 = kx - 2$ will have 3 real and distinct roots.

Question 7

Marks

(a) 9



In the diagram MCN is a straight line. Circles are drawn with centre M radius MC and centre N radius NC. AB is a common tangent to the circles with points of contact at A and B respectively. CD is a common tangent at C and meets AB at D.

(i) Copy the diagram

(ii) Explain why AMCD and BNCD are cyclic quadrilaterals

(iii) Show that Δ ACD is similar to Δ CBN

(iv) Show that MD is parallel to CB

(b) For the function $f(x) = \log_e\left(x + \sqrt{x^2+1}\right)$ 6

(i) State the domain of f. Explain your answer.

(ii) Show that $f(x)$ is odd

(iii) Evaluate $\int_{-2}^{2} \log_e(x + \sqrt{x^2+1})dx$

Question 8

Marks

(a) (i) Show that $\frac{(1-t)^n}{t} = t^{n-1}\left(\frac{1}{t} - 1\right)^n$ 6

(ii) Let $I_n = \int_1^x \frac{(1-t)^n}{t}dt, \ n = 1,2,3......$

Show that $I_n = \frac{(1-x)^n}{n} + I_{n-1}$

(b) The point $P(aCos\theta, bSin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ The tangent at P cuts the $y-axis$ at Q and the normal at P cuts the $y-axis$ at R. Find the co-ordinates at Q and R. 9

If S is the focus $(ae, 0)$prove that the points P, Q, R, S are concyclic.

...tion 1

$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

$\boxed{\frac{1}{2}(\sin^{-1} x)^2 + c}$

let $u = \sin^{-1} x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$ ②

$= \int u\,du$

$= \frac{1}{2}u^2 + c$

$= \frac{1}{2}[\sin^{-1} x]^2 + c$

$\int \sin^3 x\,dx$

$= \int \sin x(1-\cos^2 x)\,dx$ ②

$= \int \sin x - \sin x\cos^2 x\,dx$

$\boxed{= -\cos x + \frac{1}{3}\cos^3 x + c}$

$\int (\log_e x)^2\,dx$

$= \int \frac{d}{dx}(x)\cdot(\log_e x)^2\,dx$ ③

$= [x(\ln x)^2] - \int x \cdot 2 \cdot \frac{1}{x}\ln x\,dx$

$= [x(\ln x)^2] - 2\int \frac{d}{dx}(x)\cdot \ln x\,dx$

$= x(\ln x)^2 - 2\{[x\ln x] - \int x\cdot\frac{1}{x}\,dx\}$

$\boxed{= x(\ln x)^2 - 2x\ln x + 2x + c}$

$\int_0^{\frac{\pi}{3}} \frac{d\theta}{2+2\cos\theta}\,d\theta$

$t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1} t$

$d\theta = \frac{2\,dt}{1+t^2}$

$\theta = 0,\ t = 0$ ④

$\theta = \frac{\pi}{3},\ t = \frac{1}{\sqrt{3}}$

$\therefore \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{2+2\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2\,dt}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{2(1+t^2)+2(1-t^2)}\,dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{2}$

$= \left[\frac{t}{2}\right]_0^{\frac{1}{\sqrt{3}}}$

$\boxed{= \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}}$

(e) $\int_0^2 \frac{dx}{(x+1)(x^2+4)}$

$\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

$1 = a(x^2+4) + (bx+c)(x+1)$

$\therefore 5a = 1$

$\therefore a = \frac{1}{5}$

$a + b = 0$

$\therefore b = -\frac{1}{5}$

$4a + c = 1$

$c = \frac{1}{5}$ ④

$= \frac{1}{5}\left[\int_0^2 \frac{dx}{x+1} + \int_0^2 \frac{-x+1}{x^2+4}\,dx\right]$

$= \frac{1}{5}\left[\int_0^2 \frac{1}{x+1} - \frac{x}{x^2+4} + \frac{1}{x^2+4}\right]dx$

$= \frac{1}{5}\left[\ln(x+1) - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{5}\left[\left(\ln 3 - \frac{1}{2}\ln 8 + \frac{1}{2}\tan^{-1} 1\right) - \left(\ln 1 - \frac{1}{2}\ln 4\right)\right]$

$= \frac{1}{5}\left[\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{8}\right]$

Question 2

(a) (i) $(a+bi)(\overline{2-i})$

$= (a+bi)(2+i)$

$\boxed{= (2a-b) + i(a+2b)}$ ②

(ii) $\frac{(a+bi)}{(4+3i)} \times \frac{(4-3i)}{(4-3i)}$

$= \frac{(4a+3b) + i(4b-3a)}{16+9}$ ②

$\boxed{= \left(\frac{4a+3b}{25}\right) + i\left(\frac{4b-3a}{25}\right)}$

(b) (i) $(x+iy)^2 = -3-4i$

$(x^2-y^2) + 2xyi = -3-4i$

$x^2 - y^2 = -3 \quad xy = -2$

$x^2 - \left(\frac{-2}{x}\right)^2 = -3$

$x^4 - 4 = -3x^2$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0$

$\therefore x = \pm 1$

$\therefore y = \mp 2$ ②

i.e. $\boxed{(1-2i),\ (-1+2i)}$

(ii) $z^2 - 3z + (3+i) = 0$

$z = \frac{3 \pm \sqrt{9-4(3+i)}}{2}$

$= \frac{3 \pm \sqrt{-3-4i}}{2}$

$= \frac{3 \pm (1-2i)}{2}$

$= \frac{4-2i}{2},\ \frac{2+2i}{2}$

$\boxed{z = 2-i,\ 1+i}$ ③

Q2 CTD

(c) $z = \sqrt{2} - i\sqrt{2}$

$|z| = \sqrt{2+2} = 2$

$\arg z = \tan^{-1}(-1)$ in 4th quadrant

$\therefore \arg z = -\frac{\pi}{4}$

$\therefore z = \boxed{2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]}$

$z^{22} = 2^{22}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]^{22}$

$= 2^{22}\left[\cos\left(\frac{-11\pi}{2}\right) + i\sin\left(\frac{-11\pi}{2}\right)\right]$

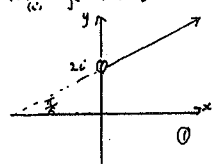
$= 2^{22}\left[\cos\frac{11\pi}{2} - i\sin\frac{11\pi}{2}\right]$

$= 2^{22}\left[\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}\right]$

$= 2^{22}[0 - i\times -1]$ ③

$= \boxed{2^{22}i = 4194304i}$

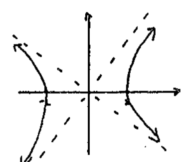
(d) (i) $\arg(z - 2i) = \frac{\pi}{6}$



①

(ii) $\text{Re}\ z^2 = 4$

$\therefore x^2 - y^2 = 4$

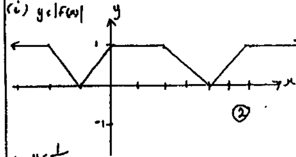


②

QUESTION 3

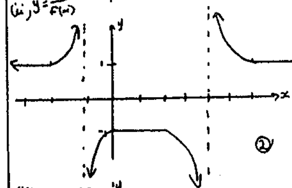
(a)

(i) $y = |f(x)|$



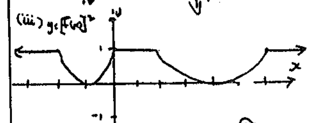
②

(ii) $y = \frac{1}{f(x)}$



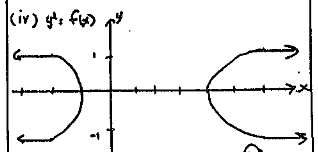
②

(iii) $y = [f(x)]^2$



②

(iv) $y^2 = f(x)$



②

(b) $f(x) = \ln(1+x) - \ln(1-x) \quad -1 < x < 1$

$f'(x) = \frac{1}{1+x} - \frac{-1}{1-x}$

$= \frac{2}{1-x^2}$

$1 - x^2 > 0$ for $-1 < x < 1$ ②

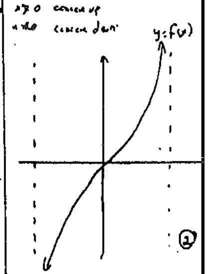
$\therefore f'(x) > 0$ for all $-1 < x < 1$

(ii) $-1 < x < 1$

$f''(x) = \frac{4x}{(1-x^2)^2}$

$x > 0$ concave up

$x < 0$ concave down



②

(iii) $y = \ln\left(\frac{1+x}{1-x}\right)$

Inverse exists for all x (from ...

Inverse $x = \ln\left(\frac{1+y}{1-y}\right)$

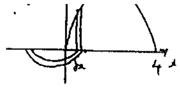
$\frac{1+y}{1-y} = e^x$

$1 + y = e^x - ye^x$

$y(e^x + 1) = e^x - 1$

$\therefore y = \frac{e^x - 1}{e^x + 1}$ ③

$\therefore \boxed{f^{-1}(x) = \frac{e^x - 1}{e^x + 1}}$



Area base (annulus)

$= \pi(x+\delta x)^2 - \pi x^2$

$= 2\pi x\delta x + \pi \delta x^2 \quad \delta x^2 \to 0$

$= 2\pi x \delta x$

$V = 2\pi x \delta x y$

$\lim_{\delta x \to 0} \sum_{x=0}^{4} 2\pi x \delta x y$

$\int_0^4 2\pi x y \, dx$

$\int_0^4 x(bx - x^2)dx$

$2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^4$

$2\pi\left[\frac{4}{3} \cdot 64 - 64\right]$

$2\pi \times \frac{64}{3}$

$\boxed{\frac{128\pi}{3} \text{ units}^3}$ ⑤

$\frac{3^n + 5^n}{2} \geq 4^n$

LHS: $\frac{3+5}{2} = 4$

RHS: $4^1 = 4$

true for $n=1$

Assume true for $n=k$

$\frac{3^k + 5^k}{2} \geq 4^k$

$\frac{3^k + 5^k}{2} - 4^k \geq 0$

and to show it is true for the term i.e. $n = k+1$

$\frac{3^{k+1} + 5^{k+1}}{2} - 4^{k+1} \geq 0$

$= \frac{3 \cdot 3^k + 5 \cdot 5^k}{2} - 4 \cdot 4^k$ ⑤

$= \frac{3 \cdot 3^k + 5 \cdot 5^k - 8 \cdot 4^k}{2}$

$= \frac{3[3^k + 5^k - 2 \cdot 4^k] + 2 \cdot 5^k - 2 \cdot 4^k}{2}$

$= 3\left[\frac{3^k + 5^k}{2} - 4^k\right] + (5^k - 4^k)$

True by assumption. Since $4 < 5$, $5^k > 4^k$, $\therefore 5^k - 4^k > 0$

∴ True for next term.

∴ True for all $n \geq 1$ by induction

(c) $\cos x + \sin x = 1 + \sin 2x$

$= 1 + 2\sin x \cos x$

$\cos x + \sin x = (\cos x + \sin x)^2$

∴ ① $\cos x + \sin x = 1$ or ② $\cos x + \sin x = 0$

② $\sin x = -\cos x$

$\tan x = -1$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

① $\cos x + \sin x = 1$

$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$

$1 - t^2 + 2t = 1 + t^2$

$2t^2 - 2t = 0$

$2t(t-1) = 0$

∴ $t = 0, \ t = 1$

∴ $\tan\frac{x}{2} = 0, \ \tan\frac{x}{2} = 1 \quad 0 \leq \frac{x}{2} \leq \pi$

∴ $\frac{x}{2} = 0, \pi, \frac{\pi}{4}$

$x = 0, 2\pi, \frac{\pi}{2}$ ⑤

∴ $\boxed{x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi}$

$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$

∴ $\frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{b \sec\theta}{a \tan\theta}$

Eqn: $y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$\boxed{bx\sec\theta - ay\tan\theta = ab}$ ②

(ii) $y = \frac{bx}{a}$

at Q: $bx\sec\theta - a\tan\theta \cdot \frac{bx}{a} = ab$

$bx(\sec\theta - \tan\theta) = ab$

∴ $x = \frac{a}{\sec\theta - \tan\theta} \times \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right)$

$= \frac{a(\sec\theta + \tan\theta)}{\sec^2\theta - \tan^2\theta}$

$= a(\sec\theta + \tan\theta)$

∴ $y = \frac{b}{a} \cdot a(\sec\theta + \tan\theta)$

$= b(\sec\theta + \tan\theta)$ ③

∴ $\boxed{Q\left[a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)\right]}$

(iii) at R: $y = -\frac{bx}{a}$

$bx\sec\theta - a\tan\theta \cdot \frac{-bx}{a} = ab$

∴ $bx(\sec\theta + \tan\theta) = ab$

∴ $x = \frac{a}{\sec\theta + \tan\theta} \times \left(\frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}\right)$

∴ $x = a(\sec\theta - \tan\theta)$

∴ $y = -\frac{b}{a} \cdot a(\sec\theta - \tan\theta)$

∴ $R\left[a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)\right]$

... midpoint QR

∴ $\left[\frac{a(\sec\theta + \tan\theta + \sec\theta - \tan\theta)}{2}, \frac{b(\sec\theta + \tan\theta - \sec\theta + \tan\theta)}{2}\right]$

$= \left(\frac{2a\sec\theta}{2}, \frac{2b\tan\theta}{2}\right)$

$= (a\sec\theta, b\tan\theta)$ ③

i.e. P is midpoint of QR.

(iv) $OP^2 = a^2\sec^2\theta + b^2\tan^2\theta$

$PQ^2 = (a\tan\theta)^2 + (b\sec\theta)^2$

$= a^2\tan^2\theta + b^2\sec^2\theta$

$OP = PQ$

∴ $a^2\sec^2\theta + b^2\tan^2\theta = a^2\tan^2\theta + b^2\sec^2\theta$

$a^2(\sec^2\theta - \tan^2\theta) = b^2(\sec^2\theta - \tan^2\theta)$

$\sec^2\theta - \tan^2\theta = 1$

∴ $a^2 = b^2$

∴ $a = b$ since $a, b > 0$

∴ hyperbola rectangular. ③

(b) $y = x^3$

$y' = 3x^2$ at (t, t^3)

$y' = 3t^2$

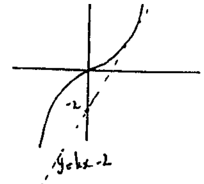
Eqn: $y - t^3 = 3t^2(x - t)$

$y - t^3 = 3t^2x - 3t^3$

$\boxed{y = 3t^2x - 2t^3}$ ①

For 3 solutions the graphs $y = x^3$ and $y = kx - 2$ must have 3 points of intersection

$y = kx - 2$ passes through $(0, -2)$



3 solutions if gradient (k) > gradient of tangent

$(0,-2) \to y = 3t^2x - 2t^3$

$-2 = -2t^3$

∴ $\boxed{t = 1}$ for tangent through $(0,-2)$

i.e. $y = 3x - 2$ ③

i.e. $\boxed{k > 3 \text{ for 3 solutions}}$

Question 6

(a)(i) $OL^2 = OK^2 + KL^2$

∴ $KL^2 = OL^2 - OK^2$

$= a^2 - x^2$

∴ $KL = \sqrt{a^2 - x^2}$

∴ $\frac{h}{KL} = \tan\alpha$

$h = KL\tan\alpha$

$\boxed{h = \sqrt{a^2 - x^2}\tan\alpha}$ ③

(ii) $\text{Area}_\Delta = \frac{KL \cdot h}{2}$

$= \frac{\sqrt{a^2 - x^2}}{2} \cdot \sqrt{a^2 - x^2}\tan\alpha$

$= \frac{1}{2}(a^2 - x^2)\tan\alpha$

∴ $\delta V = \text{area} \times \delta x$ ③

$= \boxed{\frac{1}{2}(a^2 - x^2)\tan\alpha \, \delta x}$

$\lim_{\delta x \to 0} \sum_{x=-a}^{a} \frac{\tan\alpha}{2}(a^2 - x^2)\delta x$

$= \int_{-a}^{a} \frac{\tan\alpha}{2}(a^2 - x^2)\,dx$

$= 2 \cdot \frac{\tan\alpha}{2} \int_0^a (a^2 - x^2)dx$

$= \tan\alpha\left[a^2x - \frac{x^3}{3}\right]_0^a$

$= \tan\alpha\left[\left(a^3 - \frac{a^3}{3}\right) - (0)\right]$

$= \boxed{\frac{2a^3\tan\alpha}{3}}$ ③

(b)

(i) $\frac{r\binom{n}{r}}{\binom{n}{r-1}} = r \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}}$

$= \frac{r \cdot n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!}$

$= \frac{(n-r+1)(n-r)!}{(n-r)!}$

$= \boxed{n - r + 1}$ ③

(ii) $\frac{\binom{n}{1}}{\binom{n}{0}} + \frac{2\binom{n}{2}}{\binom{n}{1}} + \frac{3\binom{n}{3}}{\binom{n}{2}} + \cdots + \frac{n\binom{n}{n}}{\binom{n}{n-1}}$

$= \sum_{r=1}^{n} \frac{r\binom{n}{r}}{\binom{n}{r-1}}$

$= \sum_{r=1}^{n} (n - r + 1)$

$= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$

i.e. AP. n terms $a = n, \ell = 1$

∴ $= \boxed{\frac{n}{2}(n+1)}$ ③

Question 7

(a)(i) $\angle MAO = 90°$ (∠ between tangent, radius)

$\angle MCO = 90°$ (" ")

∴ AMCO is cyclic (opposite ∠'s = 180°)

Similarly $\angle DCN = \angle DBN = 90°$ ②

∴ BNCD is cyclic

(ii) In Δ's ACD and CBN

$\angle ADC = \angle CNB$ (exterior ∠ of cyclic quad = opposite interior)

$CN = BN$ (radii of circle)

$AD = CD$ (= tangents from D)

∴ ΔACD ||| ΔCBN (2 sides in same ratio, 1:1, included angle equal) ④

(iv) let $\angle BCN = x$

$\angle CAD = x$ (corresponding ∠'s in ≡ Δ's)

$\angle CMO = \angle CAO = x$ (angles in same segment of chord CO, AMCO cyclic)

∴ $\angle BCN = \angle DMC$

∴ MO || BC (corresponding ∠s equal) ③

(b)(i) Domain $x + \sqrt{x^2+1} > 0$

but $x^2 + 1 > x^2$

∴ $\sqrt{x^2+1} > x$ for all x

∴ Domain: all real x ②

(ii) $f(-x) = \log(-x + \sqrt{x^2+1})$

$= \log\left[(\sqrt{x^2+1} - x) \times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right]$

$= \log\left[\frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x}\right]$

$= \log\left[\frac{1}{x + \sqrt{x^2+1}}\right]$

$= \log\left[(x + \sqrt{x^2+1})^{-1}\right]$

$= -\log(x + \sqrt{x^2+1})$

$= -f(x)$

∴ $f(-x) = -f(x)$ ∴ odd ③

∴ $\int_{-1}^{1} \log(x + \sqrt{x^2+1})\,dx = 0$

Since odd function ①

Question 8

(a)(i) $\frac{(1-t)^n}{t} = \frac{\left[t\left(\frac{1}{t} - 1\right)\right]^n}{t}$

$= \frac{t^n}{t}\left(\frac{1}{t} - 1\right)^n$

$= t^{n-1}\left(\frac{1}{t} - 1\right)^n$ ①

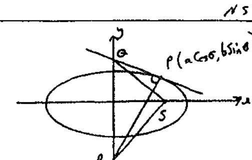
(ii) $I_n = \int_1^x \frac{(1-t)^n}{t}\,dt$

$= \int_1^x t^{n-1}\left(\frac{1}{t} - 1\right)^n dt$

$= \int_1^x \frac{d}{dt}\left(\frac{t^n}{n}\right)\left(\frac{1}{t} - 1\right)^n dt$

$= \left[\left(\frac{t^n}{n}\right)\left(\frac{1}{t} - 1\right)^n\right]_1^x - \int_1^x \left(\frac{t^n}{n}\right) n \left(\frac{1}{t} - 1\right)^{n-1} \cdot \frac{-1}{t^2}$

$\left[\frac{(1-t)^n}{n}\right]_1^x + \int_1^x t^{n-2}\left(\frac{1}{t} - 1\right)^{n-1} dt$



$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b\cos\theta}{a\sin\theta}$

Eqn PQ: $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

at Q, $x = 0$ ∴ $y = b\sin\theta + \frac{b\cos^2\theta}{\sin\theta}$

$= \frac{b}{\sin\theta}$

∴ $\boxed{Q\left(0, \frac{b}{\sin\theta}\right)}$ ②

Normal PR: $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

at R, $x = 0$: $y = b\sin\theta - \frac{a^2\sin\theta}{b}$

∴ $\boxed{R\left(0, \frac{(b^2 - a^2)\sin\theta}{b}\right)}$ ②

$m_{QS} = \frac{\frac{b}{\sin\theta} - 0}{0 - ae} = \frac{-b}{ae\sin\theta}$

$m_{RS} = \frac{\frac{(b^2 - a^2)\sin\theta}{b} - 0}{0 - ae} = \frac{(b^2 - a^2)\sin\theta}{-abe}$

but $b^2 = a^2(1 - e^2)$

$b^2 - a^2 = -a^2e^2$

∴ $m_{RS} = \frac{-a^2e^2\sin\theta}{-abe} = \frac{ae\sin\theta}{b}$

$m_{QS} \times m_{RS} = \frac{-b}{ae\sin\theta} \times \frac{ae\sin\theta}{b} = -1$ ③

∴ QS ⊥ RS

∴ $\angle QRS = 90°$