NORTH SYDNEY GIRLS HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

At the end of the examination, place your writing booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

NAME:______________________________ **TEACHER:**________________________

NUMBER:__________________________

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

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Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{6x}{\sqrt{1+x^2}}\,dx$. **2**

(b) By completing the square, or otherwise, evaluate $\displaystyle\int_{-1}^{5} \frac{dx}{\sqrt{32+4x-x^2}}$. **3**

(c) (i) Use integration by parts to find $\displaystyle\int (t-1)\ln t\,dt$. **3**

(ii) Using the substitution $t=2x+1$, evaluate $\displaystyle\int_0^1 4x\ln(2x+1)\,dx$. **3**

(d) Use the substitution $t=\tan\frac{1}{2}\theta$ to show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta-2\sin\theta+3}=\frac{\pi}{4}$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w = 2 + i$.

(i) Find w^2 in the form $x + iy$. **1**

(ii) Find $\operatorname{Im}\left(\dfrac{1}{w}\right)$. **1**

(iii) Find the real numbers x and y such that $x + 3iy = w + 4i\overline{w}$. **2**

(b) (i) If $z = \cos\theta + i\sin\theta$ show that **2**

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(ii) Given further that $z + \dfrac{1}{z} = \sqrt{2}$, find the value of **2**

$$z^{10} + \frac{1}{z^{10}}.$$

(c) A circle C and a ray L have equations $\left| z - 2\sqrt{3} - i \right| = 4$ and $\arg(z + i) = \dfrac{\pi}{6}$ respectively.

(i) Show that:

(1) the circle C passes through the point where $z = -i$ **1**

(2) the ray L passes through the centre of C. **2**

(ii) Sketch C and L on the same Argand diagram. **2**

(iii) Shade on your sketch the region satisfying both **2**

$$\left| z - 2\sqrt{3} - i \right| \le 4 \text{ and } 0 \le \arg(z + i) \le \frac{\pi}{6}.$$

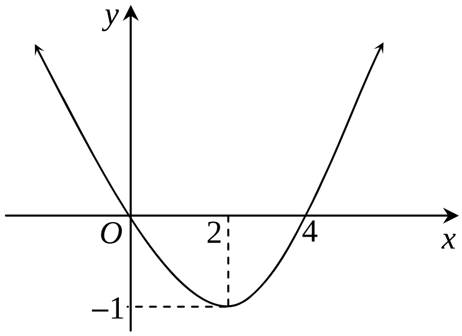
Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The sketch below shows the curve $y = f(x)$ where

$$f(x) = \frac{x(x-4)}{4}.$$

Without the use of calculus, draw sketches of the following, showing where necessary any intercepts, asymptotes and turning points

Use pages 14 and 15 to complete this question.



(i) $|y| = f(x)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \frac{x}{4}|x-4|$ **2**

(iv) $y = \tan^{-1}\left[f(x)\right]$. **2**

(b) (i) If $x \geq 0$, show that $\frac{x}{x^2+4} \leq \frac{1}{4}$. **2**

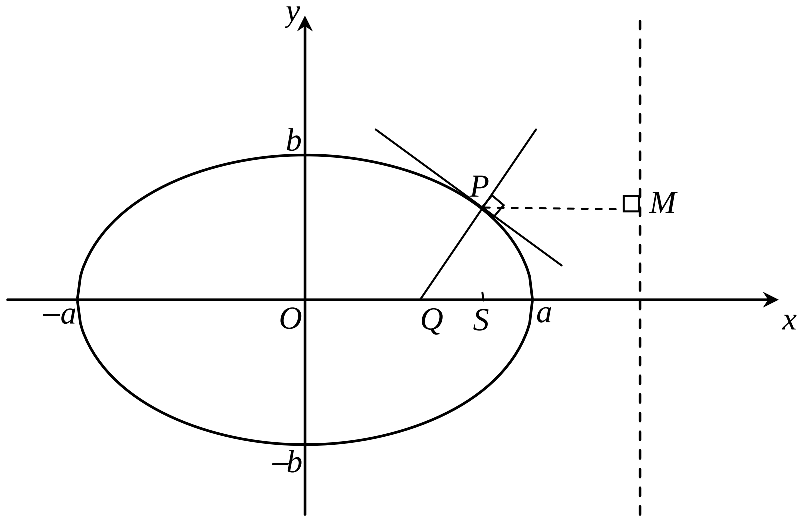
(ii) By integrating both sides of this inequality with respect to x between the limits $x = 0$ and $x = \alpha$, show that **2**

$$e^{\frac{1}{2}\alpha} \geq \frac{1}{4}\alpha^2 + 1 \text{ for } \alpha \geq 0.$$

(c) The region between the curve $y = 8x\sqrt{\sin 2x}$ and the x-axis for $0 \leq x \leq \frac{\pi}{2}$, is rotated about the x-axis. **4**
By using integration by parts twice, find the volume of the solid generated.
Leave your answer correct to 3 significant figures.

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$



(i) Write down the coordinates of the focus S and the equation of the associated directrix. **2**

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by **2**

$$\frac{9x}{x_1}-\frac{5y}{y_1}=4.$$

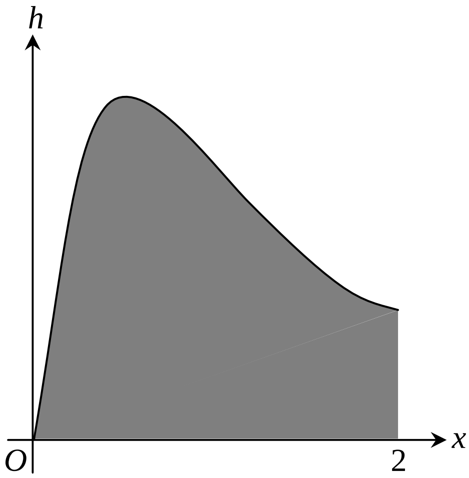
(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. Show that $QS=\frac{4}{9}PM$. **2**

(b) A curve has equation $x^3y+\cos(\pi y)=7$. **3**
Find the gradient of the curve at the point where $y = 1$.

Question 4 continues on page 7

(c) NSGHS is planning to construct an artwork in the Senior Lawn from pre-made panels. One side of each panel needs to be painted.

To determine the amount of paint needed, the area of one side of each panel needs to be calculated.
Each panel is 2 m wide. An artist's sketch of a panel is given below.



Rekrap examines the artist's sketch and decides that the height of each panel can be modeled by the following function:

$$h(x) = \frac{10x}{(x^2+1)(3x+1)}, \quad 0 \le x \le 2$$

(i) Given that $\dfrac{10x}{(x^2+1)(3x+1)}$ can be written in the form $\dfrac{x+A}{x^2+1} + \dfrac{B}{3x+1}$, find the values of A and B. **3**

(ii) Hence, find the area of one of the panels correct to 2 decimal places. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The cubic equation $z^3 + iz^2 + 3z - i = 0$ has roots α, β, and γ.

Write down a polynomial equation that has roots

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$, and $\dfrac{1}{\gamma}$. **1**

(ii) α^2, β^2, and γ^2. **2**

(b) A curve has equation

$$y = \frac{x^2}{(x-1)(x-5)}.$$

(i) Show that, if the curve intersects the line $y = k$, then the x-coordinates of the points of intersection must satisfy the equation **1**

$$(k-1)x^2 - 6kx + 5k = 0.$$

(ii) Show that if the equation in (i) has equal roots then **2**

$$k(4k+5) = 0.$$

(iii) Hence find the coordinates of the two stationary points on the curve. **3**

(iv) Sketch the curve showing intercepts, asymptotes and stationary points. **3**

(c) (i) Differentiate $x(1+x)^n$. **1**

(ii) Hence, show that **2**

$$\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} - 4\binom{n}{3} + \ldots + (-1)^n (n+1)\binom{n}{n} = 0.$$

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.
It is given that $\alpha = 2 + 2\sqrt{3}i$.

(i) Write down another root, β, of the equation. **1**

(ii) Find the third root, γ. **2**

(iii) Find the values of p and q. **2**

(iv) By expressing α in modulus-argument form, show that **2**

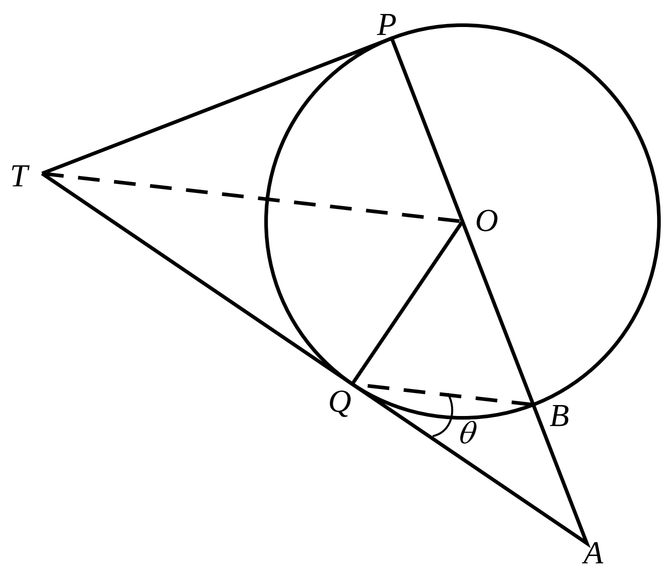
$$\left(2 + 2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right).$$

(v) Hence, show that **2**

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n.$$

where n is an integer.

(b) From an external point T, tangents are drawn to a circle with centre O, touching the circle at P and Q. Angle PTQ is acute.
The diameter PB produced meets the tangent TQ at A. Let $\angle AQB = \theta$.



(i) Copy the diagram above into your answer booklet.

(ii) Prove that $\angle PTQ = 2\theta$. **2**

(iii) Prove that ΔPBQ and ΔTOQ are similar. **2**

(iv) Hence show that $BQ \times OT = 2 \times OP^2$. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola $xy = c^2$ and the distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$

(i) Show that the equation of the tangent at $P\left(cp, \frac{c}{p}\right)$, where $p \neq 0$ is **2**

$$x + p^2 y = 2cp.$$

(ii) Show that the tangents at P and Q intersect at **2**

$$M\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right).$$

(iii) Show that if $pq = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin. **2**

(b) For each integer $n \geq 0$ let

$$I_n = \int_0^1 x\left(x^2 - 1\right)^n \, dx$$

(i) Using the method of integration by parts, show that for $n \geq 1$ **3**

$$I_n = -\frac{n}{n+1} I_{n-1}.$$

(ii) Hence, or otherwise, show that for $n \geq 0$ **2**

$$I_n = \frac{(-1)^n}{2(n+1)}.$$

(iii) Show that for n odd that $I_n < I_{n+2}$ for all $n \geq 0$. **2**

(iv) Deduce that $-\frac{1}{4n} < I_{2n+1} < -\frac{1}{4(n+2)}$. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2 \text{ and } u_{k+1} = 2u_k + 1$$

(i) Prove by induction that, for all integers $n \geq 1$, **3**

$$u_n = 3 \times 2^{n-1} - 1.$$

(ii) Show that **2**

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2).$$

(b) (i) If $a > 0$, $b > 0$ and $c > 0$, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

(ii) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and **3**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{27}{abc}.$$

(c) (i) Show that if n is any even positive integer, then **2**

$$(1+x)^n + (1-x)^n = 2\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} x^{2k}.$$

(ii) An alphabet consists of the three letters A, B and C.

(1) Show that the number of words of 4 letters containing exactly 2 As is **1**

$$\binom{4}{2} \times 2^2.$$

(2) Hence, or otherwise, show that if n is an even positive integer, then the number of words of n letters with zero or an even number of As is given by **2**

$$\frac{1}{2}\left(3^n + 1\right).$$

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

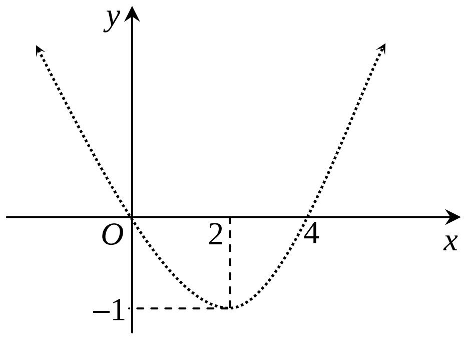
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

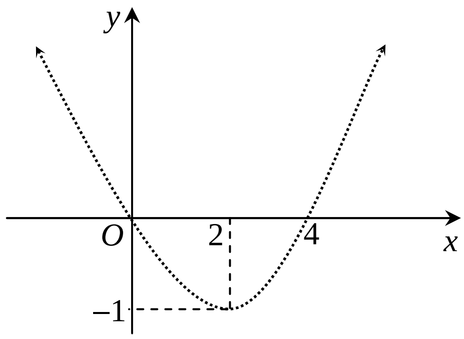
NSGHS 2011 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 3 (a)

Number______________________________ **Teacher:_______________________**

(i)



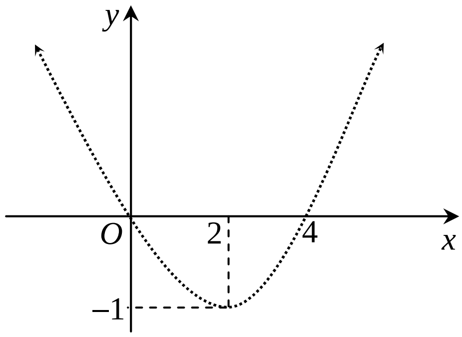
(ii)



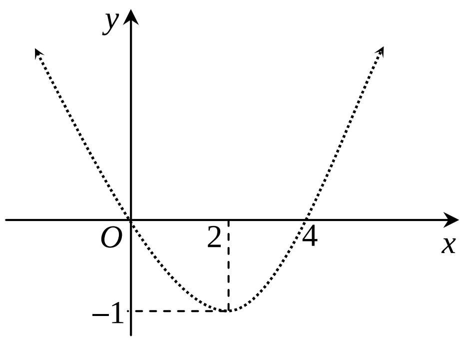
Turn over for parts (iii) and (iv)

Answer Sheet for Question 3 (a) continued

(iii)



(iv)



Now place this sheet <u>INSIDE</u> your booklet for Question 3

NORTH SYDNEY GIRLS HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

Mathematics Extension 2

2011 Trial HSC Mathematics Extension 2

Sample Answers

Question 1

(a) Find $\int \frac{6x}{\sqrt{1+x^2}}\,dx$. **2**

Let $u = 1+x^2$
$\therefore du = 2x\,dx$

$$\int \frac{6x}{\sqrt{1+x^2}}\,dx = 3\int \frac{2x}{\sqrt{1+x^2}}\,dx$$
$$= 3\int \frac{du}{\sqrt{u}} = 3\int u^{-\frac{1}{2}}\,du$$
$$= 3\left(2\sqrt{u}\right)+C$$
$$= 6\sqrt{1+x^2}+C$$

(b) By completing the square, or otherwise, evaluate $\int_{-1}^{5} \frac{dx}{\sqrt{32+4x-x^2}}$. **3**

$$32+4x-x^2 = -\left(x^2-4x\right)+32$$
$$= -\left(x^2-4x+4\right)+32+4$$
$$= 36-(x-2)^2$$

$$\int_{-1}^{5} \frac{dx}{\sqrt{32+4x-x^2}} = \int_{-1}^{5} \frac{dx}{\sqrt{36-(x-2)^2}}$$
$$= \left[\sin^{-1}\left(\frac{x-2}{6}\right)\right]_{-1}^{5}$$
$$= \sin^{-1}\left(\frac{3}{6}\right)-\sin^{-1}\left(-\frac{3}{6}\right)$$
$$= 2\sin^{-1}\left(\frac{1}{2}\right) = 2\times\frac{\pi}{6} = \frac{\pi}{3}$$

(c) (i) Use integration by parts to find $\int (t-1)\ln t\,dt$. **3**

Don't expand the brackets

$$\int (t-1)\ln t\,dt = \int \frac{d}{dt}\left(\tfrac{1}{2}t^2-t\right)\ln t\,dt$$
$$= \left(\tfrac{1}{2}t^2-t\right)\ln t - \int \left(\tfrac{1}{2}t^2-t\right)\times\frac{1}{t}\,dt$$
$$= \left(\tfrac{1}{2}t^2-t\right)\ln t - \int \left(\tfrac{1}{2}t-1\right)dt$$
$$= \left(\tfrac{1}{2}t^2-t\right)\ln t - \tfrac{1}{4}t^2+t+C$$

❖ An alternative would be to start with $\int \frac{d}{dt}\left[\tfrac{1}{2}(t-1)^2\right]\ln t\,dt$

$$\int \frac{d}{dt}\left[\tfrac{1}{2}(t-1)^2\right]\ln t\,dt = \tfrac{1}{2}(t-1)^2\ln t - \int \tfrac{1}{2}(t-1)^2\times\frac{1}{t}\,dt$$
$$= \tfrac{1}{2}(t-1)^2\ln t - \tfrac{1}{2}\int\left(t-2+\frac{1}{t}\right)dt$$
$$= \tfrac{1}{2}(t-1)^2\ln t - \tfrac{1}{4}t^2+t-\tfrac{1}{4}\ln|t|+C$$

(c) (ii) Using the substitution $t = 2x+1$, evaluate $\int_0^1 4x\ln(2x+1)\,dx$. **3**

$t = 2x+1$

$\therefore dt = 2dx$

$\therefore dx = \frac{1}{2}dt$

$t = 2x+1 \Rightarrow 2x = t-1$

$\therefore 4x = 2(t-1)$

$x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 3$

$$\int_0^1 4x\ln(2x+1)\,dx = \int_1^3 (t-1)\ln t\,dt$$
$$= \left[\left(\tfrac{1}{2}t^2 - t\right)\ln t - \tfrac{1}{4}t^2 + t\right]_1^3$$
$$= \left[\left(\tfrac{1}{2}\times 9 - 3\right)\ln 3 - \tfrac{1}{4}\times 9 + 3\right] - \left(-\tfrac{1}{4}+1\right)$$
$$= \tfrac{3}{2}\ln 3$$

(d) Use the substitution $t = \tan\frac{1}{2}\theta$ to show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta - 2\sin\theta + 3} = \frac{\pi}{4}$. **4**

$t = \tan\frac{1}{2}\theta$

$\therefore \theta = 2\tan^{-1} t$

$\therefore d\theta = \frac{2dt}{1+t^2}$

$\theta = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\cos\theta - 2\sin\theta + 3 = \frac{1-t^2}{1+t^2} - 2\left(\frac{2t}{1+t^2}\right) + 3$$
$$= \frac{1-t^2-4t+3(1+t^2)}{1+t^2}$$
$$= \frac{2t^2-4t+4}{1+t^2}$$
$$= \frac{2(t^2-2t+2)}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta - 2\sin\theta + 3} = \int_0^1 \frac{1+t^2}{2(t^2-2t+2)} \times \frac{2dt}{1+t^2}$$
$$= \int_0^1 \frac{dt}{t^2-2t+2}$$
$$= \int_0^1 \frac{dt}{(t-1)^2+1}$$
$$= \left[\tan^{-1}(t-1)\right]_0^1$$
$$= \left[\tan^{-1} 0 - \tan(-1)\right]$$
$$= \tan^{-1} 1 = \frac{\pi}{4}$$

End of Question 1 Solutions

Question 2

(a) Let $w = 2 + i$.

(i) Find w^2 in the form $x + iy$. **1**

$w^2 = (2+i)^2 = 4 - 1 + 4i = 3 + 4i$

(ii) Find $\operatorname{Im}\left(\dfrac{1}{w}\right)$. **1**

$\dfrac{1}{w} = \dfrac{1}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{2-i}{5}$

$\therefore \operatorname{Im}\left(\dfrac{1}{w}\right) = -\dfrac{1}{5}$

(iii) Find the real numbers x and y such that $x + 3iy = w + 4i\overline{w}$. **2**

$$\begin{aligned} x + 3iy &= w + 4i\overline{w} \\ &= 2 + i + 4i(2 - i) \\ &= 2 + i + 8i + 4 \\ &= 6 + 9i \end{aligned}$$

$\therefore x + 3iy = 6 + 9i$

$\therefore x = 6,\ y = 3$ [Equating real and imaginary terms]

(b) (i) If $z = \cos\theta + i\sin\theta$ show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **2**

$z = \cos\theta + i\sin\theta \Rightarrow z^n = \cos n\theta + i\sin n\theta$ [DMT]

$\therefore \dfrac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - \sin n\theta$

$\therefore z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(ii) Given further that $z + \dfrac{1}{z} = \sqrt{2}$, find the value of $z^{10} + \dfrac{1}{z^{10}}$ **2**

$z + \dfrac{1}{z} = \sqrt{2} \Rightarrow 2\cos\theta = \sqrt{2}$

$\therefore \cos\theta = \dfrac{\sqrt{2}}{2} \Rightarrow \theta = \pm\dfrac{\pi}{4}$ $[-\pi < \theta \le \pi]$

$z^{10} + \dfrac{1}{z^{10}} = 2\cos 10\theta = 2\cos 10\left(\pm\dfrac{\pi}{4}\right) = 2\cos\left(\pm\dfrac{5\pi}{2}\right) = 0$

(c) A circle C and a ray L have equations $\left| z - 2\sqrt{3} - i \right| = 4$ and $\arg(z+i) = \dfrac{\pi}{6}$ respectively.

(i) Show that:

(1) the circle C passes through the point where $z = -i$ **1**

Substitute $z = -i$ into the equation for C.

$\text{LHS} = \left| -i - 2\sqrt{3} - i \right| = \left| -2\sqrt{3} - 2i \right| = \left| 2\sqrt{3} + 2i \right| = 4 = \text{RHS}$

So the circle C passes through $z = -i$.

(2) the ray L passes through the centre of C. **2**

The centre of the circle is $2\sqrt{3} + i$.

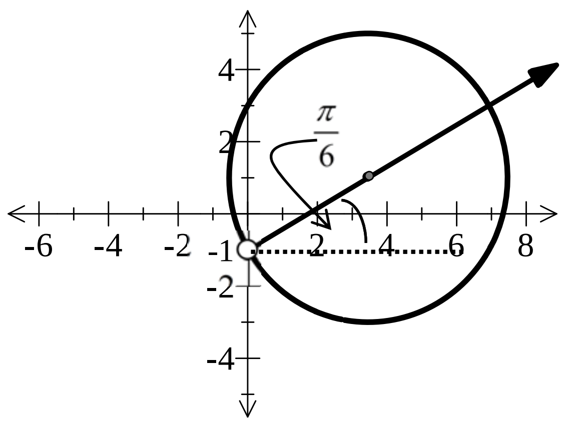
Substitute $2\sqrt{3} + i$ into the equation for L.

$\text{LHS} = \arg(z+i) = \arg\left(2\sqrt{3} + 2i\right) = \dfrac{\pi}{6} = \text{RHS}$

So the ray L passes through the centre of C.

(ii) Sketch C and L on the same Argand diagram. **2**

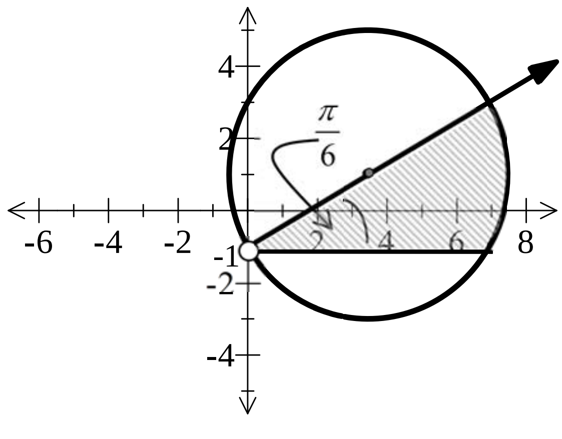
z



(iii) Shade on your sketch the region satisfying both **2**

$$\left| z - 2\sqrt{3} - i \right| \le 4 \text{ and } 0 \le \arg(z+i) \le \frac{\pi}{6}.$$

Im z



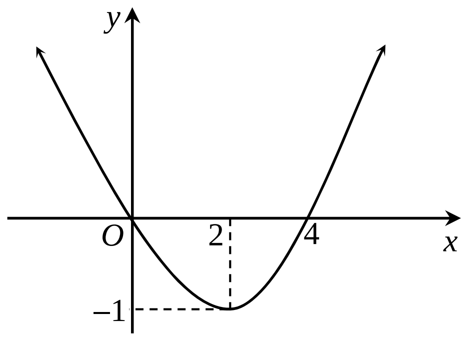
z

End of Question 2 Solutions

Question 3

(a) The sketch below shows the curve $y = f(x)$ where

$$f(x) = \frac{x(x-4)}{4}$$



(i) $|y| = f(x)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \frac{x}{4}|x-4|$ **2**

(iv) $y = \tan^{-1}\left[f(x)\right]$. **2**

See pages 23 and 24 for solutions

(b) (i) If $x \geq 0$, show that $\frac{x}{x^2+4} \leq \frac{1}{4}$. **2**

$$\frac{x}{x^2+4} - \frac{1}{4} = \frac{4-\left(x^2+4\right)}{4\left(x^2+4\right)} = -\frac{x^2}{4\left(x^2+4\right)} \leq 0$$

$$\therefore \frac{x}{x^2+4} - \frac{1}{4} \leq 0$$

$$\therefore \frac{x}{x^2+4} \leq \frac{1}{4}$$

(ii) By integrating both sides of this inequality with respect to x between the limits $x = 0$ and $x = \alpha$, show that $e^{\frac{1}{2}\alpha} \geq \frac{1}{4}\alpha^2 + 1$ for $\alpha \geq 0$. **2**

$$\int_0^\alpha \frac{x}{x^2+4}\,dx \leq \int_0^\alpha \frac{1}{4}\,dx$$

$$\therefore \left[\frac{\ln\left(x^2+4\right)}{2}\right]_0^\alpha \leq \left[\frac{x}{4}\right]_0^\alpha \Rightarrow \ln\left(\alpha^2+4\right) - \ln 4 \leq 2 \times \frac{\alpha}{4} = \frac{\alpha}{2}$$

$$\therefore \ln\left(\frac{\alpha^2+4}{4}\right) \leq \frac{\alpha}{2} \Rightarrow \frac{\alpha^2+4}{4} \leq e^{\frac{\alpha}{2}}$$

$$\therefore e^{\frac{\alpha}{2}} \geq \frac{\alpha^2+4}{4} = \frac{1}{4}\alpha^2 + 1$$

(c) The region between the curve $y = 8x\sqrt{\sin 2x}$ and the x-axis for $0 \le x \le \frac{\pi}{2}$, is rotated about the x-axis. **4**

By using integration by parts twice, find the volume of the solid generated.

Leave your answer correct to 3 significant figures.

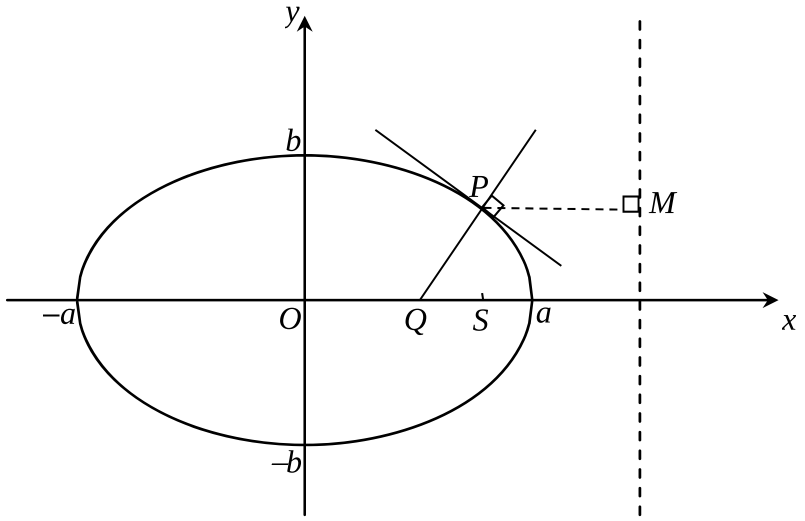
$$\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{2}} 64x^2 \sin 2x \, dx \\
&= \pi \int_0^{\frac{\pi}{2}} 32x^2 \frac{d}{dx}\left(-\cos 2x\right) dx \\
&= \pi \left[-32x^2 \cos 2x\right]_0^{\frac{\pi}{2}} - \pi \int_0^{\frac{\pi}{2}} 64x\left(-\cos 2x\right) dx \\
&= \pi \times 32\left(\tfrac{\pi}{2}\right)^2 + \pi \int_0^{\frac{\pi}{2}} 32x \frac{d}{dx}\left(\sin 2x\right) dx \\
&= 8\pi^3 + \pi\left[32x \sin 2x\right]_0^{\frac{\pi}{2}} - \pi \int_0^{\frac{\pi}{2}} 32 \sin 2x \, dx \\
&= 8\pi^3 + 16\pi\left[\cos 2x\right]_0^{\frac{\pi}{2}} \\
&= 8\pi^3 + 16\pi\left(-1-1\right) \\
&= 8\pi^3 - 32\pi
\end{aligned}$$

So the volume is approximately 148 c.u.

End of Question 3 Solutions

Question 4

(a) Consider the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$



(i) Write down the coordinates of the focus S and the equation of the associated directrix. **2**

$\dfrac{x^2}{9}+\dfrac{y^2}{5}=1 \Rightarrow a=3, b=\sqrt{5}$

$e^2=1-\dfrac{b^2}{a^2} \Rightarrow a^2e^2=a^2-b^2=9-5=4$

$\therefore S(ae,0)=(2,0)$

The associated directrix is $x=\dfrac{a}{e}$.

$a^2e^2=4 \Rightarrow ae=2$

$\therefore x=\dfrac{a}{e}=\dfrac{a^2}{ae}=\dfrac{9}{2}=4\tfrac{1}{2}$

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by $\dfrac{9x}{x_1}-\dfrac{5y}{y_1}=4$. **2**

$\dfrac{x^2}{9}+\dfrac{y^2}{5}=1 \Rightarrow \dfrac{2x}{9}+\dfrac{2yy'}{5}=0$

$\therefore \dfrac{2yy'}{5}=-\dfrac{2x}{9} \Rightarrow y'=-\dfrac{5x}{9y}$

$\therefore$ The normal at $P(x_1, y_1)$ has gradient $\dfrac{9y_1}{5x_1}$.

$\therefore y-y_1=\dfrac{9y_1}{5x_1}(x-x_1)$

$\therefore 5x_1y-5x_1y_1=9y_1x-9x_1y_1$

$\therefore 9y_1x-5x_1y=4x_1y_1$

$\therefore \dfrac{9x}{x_1}-\dfrac{5y}{y_1}=4$

(a) (iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. **2**
Show that $QS = \frac{4}{9}PM$.

Q:	M:
$\frac{9x}{x_1} = 4 \Rightarrow x = \frac{4x_1}{9}$ $\therefore \left(\frac{4x_1}{9}, 0\right)$	$\left(\frac{a}{e}, y_1\right) = \left(\frac{9}{2}, y_1\right)$

$QS = 2 - \frac{4x_1}{9} = \frac{18 - 4x_1}{9}$ [From diagram: horizontal distance]

$PM = \frac{9}{2} - x_1 = \frac{9 - x_1}{2}$ [From diagram: horizontal distance]

$\frac{4}{9}PM = \frac{4}{9}\left(\frac{9 - x_1}{2}\right) = \frac{18 - 4x_1}{9}$

$\therefore QS = \frac{4}{9}PM$

(b) A curve has equation $x^3 y + \cos(\pi y) = 7$. **3**
Find the gradient of the curve at the point where $y = 1$.

$y = 1 \Rightarrow x^3 + \cos\pi = 7$

$\therefore x^3 = 8$

$\therefore x = 2$

Differentiating implicitly:

$\therefore 3x^2 y + x^3 y' - \pi \sin(\pi y)\, y' = 0$

Substitute $(2, 1)$

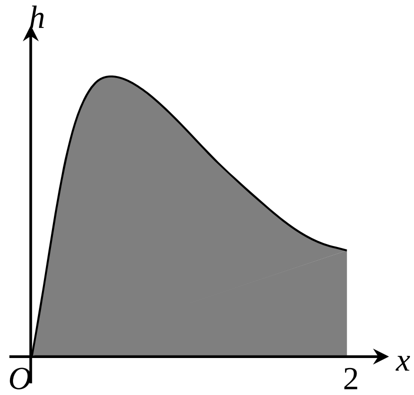
$\therefore 3 \times 4 \times 1 + 8y' - 0 = 0$

$\therefore 8y' = -12$

$\therefore y' = -\frac{3}{2}$

(c) NSGHS is planning to construct an artwork in the Senior Lawn from pre-made panels. One side of each panel needs to be painted.

To determine the amount of paint needed, the area of one side of each panel needs to be calculated.
Each panel is 2 m wide. An artist's sketch of a panel is given below.



Rekrap examines the artist's sketch and decides that the height of each panel can be modeled by the following function:

$$h(x)=\frac{10x}{(x^2+1)(3x+1)},\ \ 0\le x\le 2$$

(i) Given that $\dfrac{10x}{(x^2+1)(3x+1)}$ can be written in the form $\dfrac{x+A}{x^2+1}+\dfrac{B}{3x+1}$, find the values of A and B. **3**

$$\frac{10x}{(x^2+1)(3x+1)}\equiv\frac{x+A}{x^2+1}+\frac{B}{3x+1}$$

$$\therefore 10x=(x+A)(3x+1)+B(x^2+1)$$

$$\therefore A+B=0 \qquad \text{(matching constant terms)}$$

Substitute $x=-\frac{1}{3}$: $\quad -\frac{10}{3}=0+B\times\frac{10}{9}\Rightarrow B=-3$

$$\therefore A=3,\ B=-3$$

(ii) Hence, find the area of one of the panels correct to 2 decimal places. **3**

$$\text{Area}=\int_0^2\left(\frac{x+3}{x^2+1}-\frac{3}{3x+1}\right)dx$$

$$=\int_0^2\left(\frac{x}{x^2+1}+\frac{3}{x^2+1}-\frac{3}{3x+1}\right)dx$$

$$=\left[\frac{1}{2}\ln(x^2+1)+3\tan^{-1}x-\ln|\,3x+1\,|\right]_0^2$$

$$=\frac{1}{2}\ln 5+3\tan^{-1}2-\ln 7$$

$$\approx 2\cdot 18\ \text{m}^2$$

RADIANS!!!

End of Question 4 Solutions

Question 5

(a) The cubic equation $z^3 + iz^2 + 3z - i = 0$ has roots α, β, and γ.

Write down a polynomial equation that has roots

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$, and $\dfrac{1}{\gamma}$. **1**

Let $y = \dfrac{1}{z} \Rightarrow z = \dfrac{1}{y}$

$\therefore \dfrac{1}{z^3} + \dfrac{i}{z^2} + \dfrac{3}{z} - i = 0$

$\therefore 1 + iz + 3z^2 - iz^3 = 0$

(ii) α^2, β^2, and γ^2. **2**

Let $y = z^2$

$\therefore z\left(z^2+3\right) = i\left(z^2+1\right)$

$\therefore z^2\left(z^2+3\right)^2 = i^2\left(z^2+1\right)^2$

$\therefore y\left(y+3\right)^2 = -\left(y+1\right)^2$

$$\left[\begin{array}{l} y^3 + 6y^2 + 9y = -y^2 - 2y - 1 \\ \therefore y^3 + 7y^2 + 11y + 1 = 0 \end{array}\right]$$

(b) A curve has equation

$$y = \frac{x^2}{(x-1)(x-5)}.$$

(i) Show that, if the curve intersects the line $y = k$, then the x-coordinates of the points of intersection must satisfy the equation **1**

$$(k-1)x^2 - 6kx + 5k = 0.$$

$\dfrac{x^2}{(x-1)(x-5)} = k \Rightarrow x^2 = k(x-1)(x-5)$

$\therefore x^2 = kx^2 - 6kx + 5k$

$\therefore (k-1)x^2 - 6kx + 5k = 0$

(ii) Show that if the equation in (i) has equal roots then **2**

$$k(4k+5) = 0.$$

$\Delta = 36k^2 - 4\times(k-1)\times 5k$

$\quad = 36k^2 - 20k^2 + 20k$

$\quad = 4k(4k+5)$

For equal roots then $\Delta = 0$.

$\therefore k(4k+5) = 0$.

(iii) Hence find the coordinates of the two stationary points on the curve. **3**

The stationary points will occur when the intersection of $y = k$ and $y = \dfrac{x^2}{(x-1)(x-5)}$ produce *equal roots* i.e. $y = k$ is a tangent.

$\therefore k(4k+5) = 0$

$\therefore k = 0, -\dfrac{5}{4}$

NB k is the y-coordinate of the turning points.

$k = 0:$ $\quad -x^2 = 0 \Rightarrow x = 0$

$\therefore (0, 0)$

$k = -\dfrac{5}{4}:$ $\quad \left(-\dfrac{5}{4} - 1\right)x^2 - 6\left(-\dfrac{5}{4}\right)x + 5\left(-\dfrac{5}{4}\right) = 0$

$\therefore -9x^2 + 30x - 25 = 0$

$\therefore -(3x-5)^2 = 0$

$\therefore x = \dfrac{5}{3}$

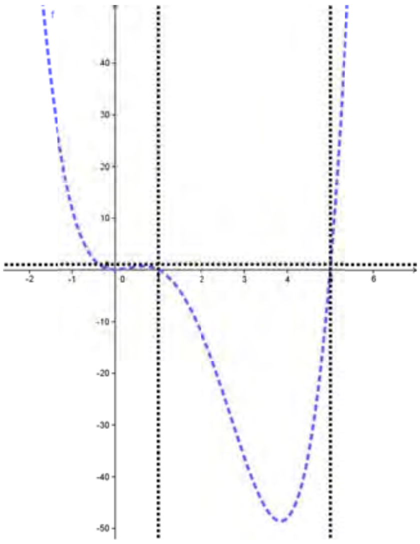
$\therefore \left(\dfrac{5}{3}, -\dfrac{5}{4}\right)$

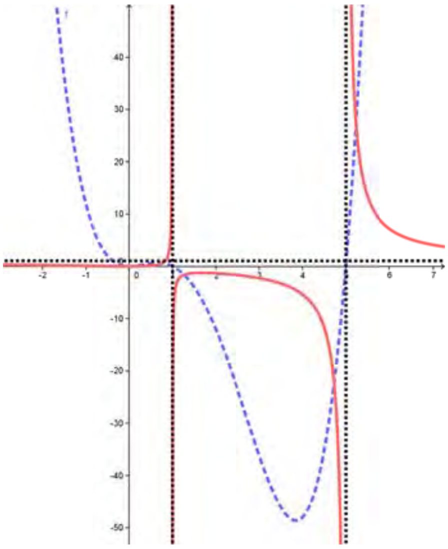
(iv) Sketch the curve showing intercepts, asymptotes and stationary points. **3**

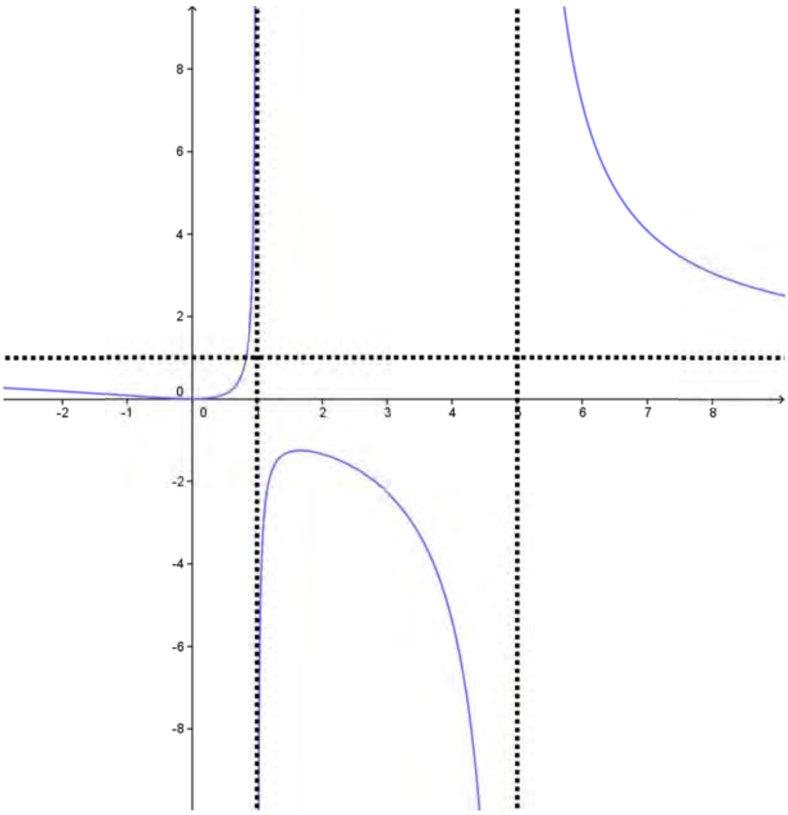
$$y = \frac{x^2}{(x-1)(x-5)}$$

Intercepts: Double root at $x = 0$; y-intercept 0
Asymptotes: $x = 1, 5$; $y = 1$.

In order to find where the graph is positive or negative sketch FIRST the polynomial $y = x^2(x-1)(x-5)$.







(c) (i) Differentiate $x(1+x)^n$. **1**

$$\frac{d}{dx}\left[x(1+x)^n\right]=(1+x)^n+nx(1+x)^{n-1}$$

(ii) Hence, show that $\binom{n}{0}-2\binom{n}{1}+3\binom{n}{2}+\ldots+(n+1)\binom{n}{n}(-1)^n=0$ **2**

$$x(1+x)^n=x\sum_{r=0}^{n}\binom{n}{r}x^r$$

$$=\sum_{r=0}^{n}\binom{n}{r}x^{r+1}$$

$$\frac{d}{dx}\left[x(1+x)^n\right]=\sum_{r=0}^{n}\binom{n}{r}(r+1)x^r$$

$$\therefore\frac{d}{dx}\left[x(1+x)^n\right]=\binom{n}{0}+2\binom{n}{1}x+3\binom{n}{2}x^2+\ldots+(n+1)\binom{n}{n}x^n$$

$$\therefore\binom{n}{0}+2\binom{n}{1}x+3\binom{n}{2}x^2+\ldots+(n+1)\binom{n}{n}x^n=(1+x)^n+nx(1+x)^{n-1}$$

Substitute $x=-1$:

$$\therefore\binom{n}{0}-2\binom{n}{1}+3\binom{n}{2}+\ldots+(n+1)\binom{n}{n}(-1)^n=(0)^n+nx(0)^{n-1}$$

$$=0$$

End of Question 5 Solutions

Question 6

(a) The cubic equation $2z^3 + pz^2 + qz + 16 = 0$ where p and q are real, has roots α, β and γ. It is given that $\alpha = 2 + 2\sqrt{3}i$.

(i) Write down another root, β, of the equation. **1**

$$\beta = \overline{\alpha} = 2 - 2\sqrt{3}i \qquad [\text{real coefficients}]$$

(ii) Find the third root, γ. **2**

$$\begin{aligned}\alpha\beta\gamma &= -8 \qquad [\text{product of roots}]\\ &= \left(2+2\sqrt{3}i\right)\left(2-2\sqrt{3}i\right)\gamma\\ &= 16\gamma\\ \therefore \gamma &= -\frac{1}{2}\end{aligned}$$

(iii) Find the values of p and q. **2**

$$\begin{aligned}\alpha+\beta+\gamma &= -\frac{p}{2} \qquad [\text{sum of roots}]\\ &= 2+2\sqrt{3}i+2-2\sqrt{3}i+\left(-\frac{1}{2}\right)\\ &= \frac{7}{2}\\ \therefore -\frac{p}{2} &= \frac{7}{2}\\ \therefore p &= -7\end{aligned}$$

$$\begin{aligned}\alpha\beta+\beta\gamma+\gamma\alpha &= \frac{q}{2} \qquad [\text{sum of roots 2 at a time}]\\ &= \alpha\beta+\gamma(\alpha+\beta)\\ &= \alpha\overline{\alpha}+\left[-\frac{1}{2}\times 2\,\text{Re}\,\alpha\right]\\ &= 16-\text{Re}\,\alpha = 16-2\\ &= 14\\ \therefore \frac{q}{2} &= 14\\ \therefore q &= 28\end{aligned}$$

(iv) By expressing α in modulus-argument form, show that **2**

$$\left(2+2\sqrt{3}i\right)^n = 4^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right).$$

$$\alpha = 2+2\sqrt{3}i = 4\text{cis}\frac{\pi}{3}$$

$$\therefore \alpha^n = \left(4\text{cis}\frac{\pi}{3}\right)^n = 4^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) \qquad [\text{DMT}]$$

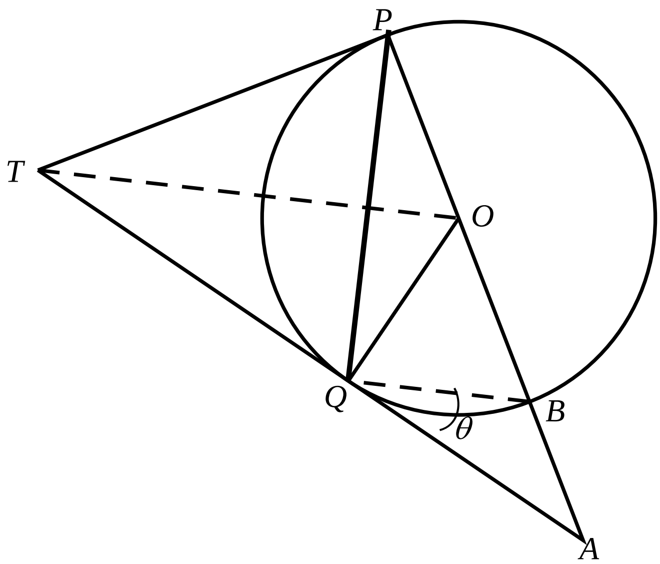
(v) Hence, show that $\alpha^n+\beta^n+\gamma^n = 2^{2n+1}\cos\frac{n\pi}{3}+\left(-\frac{1}{2}\right)^n$ **2**

where n is an integer.

Similarly $\beta^n = 4^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$

$$\begin{aligned}\alpha^n+\beta^n+\gamma^n &= 4^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)+4^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)+\left(-\frac{1}{2}\right)^n\\ &= 4^n\left(2\cos\frac{n\pi}{3}\right)+\left(-\frac{1}{2}\right)^n = 2^{2n+1}\cos\frac{n\pi}{3}+\left(-\frac{1}{2}\right)^n\end{aligned}$$

(b) From an external point T, tangents are drawn to a circle with centre O, touching the circle at P and Q. Angle PTQ is acute.
The diameter PB produced meets the tangent TQ at A. Let $\angle AQB = \theta$.



(i) Copy the diagram above into your answer booklet.

(ii) Prove that $\angle PTQ = 2\theta$. **2**

$TPOQ$ is a cyclic quadrilateral as $\angle TPO + \angle TQO = 180°$ [tangents perp. radius]

$\angle OQB = 90° - \theta$ [$OQ \perp TA$]

$\angle OBQ = 90° - \theta$ [ΔOBQ isos.]

$\therefore \angle POQ = 180° - 2\theta$ [ext. angle ΔOBQ]

$\therefore \angle TPQ = 2\theta$ [opp. angles cyclic quad. $TPOQ$]

(iii) Prove that ΔPBQ and ΔTOQ are similar. **2**

In ΔPBQ :

$\angle PQB = 90°$ [angle in semicircle]

$\angle QPB = \theta$ [angles alt. segment]

$\therefore \angle QTO = \theta$ [angles same segment]

$\therefore \angle QTO = \angle QPB$

$\angle OQT = \angle PQB = 90°$ [proven above]

$\therefore \Delta TOQ \; ||| \; \Delta PBQ$ [equiangular]

(iv) Hence show that $BQ \times OT = 2 \times OP^2$. **2**

$\therefore \dfrac{OT}{PB} = \dfrac{OQ}{BQ} = \dfrac{TQ}{PQ}$ [matching sides sim. Δs]

$\therefore \dfrac{OT}{2 \times OP} = \dfrac{OP}{BQ}$ [$PB = 2OP = 2OQ$; diameter]

$\therefore BQ \times OT = 2 \times OP^2$

End of Question 6 Solutions

Question 7

(a) Consider the hyperbola $xy = c^2$ and the distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$

(i) Show that the equation of the tangent at $P\left(cp, \frac{c}{p}\right)$, where $p \neq 0$ is **2**

$$x + p^2 y = 2cp.$$

$y = c^2 x^{-1}$

$\therefore y' = -c^2 x^{-2}$

$= -\frac{c^2}{x^2}$

$\therefore y'_P = -\frac{c^2}{c^2 p^2} = -\frac{1}{p^2}$

$\therefore y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$\therefore p^2 y - cp = -x + cp$

$\therefore x + p^2 y = 2cp$

(ii) Show that the tangents at P and Q intersect at **2**

$$M\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right).$$

Similarly, the tangent at Q is $x + q^2 y = 2cq$

$x + p^2 y = 2cp \qquad -(1)$

$x + q^2 y = 2cq \qquad -(2)$

(1) – (2)

$\therefore \left(p^2 - q^2\right) y = 2c(p - q)$

$\therefore y = \frac{2c(p-q)}{p^2 - q^2} = \frac{2c}{p+q}$

Sub into (1)

$\therefore x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$\therefore (p+q)x + 2cp^2 = 2cp(p+q)$

$\therefore (p+q)x = 2cpq$

$\therefore x = \frac{2cpq}{p+q}$

$\therefore M\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

(iii) Show that if $pq = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin. **2**

$pq = k \Rightarrow M\left(\frac{2ck}{p+q}, \frac{2c}{p+q}\right)$

$x = \frac{2ck}{p+q}, \; y = \frac{2c}{p+q} \Rightarrow \frac{y}{x} = \frac{1}{k}$

$\therefore y = \frac{1}{k} x$ [This is a straight line passing through the origin]

(b) For each integer $n \geq 0$ let

$$I_n = \int_0^1 x\left(x^2-1\right)^n \, dx$$

(i) Using the method of integration by parts, show that for $n \geq 1$, **3**

$I_n = -\dfrac{n}{n+1} I_{n-1}$.

$$I_n = \int_0^1 \frac{d}{dx}\left(\frac{x^2}{2}\right)\left(x^2-1\right)^n \, dx$$

$$= \left[\tfrac{1}{2}x^2\left(x^2-1\right)^n\right]_0^1 - \int_0^1 \left(\frac{x^2}{2}\right) \times n\left(x^2-1\right)^{n-1} \times 2x \, dx$$

$$= -n\int_0^1 x^2 \times x\left(x^2-1\right)^{n-1} \, dx$$

$$= -n\int \left[\left(x^2-1\right)+1\right] x\left(x^2-1\right)^{n-1} \, dx$$

$$= -n\int \left[x\left(x^2-1\right)^n + x\left(x^2-1\right)^{n-1}\right] dx$$

$$I_n = -nI_n - nI_{n-1}$$

$$\therefore (n+1)I_n = -nI_{n-1}$$

$$\therefore I_n = -\frac{n}{n+1} I_{n-1}$$

(ii) Hence, or otherwise, show that for $n \geq 0$, $I_n = \dfrac{(-1)^n}{2(n+1)}$. **2**

<u>**Hence**</u>

$$I_n = -\frac{n}{n+1} I_{n-1}$$

$$= -\left(\frac{n}{n+1}\right) \times \left[-\left(\frac{n-1}{n}\right)\right] I_{n-2}$$

$$= (-1)^3 \left(\frac{n}{n+1}\right) \times \left(\frac{n-1}{n}\right) \times \left(\frac{n-2}{n-1}\right) I_{n-3}$$

$$\vdots$$

$$= (-1)^n \left(\frac{n}{n+1}\right) \times \left(\frac{n-1}{n}\right) \times \left(\frac{n-2}{n-1}\right) \times \ldots \times \frac{1}{2} I_0$$

$$= (-1)^n \left(\frac{\cancel{n}}{n+1}\right) \times \left(\frac{\cancel{n-1}}{\cancel{n}}\right) \times \left(\frac{n-2}{\cancel{n-1}}\right) \times \ldots \times \frac{1}{\cancel{2}} \times \frac{1}{2}$$

$$= \frac{(-1)^n}{2(n+1)}$$

$$I_0 = \int_0^1 x \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

Otherwise

$$I_n = \frac{1}{2}\int_0^1 2x\left(x^2-1\right)^n\,dx$$

$$= \frac{1}{2}\left[\frac{\left(x^2-1\right)^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{2(n+1)}\left[0-(-1)^{n+1}\right]$$

$$= \frac{(-1)^{n+2}}{2(n+1)} = \frac{(-1)^n}{2(n+1)} \qquad \left[(-1)^{n+2} = (-1)^2\times(-1)^n\right]$$

(iii) Show that for *n* odd that $I_n < I_{n+2}$ for all $n \geq 0$. **2**

For *n* odd: $I_n = \frac{(-1)^n}{2(n+1)} = -\frac{1}{2(n+1)}$.

$$I_{n+2} = \frac{(-1)^{n+2}}{2(n+3)} = -\frac{1}{2(n+3)}$$

As $-2(n+1) > -2(n+3)$ then $\frac{1}{-2(n+1)} < \frac{1}{-2(n+3)}$ i.e. $I_n < I_{n+2}$.

(iv) Deduce that $-\frac{1}{4n} < I_{2n+1} < -\frac{1}{4(n+2)}$. **2**

From (iii) $I_{2n-1} < I_{2n+1} < I_{2n+3}$

$$I_{2n-1} = -\frac{1}{2(2n-1+1)} = -\frac{1}{4n}$$

$$I_{2n+3} = -\frac{1}{2(2n+3+1)} = -\frac{1}{4(n+2)}$$

$$\therefore -\frac{1}{4n} < I_{2n+1} < -\frac{1}{4(n+2)}$$

End of Question 7 Solutions

Question 8

(a) The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2 \text{ and } u_{k+1} = 2u_k + 1$$

(i) Prove by induction that, for all integers $n \geq 1$, **3**

$$u_n = 3 \times 2^{n-1} - 1.$$

Test $n = 1$:

$\text{LHS} = u_1 = 2$

$\text{RHS} = 3 \times 2^{1-1} - 1 = 3 - 1 = 2$

So true for $n = 1$.

Assume true for $n = k$ i.e. $u_k = 3 \times 2^{k-1} - 1$.

Need to prove true for $n = k + 1$ i.e. $u_{k+1} = 3 \times 2^k - 1$.

$$\begin{aligned}
\text{LHS} &= u_{k+1} \\
&= 2u_k + 1 \\
&= 2\left(3 \times 2^{k-1} - 1\right) + 1 \\
&= 3 \times 2^k - 2 + 1 \\
&= 3 \times 2^k - 1 \\
&= \text{RHS}
\end{aligned}$$

So $n = k + 1$ is true if $n = k$ is true.

So by the principle of mathematical induction the formula is true for all integers $n \geq 1$.

(ii) Show that **2**

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n + 2).$$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} u_r \\
&= \sum_{r=1}^{n} \left(3 \times 2^{r-1} - 1\right) \\
&= \sum_{r=1}^{n} \left(3 \times 2^{r-1}\right) - \sum_{r=1}^{n} 1 \qquad \left[\left(3 + 6 + 12 + \ldots + 3 \times 2^{n-1}\right) - \left(1 + 1 + 1 + \ldots + 1\right)\right] \\
&= 3\sum_{r=1}^{n} \left(2^{r-1}\right) - n \\
&= 3 \times \frac{1 \times \left(2^n - 1\right)}{2 - 1} - n \qquad \left[\text{geometrical series; } a = 1, r = 2\right] \\
&= 3 \times \left(2^n - 1\right) - n \\
&= 3 \times 2^n - 1 - (n + 2) \\
&= u_{n+1} - (n + 2) \\
&= \text{RHS}
\end{aligned}$$

(b) (i) If $a > 0$, $b > 0$ and $c > 0$, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

$(a-b)^2 \geq 0 \qquad [a, b \in \mathbb{R}]$

$\therefore a^2 - 2ab + b^2 \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

So for $a, b, c \in \mathbb{R}$

$$\left.\begin{aligned} a^2 + b^2 &\geq 2ab \\ b^2 + c^2 &\geq 2bc \\ c^2 + a^2 &\geq 2ca \end{aligned}\right\} +$$

$\therefore 2\left(a^2 + b^2 + c^2\right) \geq 2(ab + bc + ca)$

$\therefore a^2 + b^2 + c^2 \geq ab + bc + ca$

(ii) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \leq \dfrac{27}{abc}$. **3**

$$\begin{aligned} 81 &= (a+b+c)^2 \\ &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ &\geq ab + bc + ca + 2(ab + bc + ca) \qquad [\text{from (i)}] \\ &= 3(ab + bc + ca) \end{aligned}$$

$\therefore ab + bc + ca \leq 27$.

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{ab + bc + ca}{abc} \\ &\leq \frac{27}{abc} \end{aligned}$$

(c) (i) Show that if n is any even positive integer, then $(1+x)^n + (1-x)^n = 2\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} x^{2k}$. **2**

$$\begin{aligned}
\text{LHS} &= (1+x)^n + (1-x)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \left[1+(-1)^k\right] \\
&= 2\binom{n}{0} + 2\binom{n}{2} x^2 + \ldots + 2\binom{n}{n} x^n \qquad \left[\begin{array}{l} n \text{ even} \\ k \text{ odd} : 1+(-1)^k = 0 \end{array}\right] \\
&= 2\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} x^{2k} \\
&= \text{RHS}
\end{aligned}$$

(ii) An alphabet consists of the three letters A, B and C.

(1) Show that the number of words of 4 letters containing exactly 2 As is $\binom{4}{2} \times 2^2$. **1**

There are 4 places for letters i.e. __ __ __ __.

Choose 2 of these for the As i.e. __ A A __. This can be done in $\binom{4}{2}$ ways.

The remaining 2 spots can be filled with B or C. So this can be done in 2^2 ways.

So there are $\binom{4}{2} \times 2^2$ ways of doing this.

(2) Hence, or otherwise, show that if n is an even positive integer, then the number of words of n letters with zero or an even number of As is given by **2**

$$\frac{1}{2}\left(3^n + 1\right).$$

If there are words of n letters then the number of words with

0 As is $\binom{n}{0} \times 2^0$; 2 As is $\binom{n}{2} \times 2^2$; 4 As is $\binom{n}{4} \times 2^4$ and so on until

n As is $\binom{n}{n} \times 2^n$.

So the total number is $\binom{n}{0} \times 2^0 + \binom{n}{2} \times 2^2 + \ldots + \binom{n}{n} \times 2^n = \frac{1}{2}\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} 2^{2k}$.

$$\therefore \binom{n}{0} \times 2^0 + \binom{n}{2} \times 2^2 + \ldots + \binom{n}{n} \times 2^n = \frac{1}{2}\left[(2+1)^n + (2-1)^n\right] = \frac{1}{2}\left(3^n + 1\right).$$

End of Solutions

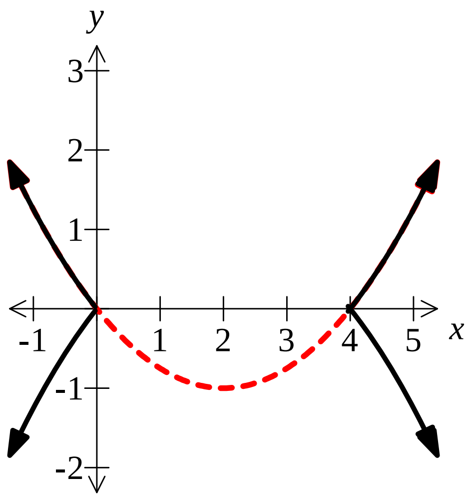
NSGHS 2011 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 3 (a)

Number SOLUTIONS

(i) $|y| = f(x)$ **1**

Where $f(x) > 0$ reflect in the x-axis.

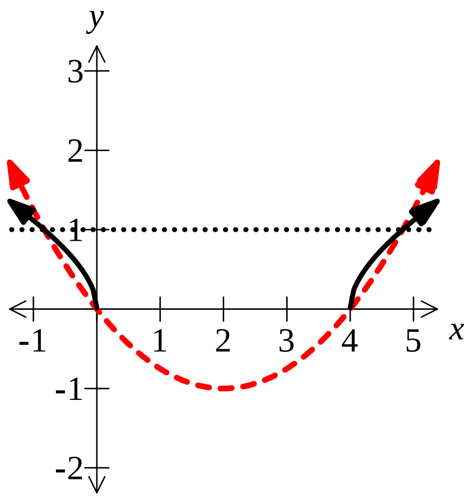
Erase where $f(x) < 0$.



(ii) $y = \sqrt{f(x)}$ **2**

NB when $0 < f(x) < 1$ then $f(x) < \sqrt{f(x)}$

At x = 0, 4 there are critical points (vertical tangents).



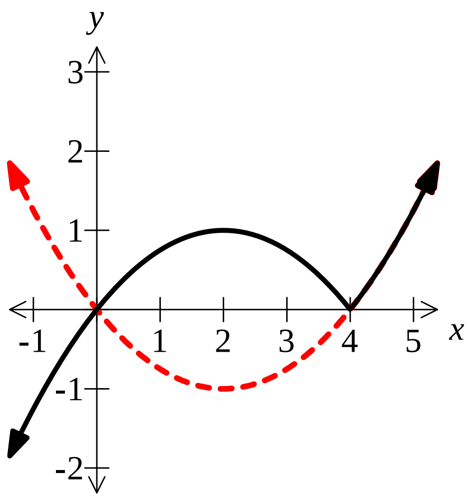
Turn over for parts (iii) and (iv)

Answer Sheet for Question 3 (a) continued

(iii) $y = \frac{x}{4}|x-4|$ **2**

When $x > 4$, then $y = \frac{x}{4}|x-4| = \frac{x(x-4)}{4}$.

When $x < 4$, then $y = \frac{x}{4}|x-4| = -\frac{x(x-4)}{4}$.



(iv) $y = \tan^{-1} f(x)$ **2**

Symmetrical around $x = 2$. The minimum is $\left(2, -\frac{\pi}{4}\right)$. Horizontal asymptote of $y = \frac{\pi}{2}$