NORTH SYDNEY GIRLS HIGH SCHOOL

2012
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- Show all necessary working in questions 11 – 16

Total Marks – 100

Section I 10 Marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II 90 marks

- Attempt Questions 11 – 16

NAME: ______________________ **TEACHER:** ______________________

NUMBER: ______________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I **Objective Response Questions**

Total marks – 10

Attempt Questions 1 – 10

Answer each question on the multiple choice answer sheet provided.

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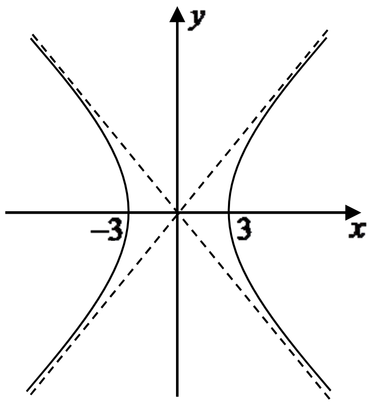
1. $\dfrac{2-i}{-2-i} = ?$

(A) $-\dfrac{3}{5}+\dfrac{4}{5}i$ (B) -1

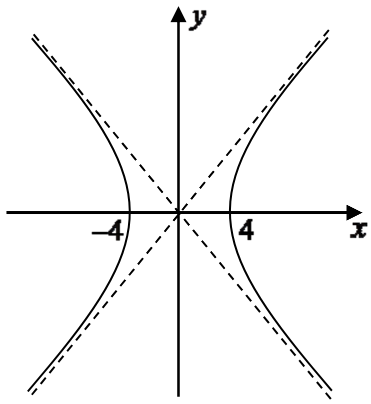
(C) $-1+\dfrac{4}{3}i$ (D) $-\dfrac{5}{3}$

2. Which of the following is the graph of $9x^2 - 16y^2 = 144$?

(A)



(B)



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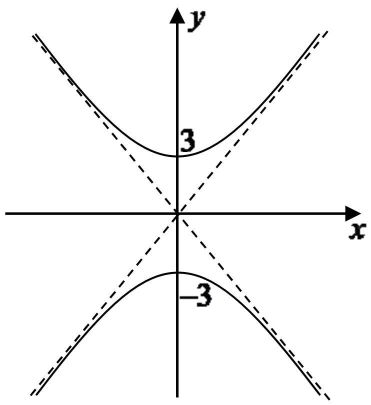
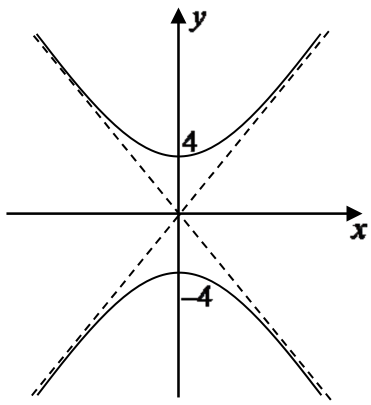
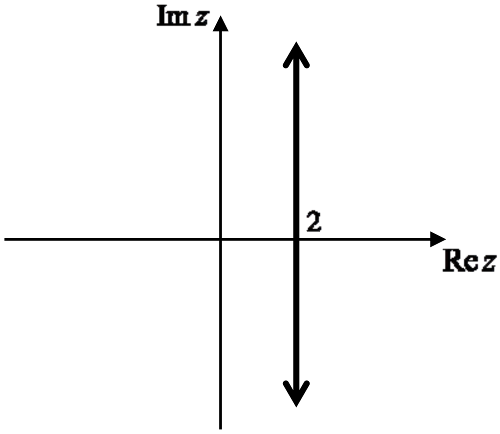


(D)



3.



Which of the following is NOT a valid algebraic description of this locus?

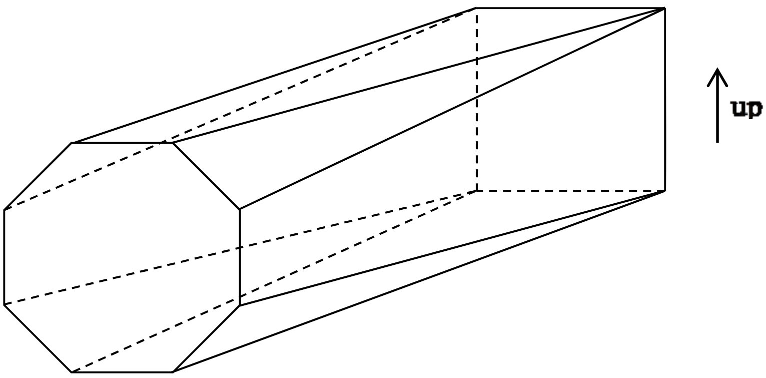
(A) $\operatorname{Re} z = 2$

(B) $|z| = |z-4|$

(C) $\arg(z-4) + \arg z = \pi$

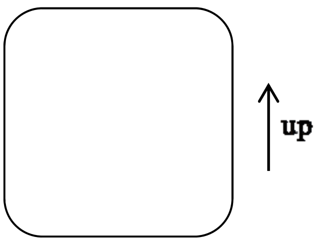
(D) $z + \overline{z} = 4$

4. The solid shown in the diagram has a pair of parallel faces, one a regular octagon, and one a square, with vertices of each end joined by straight lines.

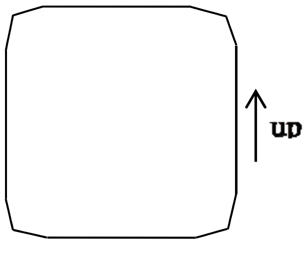


Which of the following diagrams shows a typical cross-section taken parallel to the two end faces?

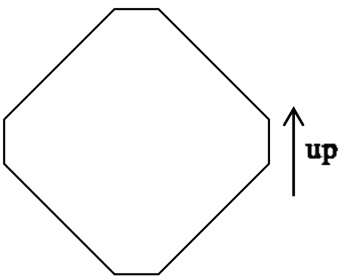
(A)



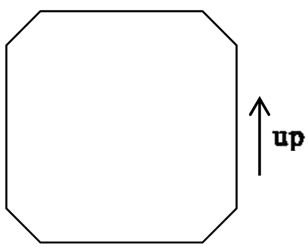
(B)



(C)



(D)



5. The polynomial equation $P(x)=0$ has roots α, β and γ.

What are the roots of the polynomial equation $P(3x+2)=0$?

(A) $\frac{\alpha}{3}-2, \frac{\beta}{3}-2, \frac{\gamma}{3}-2$

(B) $\frac{\alpha-2}{3}, \frac{\beta-2}{3}, \frac{\gamma-2}{3},$

(C) $3\alpha+2, 3\beta+2, 3\gamma+2$

(D) $\alpha+\frac{2}{3}, \beta+\frac{2}{3}, \gamma+\frac{2}{3}$

6. Consider a polynomial $P(x)$ of degree 3.

You are given 2 numbers a and b such that

- $a<b$
- $P(a)>P(b)>0$
- $P'(a)=P'(b)=0$

The polynomial has

(A) 3 real zeros

(B) 1 real zero γ such that $\gamma<a$

(C) 1 real zero γ such that $a<\gamma<b$

(D) 1 real zero α such that $\gamma>b$

7. Consider the following two statements:

I: $\int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}}$

II: $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$

Which of these statements are correct?

(A) Neither statement

(B) Statement I only

(C) Statement II only

(D) Both statements

8. A particle is moving along the x-axis, initially moving to the left from the origin.
Its velocity and acceleration are given by $v^2 = 2\ln\left(3+\cos x\right)$ and $\ddot{x} = \dfrac{-\sin x}{3+\cos x}$.

Which of the following describe its subsequent motion?

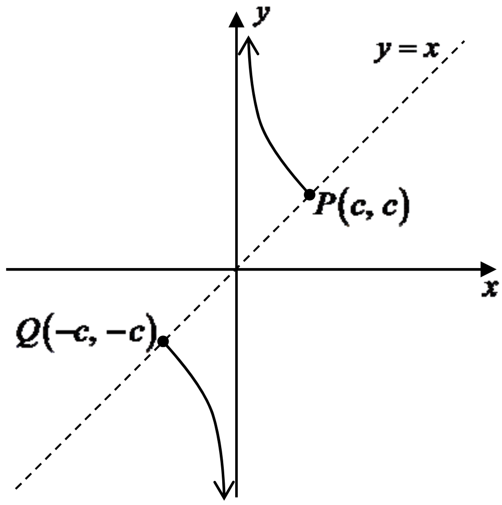
(A) Heads only to the left, alternately speeding up and slowing down, without becoming stationary.

(B) Heads only to the left, alternately slowing to a stop then speeding up.

(C) Slows to a stop, then heads to the right forever.

(D) Oscillates between two points.

9. The graph shows a part of the hyperbola $x = ct,\ y = \dfrac{c}{t}$.



Which pair of parametric equations precisely describe the sections of the hyperbola shown?

(A) $x = c\left(t^2+1\right),\ y = \dfrac{c}{t^2+1}$

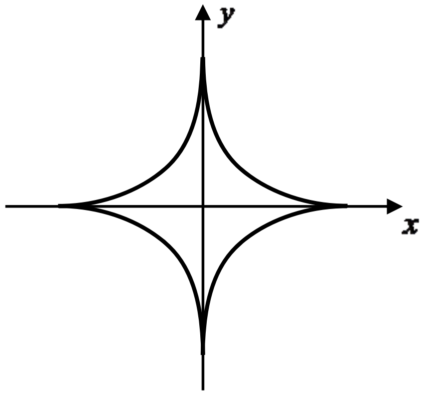
(B) $x = c\left(1-t^2\right),\ y = \dfrac{c}{1-t^2}$

(C) $x = c\sqrt{1-t^2},\ y = \dfrac{c}{\sqrt{1-t^2}}$

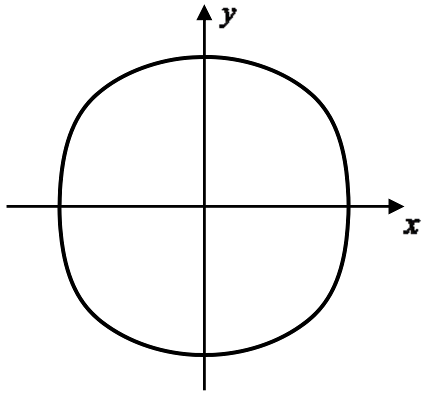
(D) $x = c\sin t,\ y = \dfrac{c}{\sin t}$

10. After differentiating a relation implicitly, we find that $\frac{dy}{dx} = \frac{y}{x}$.
Which of the following could be a graph of this relation?

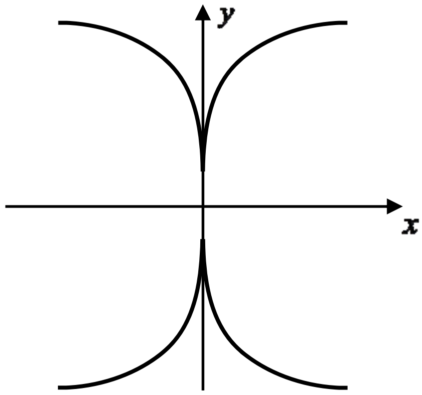
(A)



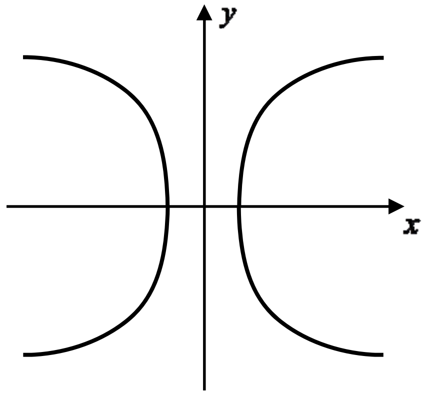
(B)



(C)



(D)



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Section II **Free Response Questions**
Total marks – 90
Attempt Questions 11 – 16

Answer each question on the multiple choice answer sheet provided.

Question 11 (15 marks)

(a) Find the exact value of $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Find $\int \frac{dx}{x^2+6x+10}$. **1**

(c) Evaluate $\int_0^1 \sin^{-1} x\;dx$. **3**

(d) (i) Show that $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x\;dx}{1+2\sin 2x+\cos 2x}$ using the substitution $t = \tan x$. **3**

(e) (i) Show that $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$. **1**

(ii) Hence, or otherwise, evaluate $\int_{-4}^{4} \left(e^x - e^{-x}\right)\cos x\,dx$. **2**

Question 12 (15 marks) Start a new booklet

(a) Given that $z = 6i - 8$, find the square roots of z in the form $a + ib$. **3**

(b) (i) Write $2 + 2\sqrt{3}i$ in modulus-argument form. **1**

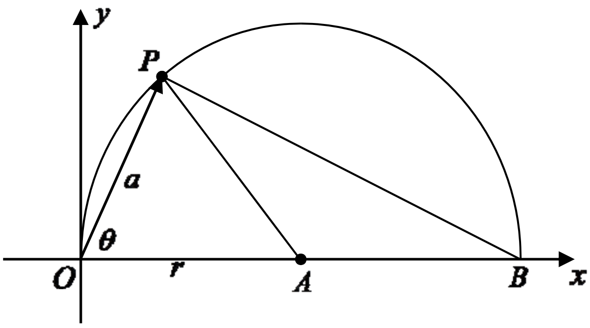
Hence:

(ii) Express $\left(2 + 2\sqrt{3}i\right)^3$ in the form $x + iy$. **1**

(iii) Find all unique solutions to the equation $z^4 = 2 + 2\sqrt{3}i$, giving answers in modulus-argument form. **2**

(c) Given z is a complex number, sketch on a number plane the locus of a point P representing z such that $\arg z = \arg\left[z - (1 + i)\right]$ **2**

(d) In the diagram, a semi-circle has diameter OB and centre A, with $OA = r$. P is a point on the semicircle, and the vector OP represents the complex number $a\,\text{cis}\,\theta$.



Write in simplest modulus-argument form the complex number represented by the vector

(i) AP **1**

(ii) BP **2**

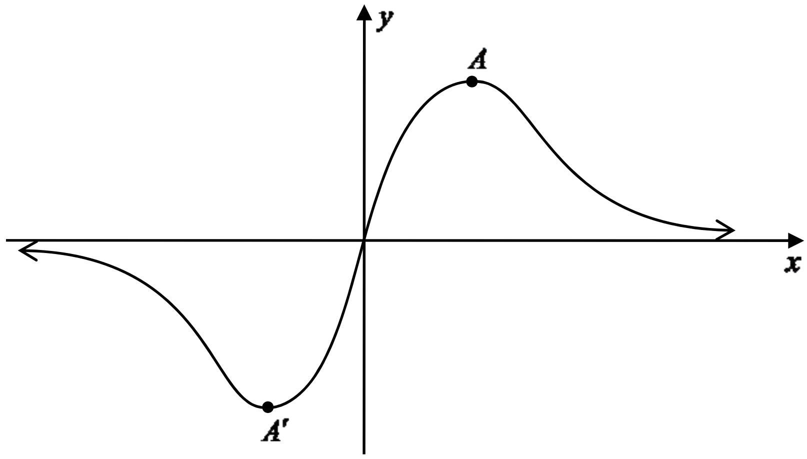
(e) In a bank of 12 switches, each switch can be set to one of three positions.

(i) Write down the total number of ways all the switches in the bank can be arranged. **1**

(ii) Find the probability that if all the switches are set randomly, there will be equal numbers in each position. **2**

Question 13 (15 marks) Start a new booklet

(a) Drawn below is the graph of $y = \dfrac{2x}{1+x^2}$.



(i) Find the coordinates of the turning points A and A'. **2**
(There is no need to test their nature.)

(ii) On separate diagrams draw graphs of the following functions:

1. $y = \dfrac{|2x|}{1+x^2}$ **1**

2. $y = \dfrac{1+x^2}{2x}$ **2**

3. $y^2 = \dfrac{2x}{1+x^2}$ **2**

4. $y = \ln\left(\dfrac{2x}{1+x^2}\right)$ **2**

(b) (i) The polynomial equation $P(x) = 0$ has a double root $x = \alpha$. **2**
Show that $x = \alpha$ is also a root of the equation $P'(x) = 0$.

(ii) You are given that $y = mx$ is a tangent to the curve $y = 3 - \dfrac{1}{x^2}$. **1**
Show that the equation $mx^3 - 3x^2 + 1 = 0$ has a double root.

(iii) Hence find the equations of any such tangents. **3**

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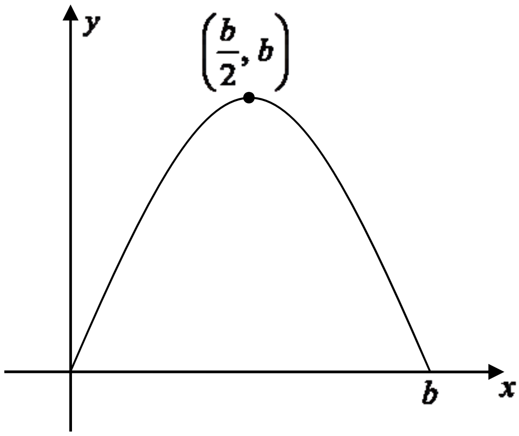
Question 14 (15 marks) Start a new booklet

(a) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = \dfrac{1}{2x+1}$, the x-axis, the y-axis and the line $x = 1$ is rotated about the line $x = 1$. **4**

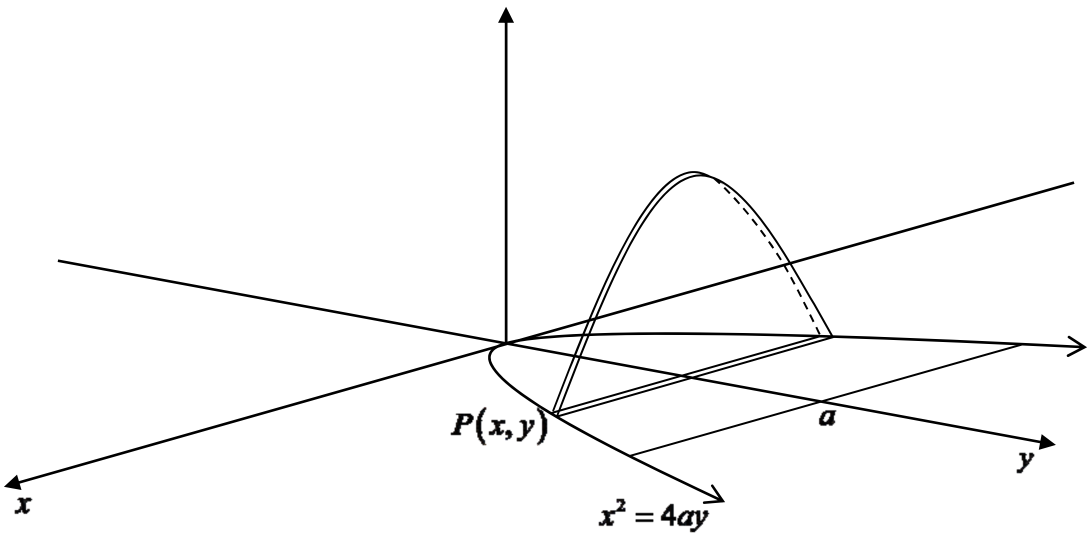
(b) The diagram shows a part of the graph of a function of the form $y = b\sin nx$.



(i) Express n in terms of b. **1**

(ii) Show that the area bounded by the curve and the x-axis is $\dfrac{2b^2}{\pi}$ units^2. **2**

(iii) The base of a solid is the region bounded by the parabola $x^2 = 4ay$ and its latus rectum. **2**

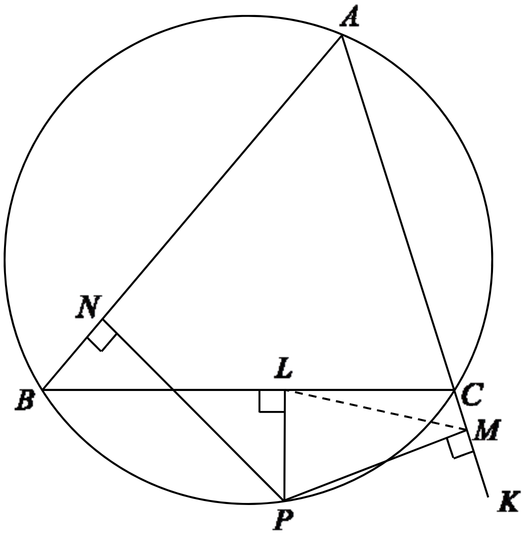


Each slice of width δy taken perpendicular to both the base and the axis of symmetry is half of a sine curve, whose amplitude is equal to its base length.

Find the volume of this solid in terms of a.

Question 14 continues on the following page

(c) In the diagram, A, B and C are three points on a circle.
P is another point on the circle, lying on the minor arc BC.
Points L, M and N are the feet of the perpendiculars from P to the sides BC, CA and AB respectively.



(i) Explain why P, L, N and B are concyclic. **1**

(ii) Explain why P, L, C and M are concyclic. **1**

Let $\angle PLM = \alpha$.

(iii) Show that $\angle ABP = \alpha$. **2**

(iv) Hence show that M, L and N are collinear. **2**

End of question 14

Question 15 (15 marks) Start a new booklet

(a) The point $P\left(a\cos\theta, b\sin\theta\right)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the ellipse meets the x-axis at the points A and A'.

(i) Prove that the tangent at P has the equation $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$. **3**

(ii) The tangent at P meets the tangents from A and A' at points Q and Q' respectively. Find the coordinates of Q and Q'. **2**

(iii) The points A, A', Q' and Q form a trapezium. Prove that the product of the lengths of the parallel sides is independent of the position of P. **2**

(b) Consider the integral $I_n=\displaystyle\int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$.

(i) Show that $I_1=\dfrac{4}{3}$. **2**

(ii) Show that $I_{n-1}-I_n=\displaystyle\int_0^1 x^{n-1}\sqrt{1-x}\,dx$. (No integration is needed.) **1**

(iii) Use integration by parts on the result of part (ii) to show that $I_n=\dfrac{2n}{2n+1}I_{n-1}$. **2**

(c) (i) Show that $a^2+b^2\geq 2ab$ for any values of a and b. **1**

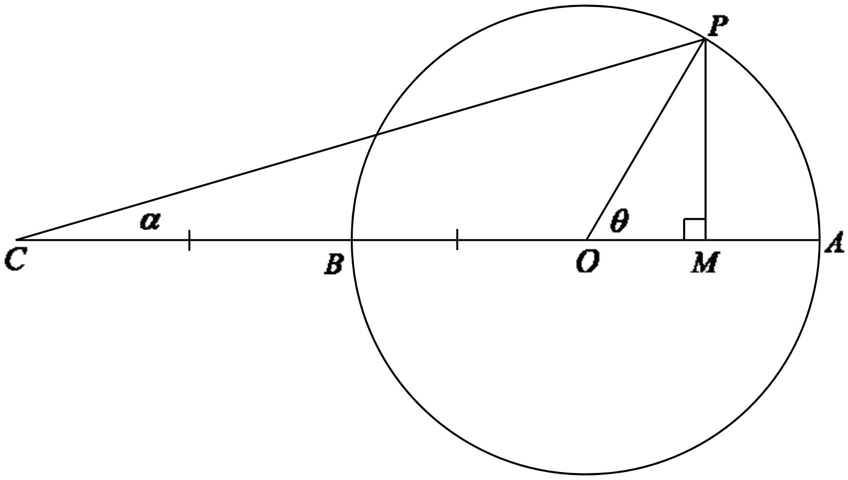
(ii) Hence show that $\tan^2\theta+\cos^2\theta+\operatorname{cosec}^2\theta\geq\sin\theta+\sec\theta+\cot\theta$ for all values of θ. **2**

Question 16 (15 marks) Start a new booklet

(a) Consider the function $f\left(x\right)=\dfrac{3\sin x}{2+\cos x}$.

(i) Show that $\dfrac{3\sin x}{2+\cos x}<x$ for $x>0$. **3**

The diagram shows a circle with centre O, where $OA=OB=BC$, $\angle POM=\theta$, $\angle PCO=\alpha$.



(ii) Show that $\tan\alpha=\dfrac{\sin\theta}{2+\cos\theta}$. **2**

(iii) Hence show that $\alpha<\dfrac{\theta}{3}$. **2**

(b) The equation $x^2+x+1=0$ has roots α and β.

A series is defined by $T_n=\alpha^n+\beta^n$ for $n=1,2,3,\ldots$.

(i) Show that $T_1=-1$ and $T_2=-1$. **2**

(ii) Show that $T_n=-T_{n-1}-T_{n-2}$ for $n=3,4,5,\ldots$. **2**

(iii) Hence use Mathematical Induction to show that $T_n=2\cos\dfrac{2n\pi}{3}$ for $n=1,2,3,\ldots$. **3**

(iv) Hence write down the value of $\displaystyle\sum_{k=1}^{2012}T_k$. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2012 Extension 2 Trial Solutions

Section I

1.	A	**2.**	B	**3.**	C	**4.**	D	**5.**	B
6.	B	**7.**	D	**8.**	A	**9.**	D	**10.**	C

Section II

Question 11

(a)
$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 (-2x)e^{-x^2}\,dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= -\frac{1}{2}\left(e^{-1}-1\right)$$
$$= \boldsymbol{\frac{e-1}{2e}}$$

(b)
$$\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1}$$
$$= \boldsymbol{\tan^{-1}(x+3) + c}$$

(c)
$$\int_0^1 \sin^{-1}x\,dx = \int_0^1 \sin^{-1}x\cdot\frac{d}{dx}(x)\,dx$$
$$= \left[x\sin^{-1}x\right]_0^1 - \int_0^1 x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \frac{\pi}{2} - 0 + \frac{1}{2}\int_0^1 (-2x)\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$
$$= \frac{\pi}{2} + \frac{1}{2}\cdot 2\left[\left(1-x^2\right)^{\frac{1}{2}}\right]_0^1$$
$$= \boldsymbol{\frac{\pi}{2} - 1}$$

OR (by subtraction of areas)

$$\int_0^1 \sin^{-1}x\,dx = 1\times\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx$$
$$= \frac{\pi}{2} + \left[\cos x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2} + (0-1)$$
$$= \frac{\pi}{2} - 1$$

(d) (i) Let $\dfrac{5-5x^2}{(1+2x)\left(1+x^2\right)} = \dfrac{a}{1+2x} + \dfrac{bx+c}{1+x^2}$

$$5-5x^2 = a\left(1+x^2\right) + (bx+c)(1+2x)$$

$\left(x=-\frac{1}{2}\right)$ $\quad \frac{15}{4} = \frac{5a}{4}$

$a = 3$

$(x=0)$ $\quad 5 = a+c$

$c = 2$

$(x=1)$ $\quad 0 = 2a + 3(b+c)$

$0 = 6 + 3(b+2)$

$b = -4$

$$\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \int_0^1 \left(\frac{3}{1+2x} - \frac{4x}{1+x^2} + \frac{2}{1+x^2}\right)dx$$

$$= \left[\frac{3}{2}\ln(1+2x) - 2\ln(1+x^2) + 2\tan^{-1}x\right]_0^1$$

$$= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2} - 0$$

$$= \frac{1}{2}(3\ln 3 - 4\ln 2 + \pi)$$

$$= \frac{1}{2}\left(\ln\frac{3^3}{2^4} + \pi\right)$$

$$= \frac{1}{2}\ln\left(\pi + \ln\frac{27}{16}\right)$$

(ii) $$\int_0^{\frac{\pi}{4}} \frac{\cos 2x\;dx}{1+2\sin 2x+\cos 2x} = \int_0^1 \frac{\frac{1-t^2}{1+t^2}\cdot\frac{dt}{1+t^2}}{1+\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{(1+t^2)^2}{(1+t^2)^2}$$

Let $t = \tan x$

$dt = \sec^2 x\,dx$

$= (1+\tan^2 x)\,dx$

$dx = \dfrac{dt}{1+t^2}$

$x = 0,\quad t = 0$

$x = \dfrac{\pi}{4},\quad t = 1$

$$= \int_0^1 \frac{(1-t^2)\,dt}{(1+t^2)\left[(1+t^2)+4t+(1-t^2)\right]}$$

$$= \int_0^1 \frac{(1-t^2)\,dt}{(1+t^2)(4t+2)}$$

$$= \frac{1}{10}\int_0^1 \frac{(5-5t^2)\,dt}{(1+t^2)(2t+1)}$$

$$= \boldsymbol{\frac{1}{20}\left(\pi + \ln\frac{27}{16}\right)}$$

(e) (i) $$\int_{-a}^{a} f(x)\,dx = \int_{a}^{-a} f(-u)(-du)$$

$$= \int_{-a}^{a} f(-x)\,dx$$

let $u = -x$

$du = -dx$

$x = -a,\quad u = a$

$x = a,\quad u = -a$

(ii) $$\int_{-4}^{4} (e^x - e^{-x})\cos x\,dx = \int_{-4}^{4} (e^{-x} - e^{x})\cos(-x)\,dx \qquad \text{(from part i)}$$

$$= -\int_{-4}^{4} (e^x - e^{-x})\cos x\,dx$$

$$2\int_{-4}^{4} (e^x - e^{-x})\cos x\,dx = 0$$

$$\int_{-4}^{4} (e^x - e^{-x})\cos x\,dx = 0$$

Question 12

(a) Let $\sqrt{6i-8}=a+ib$ (where a and b are real)

$$\left(a^2-b^2\right)+2abi=-8+6i$$

Equating real and imaginary parts:

$$2ab=6 \qquad\qquad a^2-b^2=-8$$

$$b=\frac{3}{a} \qquad\qquad a^2-\frac{9}{a^2}=-8$$

$$\left(\times a^2\right) \qquad a^4+8a^2-9=0$$

$$\left(a^2+9\right)\left(a^2-1\right)=0$$

$$a=\pm 1$$

$$b=\pm 3 \qquad\qquad \boldsymbol{\sqrt{6i-8}=\pm\left(1+3i\right)}$$

Alternatively

As $\left|6i-8\right|=10$, then $\left|a+ib\right|=\sqrt{10}$, so $a^2+b^2=10$, then solve this with the 2nd equation by elimination, substituting the answers in the 1st equation to find the second pronumeral.

(b) (i) $2+2\sqrt{3}i=\boldsymbol{4\,\mathrm{cis}\,\frac{\pi}{3}}$

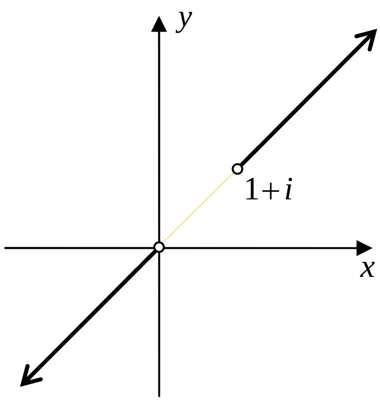
(ii) $$\left(2+2\sqrt{3}i\right)^3=\left(4\,\mathrm{cis}\frac{\pi}{3}\right)^3$$
$$=64\,\mathrm{cis}\,\pi$$
$$=\boldsymbol{-64}$$

(iii) $$z^4=4\,\mathrm{cis}\left(\frac{\pi}{3}+2n\pi\right),\text{ where } n \text{ is an integer}$$
$$=4\,\mathrm{cis}\left(\frac{6n+1}{3}\pi\right)$$
$$z=\sqrt{2}\,\mathrm{cis}\left(\frac{6n+1}{12}\pi\right)$$

Taking $n=-2,-1,0,1$:

$$\boldsymbol{z=\sqrt{2}\,\mathrm{cis}\frac{\pi}{12},\ \sqrt{2}\,\mathrm{cis}\frac{7\pi}{12},\ \sqrt{2}\,\mathrm{cis}\left(-\frac{5\pi}{12}\right),\ \sqrt{2}\,\mathrm{cis}\left(-\frac{11\pi}{12}\right)}$$

(c)



(d) (i) $\angle BAP = 2\theta$ (angle at centre is twice angle at circumference)

$\therefore\ \boldsymbol{\overrightarrow{AP} = r\,\mathbf{cis}\,2\theta}$

(ii) $\angle OPB = \dfrac{\pi}{2}$ (angle in semi-circle)

$\angle PBx = \theta + \dfrac{\pi}{2}$ (exterior angle of triangle = sum of opposite two interior angles)

$PB^2 = (2r)^2 - a^2$ (Pythagoras')

$\therefore\ \boldsymbol{\overrightarrow{BP} = \sqrt{4r^2 - a^2}\,\mathbf{cis}\left(\theta + \dfrac{\pi}{2}\right)}$

(e) (i) $\mathbf{3^{12}}$

(ii) $\boldsymbol{\dfrac{{}^{12}C_4 \times {}^{8}C_4}{3^{12}} = \dfrac{3850}{59049}}$

NB: We don't divide by 3!, as the 3 groups are considered different, and are enumerated as different cases when the sample space is calculated in part (i).

OR

Name the switch positions A, B, C.
The question is the same as forming distinct words from the letters A A A A B B B B C C C C.

ie. $\dfrac{\frac{12!}{(4!)^3}}{3^{12}} = \dfrac{3850}{59049}$

Question 13

(a) (i) $y=\dfrac{2x}{1+x^2}$

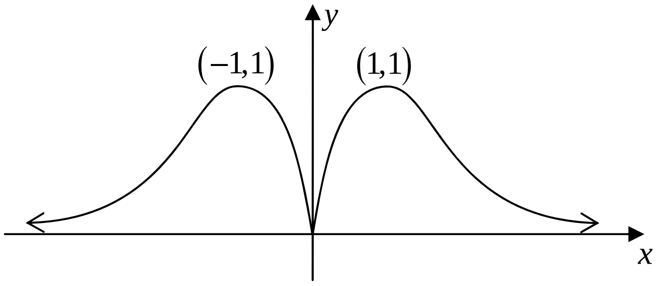
$$\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot 2-2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$=\frac{2-2x^2}{\left(1+x^2\right)^2}$$

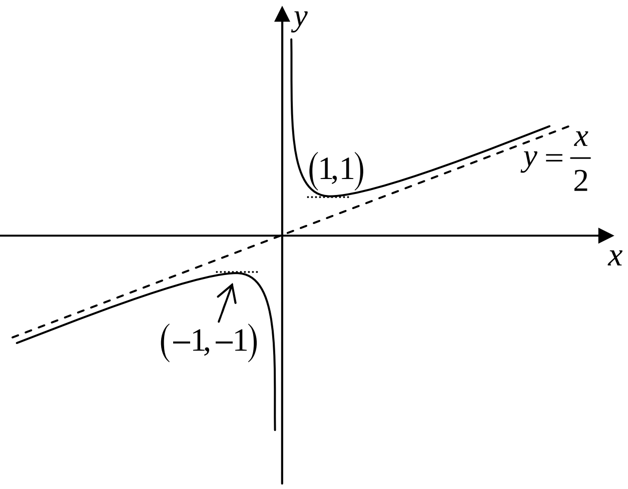
Stat Pts: $\dfrac{dy}{dx}=0 \;\Rightarrow\; x=\pm 1,\; y=\pm 1$

ie. stat points at $\mathbf{(1,1)}$ and $\mathbf{(-1,-1)}$

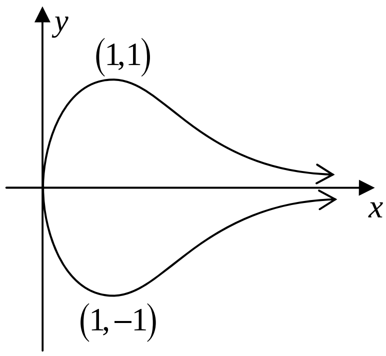
(ii) 1. $y=\dfrac{|2x|}{1+x^2}$



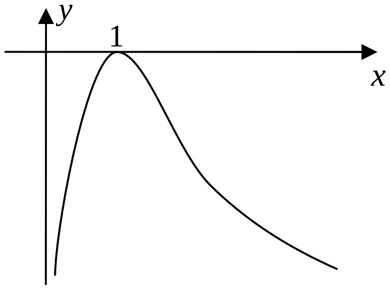
2. $y=\dfrac{1+x^2}{2x}$



3. $y^2=\dfrac{2x}{1+x^2}$



4. $y=\ln\left(\dfrac{2x}{1+x^2}\right)$



(b) (i) Let $P(x)=(x-\alpha)^2 Q(x)$.

$$P'(x)=Q(x)\cdot 2(x-\alpha)+(x-\alpha)^2\cdot Q'(x)$$
$$=(x-\alpha)\left[2Q(x)+(x-\alpha)Q'(x)\right]$$

$\therefore\ P'(\alpha)=0$

So $x=\alpha$ is a root of $P(x)=0$.

(ii) If $y=mx$ is a tangent, then $mx=3-\dfrac{1}{x^2}$ has a double root.

ie. $mx^3=3x^2-1$

$$mx^3-3x^2+1=0$$

(iii) Let $P(x)=mx^3-3x^2+1$

$$P'(x)=3mx^2-6x$$

By part (i), the double root $x=\alpha$ must be a root of

$$3mx^2-6x=0$$
$$3x(mx-2)=0$$
$$x=0 \text{ or } x=\frac{2}{m}$$

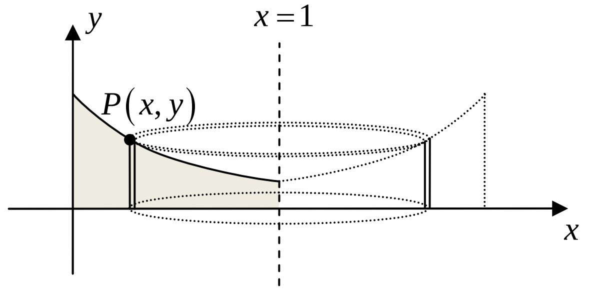
$P(0)\neq 0$, so $x=\dfrac{2}{m}$ must be the double root.

$$P\left(\frac{2}{m}\right)=0$$
$$m\left(\frac{2}{m}\right)^3-3\left(\frac{2}{m}\right)^2+1=0$$
$$\frac{8}{m^2}-\frac{12}{m^2}+1=0$$
$$\frac{4}{m^2}=1$$
$$m=\pm 2$$

So the tangents have equation $\boldsymbol{y=\pm 2x}$.

Question 14

(a)



Outer radius, $R = 1 - x$

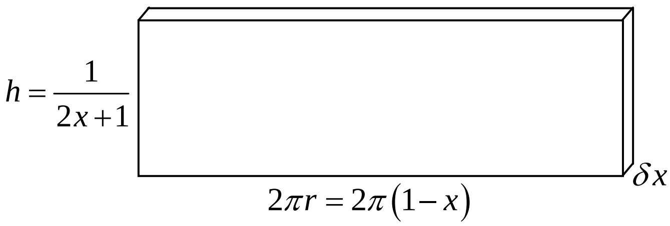
Inner radius, $r = 1 - x - \delta x$

Height, $h = y = \dfrac{1}{2x+1}$

Volume of typical slice:

$$\begin{aligned}\delta V &\approx \pi\left(R^2 - r^2\right)h \\ &= \pi(R+r)(R-r)h \\ &= \pi(2 - 2x - \delta x)(\delta x)\cdot\frac{1}{2x+1} \\ &\approx 2\pi\cdot\frac{1-x}{2x+1}\cdot\delta x \qquad \text{when } \delta x \text{ is sufficiently small}\end{aligned}$$

OR



$$\delta V \approx 2\pi(1-x)\cdot\frac{1}{2x+1}\cdot\delta x$$

$$\begin{aligned}V &= \lim_{\delta x\to 0}\sum_{x=0}^{1} 2\pi\frac{1-x}{2x+1}\delta x \\ &= 2\pi\int_0^1 \frac{1-x}{2x+1}\,dx \\ &= 2\pi\int_0^1 \frac{-\frac{1}{2}(2x+1)+\frac{3}{2}}{2x+1}\,dx \\ &= 2\pi\int_0^1\left(-\frac{1}{2}+\frac{3}{2(2x+1)}\right)dx \\ &= 2\pi\left[-\frac{x}{2}+\frac{3}{4}\ln(2x+1)\right]_0^1 \\ &= 2\pi\left(-\frac{1}{2}+\frac{3}{4}\ln 3 - 0\right) \\ &= \boldsymbol{\frac{\pi}{2}(3\ln 3 - 2)\ \textbf{units}^3}\end{aligned}$$

(b) (i) Period = $2b$

$$\frac{2\pi}{n} = 2b$$

$$\boldsymbol{n = \frac{\pi}{b}}$$

(ii) $$\text{Area} = \int_0^b b\sin\frac{\pi}{b}x\,dx$$

$$= -\frac{b^2}{\pi}\left[\cos\frac{\pi}{b}x\right]_0^b$$

$$= -\frac{b^2}{\pi}(-1-1)$$

$$= \frac{2b^2}{\pi}\ \text{units}^2$$

(iii) from part (ii), $$\delta V = \frac{2b^2}{\pi}\delta y$$

$$= \frac{2(2x)^2}{\pi}\delta y$$

$$= \frac{8}{\pi}x^2\delta y$$

$$= \frac{8}{\pi}(4ay)\delta y$$

$$= \frac{32a}{\pi}y\,\delta y$$

$$V = \lim_{\delta y\to 0}\sum_{y=0}^{a}\frac{32a}{\pi}y\,\delta y$$

$$= \frac{32a}{\pi}\int_0^a y\,dy$$

$$= \frac{16a}{\pi}\left[y^2\right]_0^a$$

$$\boldsymbol{V = \frac{16}{\pi}a^3\ \text{units}^3}$$

(c) (i) BP subtends equal angles at N and L (converse of angles in same segment)

(ii) $\angle PLC + \angle PMC = 90° + 90°$

$= 180°$

$\therefore$ $PLCM$ is cyclic (opposite angles are supplementary)

(iii) Construct BP and PM.

$\angle PCM = \angle PLM = \alpha$ (angles in same segment of circle $PLCM$)

$\angle ABP = \angle PCM = \alpha$ (exterior angle or cyclic quad $BACP$ = opposite interior angle)

(iii) Construct NL.

$\angle NLP + \angle NBP = 180°$ (opposite angles of cyclic quad $PLNB$ are supplementary)

$\angle NLP = 180° - \alpha$

$\angle NLP + \angle PLM = (180 - \alpha) + \alpha$

$\angle MLN = 180°$

ie. M, L, and N are collinear

NB: We can't call $\angle PLM$ the exterior angle of $PLNB$ **until** we know that MLN is a straight line.

Question 15

(a) (i)

$$x = a\cos\theta \qquad y = b\sin\theta \qquad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta \qquad = b\cos\theta\cdot\frac{-1}{a\sin\theta}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}\left(x - a\cos\theta\right)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab\left(\sin^2\theta + \cos^2\theta\right)$$

$$(\div ab) \quad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

(ii)

$$(x = \pm a) \quad \frac{\pm a\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$\frac{y\sin\theta}{b} = 1 \mp \cos\theta$$

$$y = \frac{b\left(1 \mp \cos\theta\right)}{\sin\theta}$$

ie. $\boldsymbol{Q\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)}$, $\boldsymbol{Q'\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)}$

(iii) Product of lengths

$$= \frac{b\left(1-\cos\theta\right)}{|\sin\theta|}\cdot\frac{b\left(1+\cos\theta\right)}{|\sin\theta|}$$

$$= \frac{b^2\left(1-\cos^2\theta\right)}{\sin^2\theta}$$

$$= \frac{b^2\sin^2\theta}{\sin^2\theta}$$

$$= b^2 \quad \text{(which is a constant)}$$

(b) (i)

$$I_1 = \int_0^1 \frac{x\,dx}{\sqrt{1-x}}$$

$$= \int_0^1 \frac{-(1-x)+1}{\sqrt{1-x}}\,dx$$

$$= \int_0^1 \left(-(1-x)^{\frac{1}{2}} + (1-x)^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{2}{3}(1-x)^{\frac{3}{2}} - 2(1-x)^{\frac{1}{2}}\right]_0^1$$

$$= (0-0) - \left(\frac{2}{3} - 2\right)$$

$$= \frac{4}{3}$$

Other options:

1. Let $u = 1-x$
2. Let $u^2 = 1-x$
3. Let $x = \sin^2\theta$ (messy)
4. Integration by parts

(ii) $$I_{n-1}-I_n=\int_0^1 \frac{x^{n-1}}{\sqrt{1-x}}\,dx-\int_0^1 \frac{x^n}{\sqrt{1-x}}\,dx$$

$$=\int_0^1 \frac{x^{n-1}-x^n}{\sqrt{1-x}}\,dx$$

$$=\int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\,dx$$

$$=\int_0^1 x^{n-1}\sqrt{1-x}\,dx$$

(iii) $$I_{n-1}-I_n=\int_0^1 x^{n-1}\sqrt{1-x}\,dx$$

$$=\int_0^1 \sqrt{1-x}\cdot\frac{d}{dx}\left(\frac{x^n}{n}\right)dx$$

$$=\frac{1}{n}\left[x^n\sqrt{1-x}\right]_0^1-\frac{1}{n}\int_0^1 x^n\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1)\,dx$$

$$=0+\frac{1}{2n}\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}}$$

$$I_{n-1}-I_n=\frac{1}{2n}I_n$$

$$2nI_{n-1}-2nI_n=I_n$$

$$2nI_{n-1}=(2n+1)I_n$$

$$I_n=\frac{2n}{2n+1}I_{n-1}$$

(c) (i) $$(a-b)^2\geq 0 \qquad \forall a,b$$

$$a^2-2ab+b^2\geq 0$$

$$a^2+b^2\geq 2ab$$

(ii) From (i), $\tan^2\theta+\cos^2\theta\geq 2\tan\theta\cos\theta=2\sin\theta$

$\tan^2\theta+\operatorname{cosec}^2\theta\geq 2\tan\theta\operatorname{cosec}\theta=2\sec\theta$

$\cos^2\theta+\operatorname{cosec}^2\theta\geq 2\cos\theta\operatorname{cosec}\theta=2\cot\theta$

Adding: $2\left(\tan^2\theta+\cos^2\theta+\operatorname{cosec}^2\theta\right)\geq 2\left(\sin\theta+\sec\theta+\cot\theta\right)$

$$\tan^2\theta+\cos^2\theta+\operatorname{cosec}^2\theta\geq\sin\theta+\sec\theta+\cot\theta$$

Question 16

(a) (i) Let $f(x) = x - \dfrac{3\sin x}{2+\cos x}$.

$$f(0) = 0$$

$$f'(x) = 1 - \frac{(2+\cos x)(3\cos x) - (3\sin x)(-\sin x)}{(2+\cos x)^2}$$

$$= \frac{(2+\cos x)^2 - 3\cos x(2+\cos x) - 3\sin^2 x}{(2+\cos x)^2}$$

$$= \frac{4+4\cos x+\cos^2 x - 6\cos x - 3\cos^2 x - 3 + 3\cos^2 x}{(2+\cos x)^2}$$

$$= \frac{1-2\cos x+\cos^2 x}{(2+\cos x)^2}$$

$$= \left(\frac{1-\cos x}{2+\cos x}\right)^2$$

$$\geq 0 \quad \forall x$$

$\therefore \; f(x) > 0 \quad \forall x > 0$ (starts at zero, and decreases monotonically)

$$\therefore x - \frac{3\sin x}{2+\cos x} > 0$$

$$\therefore \frac{3\sin x}{2+\cos x} < x \quad \text{for } x > 0$$

(ii) Let $OB = OP = r$

from ΔMOP, $OM = r\cos\theta$ and $PM = r\sin\theta$

In ΔCMP, $\tan\alpha = \dfrac{PM}{CM}$

$$= \frac{PM}{CO+OM}$$

$$= \frac{r\sin\theta}{2r + r\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{2+\cos\theta}$$

(iii) $\tan\alpha = \dfrac{1}{3}\cdot\dfrac{3\sin\theta}{2+\cos\theta}$

$$< \frac{1}{3}\theta \quad (\text{from part i, since } \theta > 0)$$

Also, for $0 < \alpha < \dfrac{\pi}{2}$, $\alpha < \tan\alpha$.

$$\therefore \quad \alpha < \frac{\theta}{3}$$

(b) (i) $T_1 = \alpha + \beta$

$$= \frac{-1}{1}$$

$$= -1$$

$T_2 = \alpha^2 + \beta^2$

$$= (\alpha+\beta)^2 - 2\alpha\beta$$

$$= (-1)^2 - 2(1)$$

$$= -1$$

(ii) Since $x^2+x+1=0$ has roots α and β, then:

$\left(\times x^{n-2}\right)$ $\quad x^n+x^{n-1}+x^{n-2}=0$ has roots α and β (and a root 0 of multiplicity $n-2$)

$$\therefore \quad \alpha^n+\alpha^{n-1}+\alpha^{n-2}=0$$
$$\text{and} \quad \beta^n+\beta^{n-1}+\beta^{n-2}=0$$

adding: $\quad \left(a^n+\beta^n\right)+\left(\alpha^{n-1}+\beta^{n-1}\right)+\left(\alpha^{n-2}+\beta^{n-2}\right)=0$

$$T_n+T_{n-1}+T_{n-2}=0$$
$$T_n=-T_{n-1}-T_{n-2}$$

OR

$$\begin{aligned}
\text{RHS} &= -T_{n-1}-T_{n-2} \\
&= -\left(\alpha^{n-1}+\beta^{n-1}\right)-\left(\alpha^{n-2}+\beta^{n-1}\right) \\
&= -\left[\left(\frac{\alpha^n}{\alpha}+\frac{\alpha^n}{\alpha^2}\right)+\left(\frac{\beta^n}{\beta}+\frac{\beta^n}{\beta^2}\right)\right] \\
&= -\left[\alpha^n\left(\frac{1+\alpha}{\alpha^2}\right)+\beta^n\left(\frac{1+\beta}{\beta^2}\right)\right] \\
&= -\left[\alpha^n(-1)+\beta^n(-1)\right] \qquad \left[1+\alpha+\alpha^2=1+\beta+\beta^2=0\right] \\
&= \alpha^n+\beta^n \\
&= T_n
\end{aligned}$$

(iii) Test $n=1$ and $n=2$: $\quad \text{RHS}=2\cos\dfrac{2\pi}{3}=-1=T_1=\text{LHS}$

$$\text{RHS}=2\cos\frac{4\pi}{3}=-1=T_2=\text{LHS}$$

$\therefore$ true for $n=1$ and $n=2$

Assume true for $n=k$ and $n=k+1$:

ie. $\quad T_k=2\cos\dfrac{2k\pi}{3}$ and $T_{k+1}=2\cos\dfrac{2(k+1)\pi}{3}$

Prove true for $n=k+2$:

ie. $\quad$ Prove $T_{k+2}=2\cos\dfrac{2(k+2)\pi}{3}$

$$\begin{aligned}
T_{k+2} &= -T_{k+1}-T_k \qquad (\text{from part ii}) \\
&= -2\cos\frac{2(k+1)\pi}{3}-2\cos\frac{2k\pi}{3} \qquad (\text{by assumption}) \\
&= -2\cos\left(\frac{2k\pi}{3}+\frac{2\pi}{3}\right)-2\cos\frac{2k\pi}{3} \\
&= -2\left(\cos\frac{2k\pi}{3}\cos\frac{2\pi}{3}-\sin\frac{2k\pi}{3}\sin\frac{2\pi}{3}+\cos\frac{2k\pi}{3}\right) \\
&= -2\left(-\frac{1}{2}\cos\frac{2k\pi}{3}-\frac{\sqrt{3}}{2}\sin\frac{2k\pi}{3}+\cos\frac{2k\pi}{3}\right) \\
&= -2\left(\frac{1}{2}\cos\frac{2k\pi}{3}-\frac{\sqrt{3}}{2}\sin\frac{2k\pi}{3}\right)
\end{aligned}$$

$$= 2\left(-\frac{1}{2}\cos\frac{2k\pi}{3} + \frac{\sqrt{3}}{2}\sin\frac{2k\pi}{3}\right)$$

$$= 2\left(\cos\frac{2k\pi}{3}\cos\frac{4\pi}{3} - \sin\frac{2k\pi}{3}\sin\frac{4\pi}{3}\right)$$

$$= 2\cos\left(\frac{2k\pi}{3} + \frac{4\pi}{3}\right)$$

$$= 2\cos\frac{2(k+2)\pi}{3}$$

$\therefore$ True for $n = k+2$ when true for $n = k$ and $n = k+1$

$\therefore$ By Mathematical Induction, $T_n = 2\cos\frac{2n\pi}{3}$ for $n = 1, 2, 3, \ldots$

(iv) $\sum_{k=1}^{2012} T_k = (-1) + (-1) + 2 + (-1) + (-1) + 2 + \ldots$

$= \mathbf{-2}$ (since 2012 is two more than a multiple of 3)