NORTH SYDNEY GIRLS HIGH SCHOOL

2008
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

Student Number: ____________________ **Teacher:** ____________________________

Student Name: __________________________________

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120
	%

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx$ **1**

(b) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. **3**

(c) (i) Express $\dfrac{3x^2+x+6}{(x^2+9)(x-1)}$ in the form $\dfrac{Ax+B}{x^2+9}+\dfrac{C}{x-1}$. **2**

(ii) Hence find $\displaystyle\int \frac{3x^2+x+6}{\left(x^2+9\right)\left(x-1\right)}\,dx$ **1**

(d) State whether each of the following is true or false giving brief reasons.

(i) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx = 0$ **1**

(ii) $\displaystyle\int_{-a}^{a} e^{-x^2}\,dx \le 2a$ **2**

(e) (i) Show that $\tan(\sin^{-1} e^x) = \dfrac{e^x}{\sqrt{1-e^{2x}}}$, $x < 0$ **2**

(ii) Hence or otherwise find $\displaystyle\int \tan(\sin^{-1} e^x)\,dx$ **3**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Given $z = \frac{1}{2}(1+i)$

(i) Find the values of

(α) $|z|$ **1**

(β) $\arg z$ **1**

(γ) $|z^2|$ **1**

(δ) $\arg(z^2)$ **1**

(ii) On an Argand diagram, show the complex number which represents **1**

$\overline{1+z+z^2}$

(b) Solve the equation $z\overline{z} - 2z + 2\overline{z} = 5 - 4i$ given $z = x + iy$. **2**

(c) On separate Argand diagrams, sketch the loci of z where

(i) $|z-1| > |z|$ **2**

(ii) $|z+1| + |z-1| = 4$ [Show the important features on the x-axis only] **3**

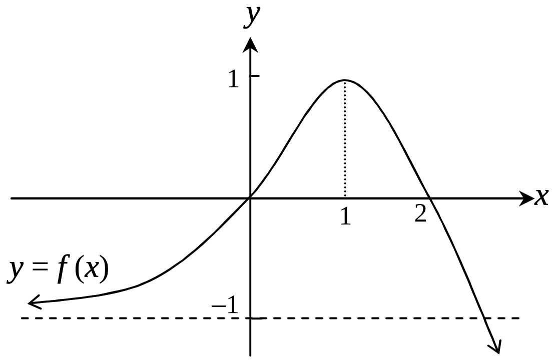
(d) Let α and β be the roots of the equation $x^2 - 2x + 4 = 0$. Prove that **3**

$$\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$$

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is illustrated. The line $y = -1$ is a horizontal asymptote.



Using the separate page of graphs provided, sketch each of the graphs below. In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = f(-x)$ **1**

(ii) $y = f\left(|x|\right)$ **2**

(iii) $y = \left[f(x)\right]^2$ **2**

(iv) $y = \tan^{-1}\left\{f(x)\right\}$ **2**

(v) $y = e^{f(x)}$ **2**

(b) Consider the function $f\left(x\right) = \cos(2\sin^{-1}\frac{x}{2})$.

(i) Express $f(x)$ in the form of a polynomial function **1**

(ii) Sketch the graph of $y = f(x)$ **2**

(c) The polynomial $1 + BX^n + AX^{n+1}$ is divisible by $\left(X-1\right)^2$. Show that

(i) $A = n$ **2**

(ii) $B + n + 1 = 0$ **1**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The area enclosed by the curve $y = e^x$, the y-axis and the line $y = 2$ is rotated about the y-axis to form a solid.

(i) Draw a diagram to illustrate the region. **1**

(ii) By taking strips parallel to the y-axis and using the method of cylindrical shells, show that the volume of a shell of the solid is given by **2**

$$\Delta V \doteqdot 2\pi x\left(2 - e^x\right)\Delta x$$

(iii) Hence, find the volume of the solid correct to 2 significant figures. **2**

(b) (i) Prove by Mathematical Induction for integers $n \geq 1$ that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots.. + \frac{1}{n^2} \leq \frac{2n-1}{n}.$$ **3**

(ii) Hence show that

$$1\tfrac{12}{25} \leq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots + \frac{1}{49^2} \leq 1\tfrac{48}{49}.$$ **2**

(c) (i) If $I_n = \int_0^1 \left(1 - x^2\right)^n dx$, show that $I_n = \frac{2n}{2n+1} I_{n-1},\ n \geq 1.$ **3**

Hint: In your response, you may use the fact that $x^2 = 1 - \left(1 - x^2\right)$.

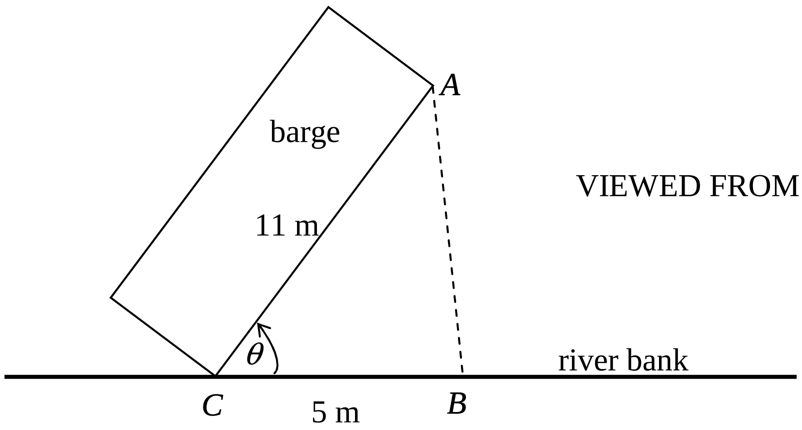
(ii) Hence find $\int_0^1 \left(1 - x^2\right)^3 dx$. **2**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A barge of side length 11 metres is moored so that it touches the edge of a river bank. The barge has an extendable yet rigid rope *AB* which attaches it to the river bank at a point *B*, 5 metres further down the bank from *C* as shown below. The point *C* is fixed.

As the barge steadily sways, the area of the triangular region *ABC* varies as angle θ increases at a rate of 0·1 radians per day.



(i) Show that the area of ΔABC is given by $A = \dfrac{55\sin\theta}{2}$. **2**

(ii) Find an expression for $\dfrac{dA}{dt}$, the rate of change of the area of ΔABC. **2**

(iii) Show that when $\theta = \dfrac{\pi}{4}$, $\dfrac{dA}{dt}$ is $\dfrac{11\sqrt{2}}{8}$ m^2 per day. **2**

(b) Given that $a^2 + b^2 \geq 2ab$ prove that for *a*, *b* and *c* positive

(i) $a^3 + b^3 \geq ab(a+b)$ **2**

(ii) $(a+b)(b+c)(c+a) \geq 8abc$ **2**

(c) (i) Prove from first principles that, in general, a function $f(x)$ which is ODD and differentiable for all *x* has a gradient function, $f'(x)$ which is EVEN. **3**

(ii) Explain with the help of a graph why the function $f(x)$, which is odd and continuous for all *x* satisfies the equality **2**

$$\int_{-a}^{b} f(x)\,dx = \int_{a}^{b} f(x)\,dx$$

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

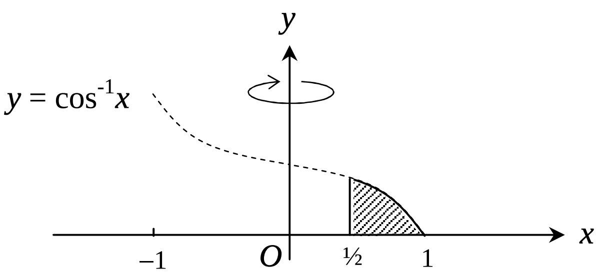
(a) (i) Show that $x^3 - 9x - 12 = 0$ can be transformed into the equation $Y^6 - 12Y^3 + 27 = 0$ by using the substitution $x = Y + \frac{3}{Y}$. **3**

(ii) Hence solve for Y^3. **1**

(iii) Show that $x = Y + Y^2$ for one of the solutions of Y^3 **1**

(iv) Hence, or otherwise, show that one of the roots of $x^3 - 9x - 12 = 0$ is $\sqrt[3]{3}\left(1 + \sqrt[3]{3}\right)$. **1**

(b) The region bounded by $y = \cos^{-1} x$, $x = \frac{1}{2}$ and the x-axis is rotated about the y-axis to generate a solid S as shown below.



(i) Show that an annular cross-section of the solid, parallel to the x-axis, with height Δy has its volume given by **2**

$$\Delta V \doteqdot \pi(\cos^2 y - \frac{1}{4})\Delta y$$

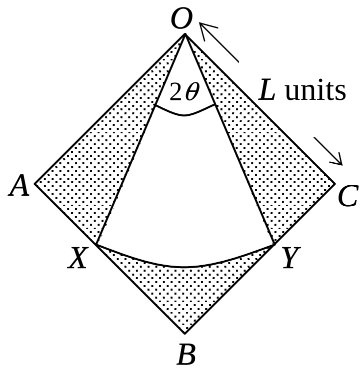
(ii) Hence find the volume of the solid S. **3**

Question 6 continues on page 8

Question 6 (continued)

(c) The diagram of the emblem below is composed of a sector inscribed in a square. The square has side length L and the circular sector has an angle of 2θ.

$\Delta OAX \equiv \Delta OCY$. The area of sector OXY is $\dfrac{L^2}{2}$.



Show that

(i) The radius of the sector is $L\sec\left(\dfrac{\pi}{4}-\theta\right)$. **2**

(ii) $2\theta = \cos^2\left(\dfrac{\pi}{4}-\theta\right)$. **2**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Eight children wish to take seats in an eight seated merry -go-round. There are four boys and four girls in this group. One girl does not want to sit next to any boy and one boy does not want to sit next to any girl. How many seating arrangements are possible? **3**

(b) Prove that

(i) $\tan x > x$ for $0 < x < \frac{\pi}{2}$ **3**

(ii) $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$ **4**

(c) Farmer Jee plants wheat in a field. By evenly spreading q kilograms of fertiliser over the wheat, a yield of Y kilograms of wheat per hectare, is produced.

The amount of wheat produced, Y kg, is a function the amount of fertiliser, q kg, spread and is given by the growth equation:

$$Y = 1650 - Ae^{kq}$$

After 10kg of fertiliser was spread, a yield of 550kg of wheat was produced. [Assume when $q = 0$, $Y = 0$]

(i) Show that $Y = 1650\left(1 - e^{-0\cdot 0405q}\right)$. **2**

(ii) Show that $\frac{dY}{dq} = 0\cdot 0405\,(1650 - Y)$. **1**

(iii) When the 550kg of wheat was produced, it was noted that the fertiliser had been spread at a rate of 10 kilograms per hour to achieve this yield. Find the corresponding rate of increase of the yield of wheat over this time. **2**

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the rectangular hyperbola with equation $xy = 16$.

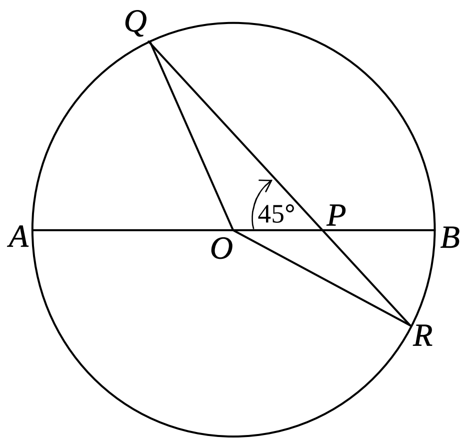
(i) Show that the equation of the tangent to the point $P\left(4p, \frac{4}{p}\right)$ on $xy = 16$ is given by $x + p^2 y = 8p$. **2**

(ii) If the tangents at $P\left(4p, \frac{4}{p}\right)$ and $Q\left(4q, \frac{4}{q}\right)$ meet at $R(x_0, y_0)$ prove that $pq = \dfrac{x_0}{y_0}$ and that $p + q = \dfrac{8}{y_0}$. **3**

(iii) Show that $PQ^2 = 16(p-q)^2\left[1 + \dfrac{1}{p^2q^2}\right]$. **1**

(iv) If the length of chord PQ is 8 units deduce that the locus of R is given by $(x^2 + y^2)(16 - xy) = x^2 y^2$. **2**

(b) Circle centre O has a diameter AB which intersects the chord QR at P at an angle of 45°, as shown.



(i) Copy or trace the diagram onto your page and construct $OT \perp AB$ given that T lies on QP.

(ii) Use congruent triangles to show that $QT = PR$. **2**

(iii) Show that $\left(\dfrac{AB}{2} - OP\right)\left(\dfrac{AB}{2} + OP\right) = PQ \times PR$. **2**

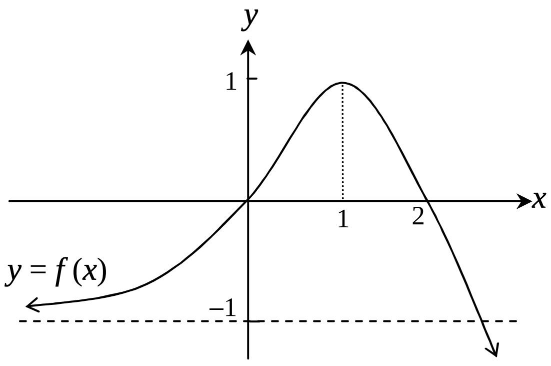
(iv) By considering the expansion of $(PQ - PR)^2$ and the results from (ii) and (iii) above show that $AB^2 = 2\left(PQ^2 + PR^2\right)$. **3**

End of paper

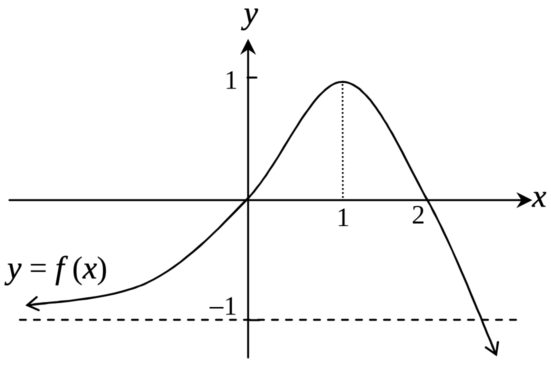
Student Number:______________________________

Use this page for your answers to Question 3a)

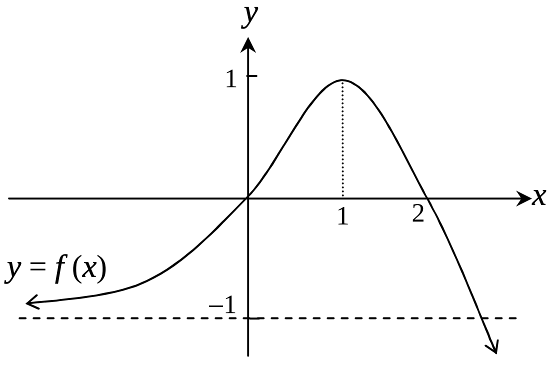
i)



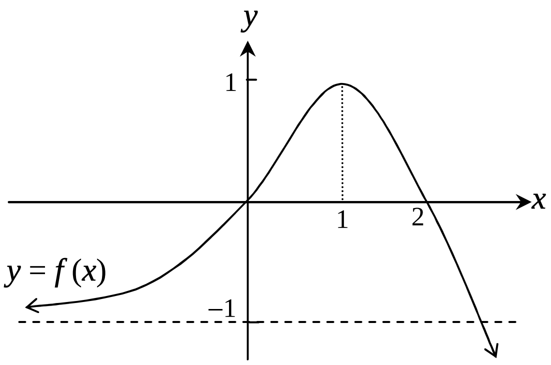
ii)



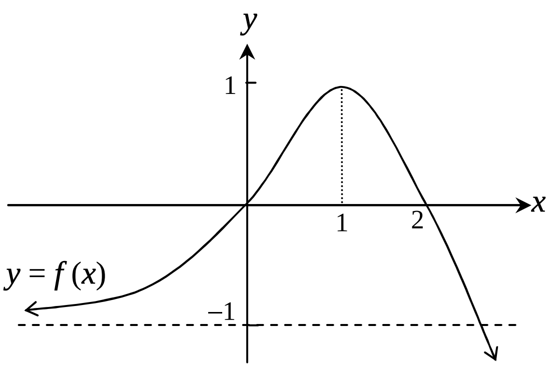
iii)



iv)



v)



Insert this page in your booklet for Question 3

Q1

a) $I = \int \frac{e^{\tan x}}{\cos^2 x}\,dx$

$= \int \sec^2 x\, e^{\tan x}\,dx$

Let $u = \tan x$

$du = \sec^2 x\,dx$

$I = \int e^u\,du$

$= e^{\tan x} + C$ (2)

b) $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$

Let $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$dx = \frac{2\,dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

When $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

$I = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$

$= \int_0^1 \frac{2\,dt}{1+t^2+1-t^2}$

$= \int_0^1 dt$

$= [t]_0^1$

$= 1$

c) (i) $\frac{3x^2+x+6}{(x^2+9)(x-1)} = \frac{Ax+B}{x^2+9} + \frac{C}{x-1}$

$3x^2+x+6 = (Ax+B)(x-1) + C(x^2+9)$

Let $x = 1$

$10 = 10C$

$C = 1$

Let $x = 0$

$6 = -B + 9C$

$6 = -B + 9$

$B = 3$

$A + C = 3$

$A + 1 = 3$

$A = 2$

$\frac{3x^2+x-6}{(x^2+9)(x-1)} = \frac{2x+3}{x^2+9} + \frac{1}{x-1}$

(ii) $\therefore I = \int\left(\frac{2x+3}{x^2+9} + \frac{1}{x-1}\right)dx$

$= \int\left(\frac{2x}{x^2+9} + \frac{3}{x^2+9} + \frac{1}{x-1}\right)dx$

$= \ln(x^2+9) + 3\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + \ln(x-1) + C$

$= \ln(x^2+9)(x-1) + \tan^{-1}\frac{x}{3} + C$

d (i) $\tan^2 x$ is an EVEN function as $\tan x$ is an odd function

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx = 2\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

which is NOT equal to zero from the graph below

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx \neq 0$ AND THE STATEMENT GIVEN

(ii)

Area of ABCD = BC.DC
= 1.2a
= 2a

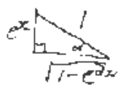
AREA of SHADED REGION $\leq$ AREA OF RECTANGLE ABCD

$\therefore \int_{-a}^{a} e^{-x^2}\,dx \leq 2a$

STATEMENT IS TRUE

e) (i) $\sin^{-1} e^x = \alpha$

$e^x = \sin\alpha$



$\therefore \tan\alpha = \frac{e^x}{\sqrt{1-e^{2x}}} \Rightarrow \alpha = \tan^{-1}\left(\frac{e^x}{\sqrt{1-e^{2x}}}\right)$

$\therefore \tan(\sin^{-1} e^x) = \tan\left(\tan^{-1}\frac{e^x}{\sqrt{1-e^{2x}}}\right)$

$= \frac{e^x}{\sqrt{1-e^{2x}}}$

Note $1 - e^{2x} > 0$ $\therefore x < 0$ and hence this is why $x<0$ was given as a restriction

(ii) $I = \int \tan(\sin^{-1} e^x)\,dx$

$= \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

Let $u = e^x$

$du = e^x\,dx$

$I = \int \frac{du}{\sqrt{1-u^2}}$

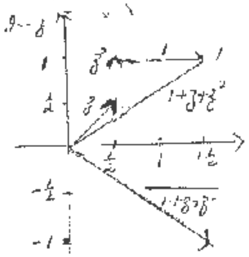
$= \sin^{-1} e^x + C$

Question 2

$z = \frac{1}{2}(1+i)$
$= \frac{1}{2} + \frac{1}{2}i$

(a)
(i) $|z| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$
$= \sqrt{\frac{1}{4} + \frac{1}{4}}$
$= \frac{1}{\sqrt{2}}$

(β) $\arg z = \frac{\pi}{4}$
(γ) $|z^2| = \frac{1}{2}$
(δ) $\frac{\pi}{2}$



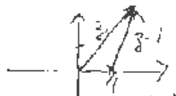
ALTERNATIVELY
$1+z+z^2 = \frac{3}{2} + i$
$\overline{1+z+z^2} = \frac{3}{2} - i$
THEN SKETCH

(b) $z\bar{z} - 2z + 2\bar{z} = 5 - 4i$
Let $z = x + iy$
$\bar{z} = x - iy$
$LHS = x^2 + y^2 - 4yi$
$= x^2 + y^2 - 4yi$
$RHS = 5 - 4i$

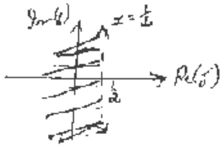
$\therefore x^2 + y^2 = 5$ ①
$y = 1$ ②
②→①
$x^2 + 1 = 5$
$x^2 = 4$
$x = \pm 2$

$z = 2 + i$
or $z = -2 + i$

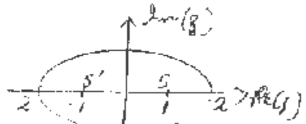
(c) (i) $|z - 1| > |z|$



If $|z-1| > |z|$ from diagram
$Re(z) < \frac{1}{2}$



(ii)



S, S′ ARE FOCI OF THIS ELLIPSE

$2a = 4$
$a = 2$

d) $x^2 - 2x + 4 = 0$
$x = \frac{2 \pm \sqrt{-12}}{2}$
$= \frac{2 \pm 2\sqrt{3}i}{2}$
$= 1 \pm \sqrt{3}i$

Let $\alpha = 1 + \sqrt{3}i$
$= 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
$\alpha^n = 2^n(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3})$

$\beta = 1 - \sqrt{3}i$
$\beta = [2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})]^n$
$\beta^n = 2^n(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^n$
$\beta^n = 2^n(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3})$ ← (De Moivre's Theorem)

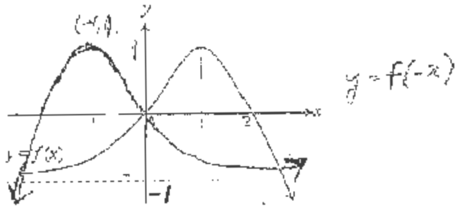
$\therefore \alpha^n + \beta^n = 2^n(2\cos\frac{n\pi}{3})$
$= 2^{n+1}\cos\frac{n\pi}{3}$

Q3

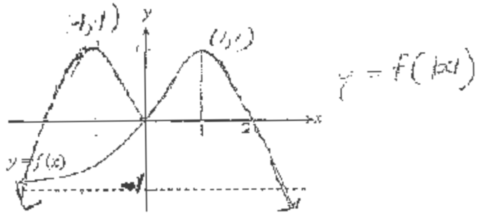
Student Number:_____ _______

Use this page for your answers to Question 3a)

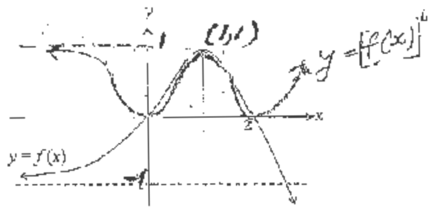
i)



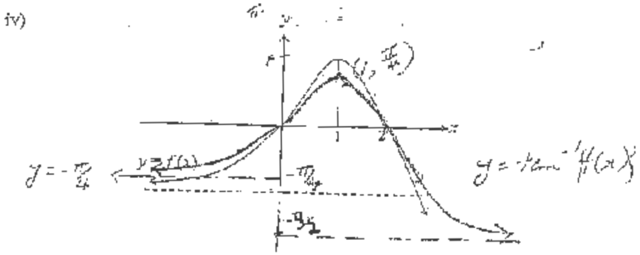
ii)



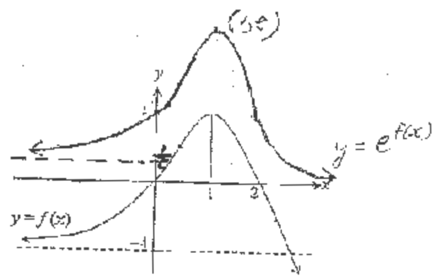
iii)



iv)



v)



Insert this page in your booklet for Question 3

5

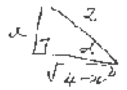
4

Q3 b) $f(x) = \cos(2\sin^{-1}\frac{x}{2})$

Let $\alpha = \sin^{-1}\frac{x}{2}$

$\frac{x}{2} = \sin\alpha$

$\cos\alpha = \frac{\sqrt{4-x^2}}{2}$



$f(x) = \cos 2\alpha$

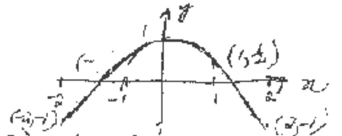
$= 2\cos^2\alpha - 1$

$= 2\left(\frac{\sqrt{4-x^2}}{2}\right)^2 - 1$

$= 2\frac{(4-x^2)}{4} - \frac{4}{4}$

$= \frac{4-2x^2}{4}$

$= \frac{2-x^2}{2}$ FOR $-2\le x\le 2$



c) $P(x) = 1 + Bx^n + Ax^{n+1}$

(i) $P(1) = 1+B+A$ (1)

AND $P(1) = 0$

$A+B = -1$

$P'(x) = nBx^{n-1} + A(n+1)x^{n+1}$

$P'(1) = nB + A(n+1)$ (2)

Since $P'(1) = 0$

$nB + nA + A = 0$ (3)

$n(A+B) + A = 0$

Since $A+B = -1$

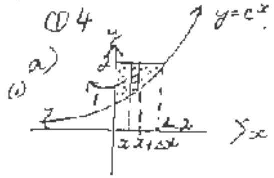
$-n + A = 0$

$A = n$ (4)

(ii) Substitute (4) → (3)

$nB + n^2 + n = 0$

Q4 a) (i)



(ii) $\Delta V = \pi(R+r)(R-r)(2-e^x)$

$= \pi(x+\Delta x+x)(\Delta x)(2-e^x)$

$= \pi(2x\Delta x + \Delta x^2)(2-e^x)$

Let $\Delta x^2 = 0$

$= 2\pi x(2-e^x)\Delta x$

$V = 2\pi\int_0^{\ln 2}(2x - xe^x)\,dx$

$= 2\pi\left[x^2 - xe^x + e^x\right]_0^{\ln 2}$

* $\int xe^x\,dx = \int x\frac{d}{dx}(e^x)\,dx$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x$

$V = 2\pi\left[((\ln 2)^2 - \ln 2\cdot 2 + 2) - (1)\right]$

$= 2\pi\left[(\ln 2)^2 - \ln 4 + 1\right]$

$\doteq 0.59$

b) (i)

Step 1

Let $n = 1$

$LHS = \frac{1}{1^2}$

$= 1$

$RHS = \frac{2\times 1 - 1}{1}$

$= 1$

Step 2

Assume true for $n = k$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} \le \frac{2k-1}{k}$

is true

Show that

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le \frac{2k+1}{k+1}$ is true

$LHS = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$

$\le \frac{2k-1}{k} + \frac{1}{(k+1)^2}$

$= \frac{(2k-1)(k+1)^2 + k}{k(k+1)^2}$

$= \frac{2k^3+3k^2-1+k}{k(k+1)^2}$

$\le \frac{2k^3+3k^2+k}{k(k+1)^2}$

$= \frac{k(2k+1)(k+1)}{k(k+1)^2}$

$= \frac{2k+1}{k+1}$ Since formula is true for $n = k+1$ if true for $n = k$ then it is true for $n = 2$ if true for $n = 1$ etc

(ii) Let $n = 49$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{49^2} \le \frac{2\times 49 - 1}{49}$

$= 1\frac{48}{49}$

Next

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{49^2} \ge 1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{49\cdot 50}$

$RHS = 1 + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots + (\frac{1}{49} - \frac{1}{50})$

$= 1 + \frac{1}{2} - \frac{1}{50}$

$= 1\frac{12}{25}$

$\therefore 1\frac{12}{25} \le 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{49^2} \le 1\frac{48}{49}$

(i) $I_n = \int_0^1 (1-x^2)^n dx$

$= \int_0^1 [\frac{d}{dx} x](1-x^2)^n dx$

$= [x(1-x^2)^n]_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1} dx$

$= 2n \int_0^1 (1-(1-x^2))(1-x^2)^{n-1} dx$

$= -2n\int_0^1 [(1-x^2)(1-x^2)^{n-1} + (1-x^2)^{n-1}] dx$

$I_n = -2n\int_0^1 (1-x^2)^n dx - 2n\int_0^1 (1-x^2)^{n-1} dx$

$= -2n I_n - 2n I_{n-1}$

$\therefore (2n+1) I_n = 2n I_{n-1}$

$I_n = \frac{2n I_{n-1}}{2n+1}$

(ii) $\int_0^1 (1-x^2) dx$

$I_3 = \frac{6}{7} I_2$

$= \frac{6}{7} \cdot \frac{4}{5} I_1$

$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I_0$

$I_0 = \int_0^1 (1-x^2)^0 dx \quad (= \int_0^1 1\, dx)$

$= [x]_0^1$

$= 1$

$\therefore I_3 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$

$= \frac{16}{35}$

Question 5

a) (i) $A = \frac{1}{2} \cdot 11 \times 5 \times \sin\theta$

$= \frac{55}{2} \sin\theta$

(ii) $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{55}{2} \cos\theta \times 0.1$

$= \frac{11}{4} \cos\theta$

(iii) $\theta = \frac{\pi}{4}$

$\frac{dA}{dt} = \frac{11}{4} \cos\frac{\pi}{4}$

$= \frac{11}{4} \cdot \frac{1}{\sqrt{2}}$

$= \frac{11}{4\sqrt{2}}$

$= \frac{11\sqrt{2}}{8}$

$\frac{11\sqrt{2}}{8}\ m^2/day$

b) (i) $a^3+b^3 = (a+b)(a^2+b^2-ab)$

$\geq (a+b)(2ab-ab)$ $\quad a^2+b^2 \geq 2ab$

$= ab(a+b)$

(ii)

$(b+c)(c+a)(a+b) = (bc+ab+c^2+ac)(a+b)$

RHS $= abc + b^2c + a^2b + ab^2 + a^2c + bc^2 + ac^2 + abc$

$= 2abc + c(b^2+a^2) + b(a^2+c^2) + a(b^2+c^2)$

$\geq 2abc + c \cdot 2ab + b \cdot 2ac + a \cdot 2bc$

$= 8abc$

c) If $f(x)$ is an ODD function

(i) $f(-x) = -f(x)$ AND $f'(-x) = f'(x)$

From

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

LHS $= f'(-x)$

$= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$

$= \lim_{h\to 0} \frac{f(-(x-h)) + f(x)}{h}$

$= \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h}$

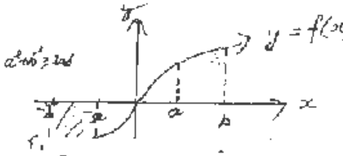
$= \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$

$= f'(x)$

Since $f'(x) = f'(-x)$ then the function $f'(x)$ is EVEN

(ii)

$\int_{-a}^{b} f(x)dx = \int_a^b f(x)dx$



Because of point symmetry, with an ODD function

$\int_{-a}^{b} f(x)dx = \int_{-a}^{a} f(x)dx + \int_a^b f(x)dx$

$= 0 + \int_a^b f(x)dx \quad [f(x)\ \text{ODD}]$

Question 6

$x^3 - 9x - 12 = 0$

Let $x = Y + \frac{3}{Y}$

$LHS = (Y+\frac{3}{Y})^3 - 9(Y+\frac{3}{Y}) - 12$

$= Y^3 + 3Y^2\cdot\frac{3}{Y} + 3Y\cdot\frac{9}{Y^2} + \frac{27}{Y^3} - 9Y - \frac{27}{Y} - 12$

$= Y^3 + 9Y + \frac{27}{Y} + \frac{27}{Y^3} - 9Y - \frac{27}{Y} - 12$

$= Y^3 + \frac{27}{Y^3} - 12$

ie. $Y^3 + \frac{27}{Y^3} - 12 = 0$

$Y^6 + 27 - 12Y^3 = 0$

$Y^6 - 12Y^3 + 27 = 0$

(ii) $(Y^3 - 9)(Y^3 - 3) = 0$

$Y^3 = 3$ or 9

(iii) If $Y^3 = 3$

then $x = Y + \frac{3}{Y}$

$= Y + \frac{3Y^2}{Y^3}$

$= Y + \frac{3Y^2}{3}$

$= Y + Y^2$

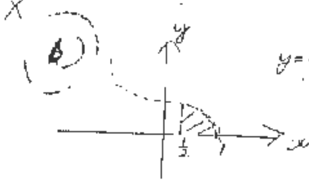
(iv) One root is

$Y + Y^2 = Y(1+Y)$

$= \sqrt[3]{3}(1+\sqrt[3]{3})$

(v)

b)



$y = \cos^{-1}x \Rightarrow x = \cos y$

$r = \frac{1}{2}$

$R = \cos y$

$\Delta V = \pi(R^2 - r^2)\Delta y$

$= \pi(\cos^2 y - \frac{1}{4})\Delta y$

$V = \pi\int_0^{\pi/3}(\cos^2 y - \frac{1}{4})dy$

$= \pi\int_0^{\pi/3}(\frac{1}{2}(\cos 2y + 1) - \frac{1}{4})dy$

$= \pi\int_0^{\pi/3}(\frac{1}{2}\cos 2y + \frac{1}{4})dy$

$= \pi\left[\frac{1}{4}\sin 2y + \frac{y}{4}\right]_0^{\pi/3}$

$= \pi\left[\left(\frac{\sin\frac{2\pi}{3}}{4} + \frac{\pi}{12}\right) - 0\right]$

$= \pi\left[\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right]$ $\pi(\frac{\sqrt{3}}{8} + \frac{\pi}{12})$ cubic units

c) (i)

$(\frac{\pi}{4} - \theta)$ $(\frac{\pi}{2} - 2\theta) / 2 = \frac{\pi}{4} - \theta$

$\cos(\frac{\pi}{4} - \theta) = \frac{L}{OY}$

$OY = \frac{L}{\cos(\frac{\pi}{4} - \theta)}$

$= L\sec(\frac{\pi}{4} - \theta)$

(ii) $A = \frac{L^2}{2}$

OY is RADIUS

$\frac{1}{2}r^2\theta \times 2 = \frac{L^2}{2}$

$r^2\theta = \frac{L^2}{2}$

$L^2\sec^2(\frac{\pi}{4} - \theta)2\theta = L^2$

$\sec^2(\frac{\pi}{4} - \theta) = \frac{1}{2\theta}$

$\cos^2(\frac{\pi}{4} - \theta) = 2\theta$

QUESTION 7

G (G_F) G — FUSSY GIRL IN MIDDLE $^3P_2$ OR $2 \times {}^3C_2$ WAYS

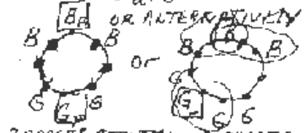
Repeat for fussy boy

G G_F G | B B_F B | BG

NUMBER $= 3! \times {}^3C_2 \times 2 \times {}^3C_2 \times 2$

$= 216$

OR ALTERNATIVELY



FIX B_F AND G_F

3 SPACES BETWEEN B_F AND G_F: $3 \times 2 \times 3 \times 2 \times 2 = 72$ — 2 BOYS ABOUT B_F, 2 GIRLS PLACED ABOUT G_F, REMAINING 2 PEOPLE

2 SPACES BETWEEN B_F AND G_F: $3 \times 2 \times 3 \times 2 \times 4 = 144$ — 2 BOYS ABOUT B_F, 2 GIRLS ABOUT G_F

TOTAL NO ARRANGEMENTS $= 72 + 144 = 216$

b) (i) Consider

$E = \tan x - x$

$\frac{dE}{dx} = \sec^2 x - 1$

Since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$

$\frac{dE}{dx} > 0$ and so

E is increasing function

Since when $x = 0$

$E = 0$

and E is increasing

then $E > 0$ for $0 < x < \frac{\pi}{2}$

ie $\tan x > x$

(ii) ie $\frac{\sin x}{x} > \frac{2}{\pi}$

Let $y = \frac{\sin x}{x}$

$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$

Since $x < \tan x$

$x < \frac{\sin x}{\cos x}$

So $\frac{dy}{dx} < 0$ and y is a decreasing function

As $x \to 0$, $\frac{\sin x}{x} \to 1$

and $f(\frac{\pi}{2}) = \frac{2}{\pi}$

$\therefore \frac{2}{\pi} < \frac{\sin x}{x} < 1$ ie $\frac{2x}{\pi} < \sin x$

c) (i) When $q = 0$, $Y = 0$; $q = 10$, $Y = 550$

From (1)

$Y = 1650 - Ae^{kq}$

$0 = 1650 - A$

$A = 1650$

$Y = 1650 - 1650e^{kq}$

From (1)

$550 = 1650 - 1650e^{10k}$

$1650e^{10k} = 1100$

$e^{10k} = \frac{1100}{1650}$

$k \doteqdot -0.0405$

$\therefore Y = 1650 - 1650e^{-0.0405q}$

$= 1650(1 - e^{-0.0405q})$

(ii) $\frac{dY}{dq} = 0.0405 \times 1650e^{-0.0405q}$

$= 0.0405(1650 - Y)$ From (1)

(iii) $\frac{dY}{dt} = \frac{dY}{dq} \times \frac{dq}{dt}$

$= 0.0405(1650 - Y) \times 10$

$\frac{dY}{dt} = 0.405(1650 - Y)$

$= 0.405(1650 - 550)$

$= 445.5$

445.5 kg/L

(17) 8a.

(i) $y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$

$y - \frac{4}{p} = -\frac{x}{p^2} + \frac{4}{p}$

$p^2 y - 4p = -x + 4p$

$x + p^2 y = 8p$

(ii) Tangents at P & Q

$x + p^2 y = 8p$ (1)

$x + q^2 y = 8q$ (2)

(1) − (2)

$(p^2 - q^2) y = 8(p-q)$

$(p+q) y = 8 \quad (p \neq q)$

$\therefore y_0 = \frac{8}{p+q}$ (3)

From (1)

Let $x_0 = 8p - p^2 q$

$= 8p - \frac{8p^2}{p+q}$

$= \frac{8p(p+q) - 8p^2}{p+q}$

$= \frac{8pq}{p+q}$ (4)

$\therefore p + q = \frac{8pq}{x_0}$

From (3)

$y_0 = \frac{8}{\left(\frac{8pq}{x_0}\right)}$

$y_0 = \frac{x_0}{pq}$

$\frac{x_0}{y_0} = pq$

(iii) $PQ^2 = 16(p-q)^2 + 16\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$= 16(p-q)^2 + 16\left(\frac{q-p}{pq}\right)^2$

$= 16(p-q)^2 + 16\frac{(p-q)^2}{p^2q^2}$

$= 16(p-q)^2\left[1 + \frac{1}{p^2q^2}\right]$

(iv) Let

$16(p-q)^2\left[1 + \frac{1}{p^2q^2}\right] = 64$

$(p-q)^2\left(1 + \frac{1}{p^2q^2}\right) = 4$

$\left((p+q)^2 - 4pq\right)\left(1 + \frac{1}{p^2q^2}\right) = 4$

$\left(\frac{64}{y_0^2} - \frac{4x_0}{y_0}\right)\left(1 + \frac{y_0^2}{x_0^2}\right) = 4$

$4\left(\frac{16}{y_0^2} - \frac{x}{y}\right)\left(1 + \frac{y^2}{x^2}\right) = 4$

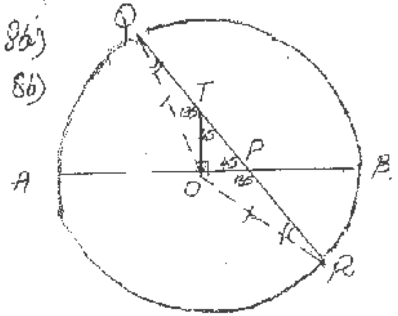
$\left(\frac{16}{y^2} - \frac{x}{y}\right)\left(1 + \frac{y^2}{x^2}\right) = 1$

$\left(\frac{16 - xy}{y^2}\right)\left(\frac{x^2 + y^2}{x^2}\right) = 1$

$(16 - xy)(x^2 + y^2) = x^2 y^2$

8b)

8b)



(ii) In $\triangle OQT$, $\triangle ORP$

$OQ = OR$ [Radii of circle]

$\angle OQT = \angle ORP$ [Base angles of isosceles triangle]

$\angle QTO = \angle RPO = 135°$

as $\angle PTO = 45°$ (Isosceles $\triangle PTO$)

$\angle QTO = 135°$ (Adjacent supplementary angles on a straight angle)

$\angle RPO = 135°$ (As above)

$\therefore \triangle OQT \equiv \triangle ORP$ [AAS congruence test]

Hence $QT = PR$, as they are corresponding sides of congruent triangles

(iii) $AP.PB = PQ.PR$ [product of intercepts of intersecting chords]

LHS = $AP.PB$

$= (AO + OP)(OB - OP)$

$= \left(\frac{AB}{2} + OP\right)\left(\frac{AB}{2} - OP\right)$

$= PQ.PR$

$= RHS$

$\therefore \left(\frac{AB}{2} + OP\right)\left(\frac{AB}{2} - OP\right) = PQ \times PR$

(iv) $(PQ - PR)^2 = PQ^2 - 2PQ.PR + PR^2$

$= PQ^2 + PR^2 - 2\left(\frac{AB}{2} + OP\right)\left(\frac{AB}{2} - OP\right)$

$= PQ^2 + PR^2 - 2\left[\frac{AB^2}{4} - OP^2\right]$

$= PQ^2 + PR^2 - \frac{AB^2}{2} + 2OP^2$

($PT = PQ - QT$
$= PQ - PR$ since $QT = PR$)

Now

$(PT)^2 = PQ^2 + PR^2 - \frac{AB^2}{2} + 2OP^2$

In $\triangle PTO$

$PT^2 = OT^2 + OP^2$

$PT^2 = 2OP^2$

$(PT = \sqrt{2}\, OP)$

$2OP^2 = PQ^2 + PR^2 - \frac{AB^2}{2} + 2OP^2$

$\frac{AB^2}{2} = PQ^2 + PR^2$

$AB^2 = 2(PQ^2 + PR^2)$