NORTH SYDNEY GIRLS HIGH SCHOOL

Mathematics Extension 2

2013 Trial HSC Examination

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section 1 – Pages 2 – 5
10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – Pages 6-14
90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

Student Number ____________________

Class__________________

Student Name _____________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Let $z = a + ib$ where a and b are real and non-zero. Which of the following is not true?

(A) $z + \overline{z}$ is real

(B) $\dfrac{z}{\overline{z}}$ is non-real

(C) $z^2 - (\overline{z})^2$ is real

(D) $z\overline{z}$ is real and positive

2 Which of the following corresponds to the set of points in the complex plane defined by $|z + 2i| = |z|$?

(A) the point given by $z = -i$

(B) the line $\mathrm{Im}(z) = -1$

(C) the circle with centre $-2i$ and radius 1

(D) the line $\mathrm{Re}(z) = -1$

3 The equation $9x^3 - 27x^2 + 11x - 7 = 0$ has roots α, β and γ.

What is the value of $\dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\alpha\beta}$?

(A) $\dfrac{27}{7}$

(B) $-\dfrac{27}{7}$

(C) $-\dfrac{11}{7}$

(D) $\dfrac{11}{7}$

4 The polynomial equation $P(x) = 0$ has real coefficients, and has roots which include $x = -2 + i$ and $x = 2$. What is the minimum possible degree of $P(x)$?

(A) 1 (B) 2 (C) 3 (D) 4

5 Using a suitable substitution, what is the correct expression for $\int_0^{\frac{\pi}{3}} \sin^3 x \cos^4 x \, dx$ in terms of u?

(A) $\int_0^{\frac{\sqrt{3}}{2}} \left(u^4 - u^6\right) du$

(B) $\int_1^{\frac{1}{2}} \left(u^6 - u^4\right) du$

(C) $\int_{\frac{1}{2}}^{1} \left(u^6 - u^4\right) du$

(D) $\int_0^{\frac{\sqrt{3}}{2}} \left(u^6 - u^4\right) du$

6 There are 5 pairs of socks in a drawer. Four socks are randomly chosen from the drawer. Which expression represents the probability that all four of the socks come from different pairs ?

(A) $1 \times \frac{1}{9} \times \frac{1}{8} \times \frac{1}{7}$

(B) $1 \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8}$

(C) $1 \times \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5}$

(D) $1 \times \frac{8}{9} \times \frac{6}{8} \times \frac{4}{7}$

7 The equation $x^3 + y^3 = 3xy$ is differentiated implicitly with respect to x. Which of the following expressions is $\frac{dy}{dx}$?

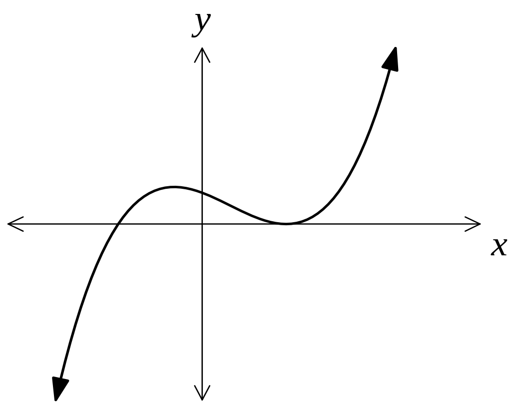
(A) $\frac{y - x^2}{y^2 - x}$

(B) $\frac{y^2 - x}{y - x^2}$

(C) $\frac{x^2 + y^2}{x}$

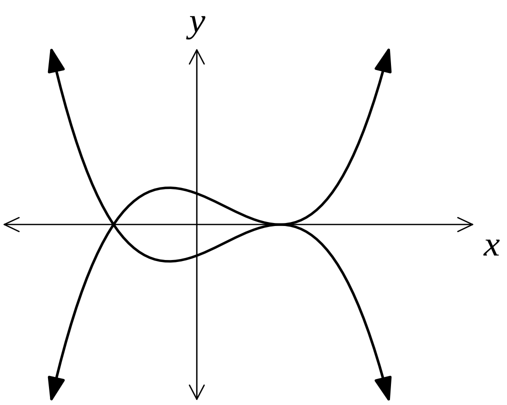
(D) $\frac{x^2}{x - y^2}$

8 The graph $y = f(x)$ is shown.

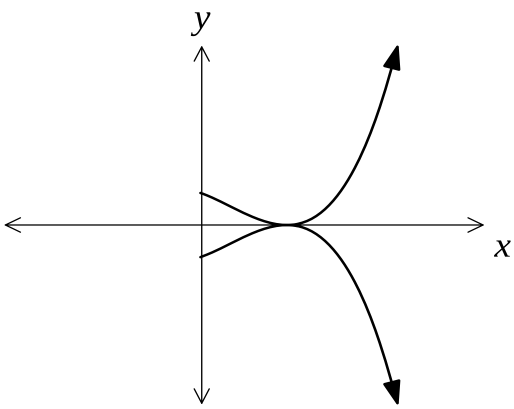


Which of the following graphs best represents $y^2 = f(x)$?

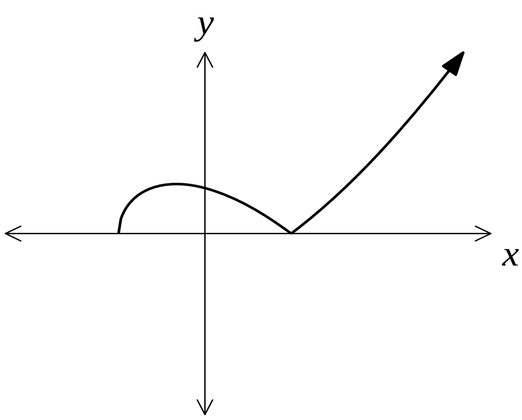
(A)



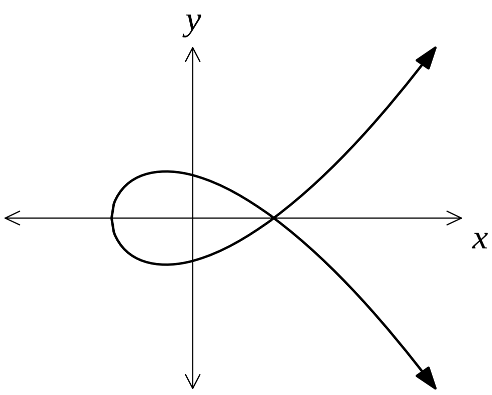
(B)



(C)



(D)



9 A body is moving in a straight line and, after t seconds, it is x metres from the origin and travelling at v ms^{-1}. Given that $v = x$ and that $t = 3$ where $x = -1$, what is the equation for x in terms of t ?

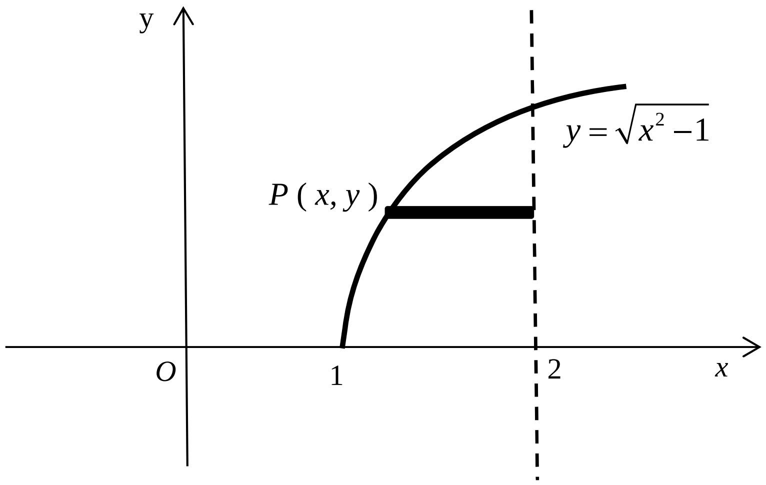
(A) $x = e^{t-3}$

(B) $x = -e^{t-3}$

(C) $x = \sqrt{2t-5}$

(D) $x = -\sqrt{2t-5}$

10



The region bounded by the x axis, the curve $y = \sqrt{x^2 - 1}$ and the line $x = 2$ is rotated around the y axis.

The slice at $P(x, y)$ on the curve is perpendicular to the axis of rotation. What is the volume δV of the annular slice formed?

(A) $\pi\left(3 - y^2\right)\delta y$

(B) $\pi\left(4 - \left(y^2 + 1\right)^2\right)\delta y$

(C) $\pi\left(4 - x^2\right)\delta x$

(D) $\pi\left(2 - x\right)^2 \delta x$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = -5 - 12i$ and $\omega = 2 - i$. Find in the form $x + iy$

(i) $(1+i)\overline{\omega}$ **1**

(ii) $\dfrac{z}{2-3i}$ **2**

(b) By first writing $w = -\sqrt{3} + i$ in modulus argument form, show that $w^3 - 8i = 0$. **2**

(c) By completing the square, find $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$. **2**

(d) Use the substitution $u = x^2 + 1$ to evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{x^2+1}}\,dx$. **3**

(e) (i) Without using calculus, sketch the curve $y = \dfrac{x+2}{(x-1)(x+3)}$ showing all important features. **2**

(ii) Find the area bounded by the curve and the x-axis between $x = 2$ and $x = 5$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Consider the complex number $z = x + iy$ where $z^2 = a + ib$.

(i) Sketch on the same set of axes, the graphs of $x^2 - y^2 = a$ and $2xy = b$ where both a and b are positive. **2**
The foci and directrices of the curves need NOT be found.

(ii) Use the graphs to explain why there are two distinct square roots of the complex number $a + ib$ if $a > 0$ and $b > 0$. **1**

(iii) Consider how the sketch changes when b is negative. What is the relationship between the new square roots and those found when b was positive? **1**

(b) The region enclosed by the curves $y = \dfrac{4}{x^2 + 4}$ and $y = \dfrac{1}{x^2 + 1}$ and the ordinates $x = 0$ and $x = 2$ is rotated about the y axis. Using the method of cylindrical shells, find the volume of the solid formed. **3**

(c) A particle's acceleration is given by $\ddot{x} = 3(1 - x)(1 + x)$ where x is the displacement in metres. Initially the particle is at the origin with velocity 2 metres per second.

(i) Show that $v^2 = 2(2 - x)(x + 1)^2$. **2**

(ii) Find the velocity and acceleration at $x = 2$. **2**

(iii) Describe the motion of the particle. **2**

(iv) Find the maximum speed and where it occurs. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $x^2 + y^2 \geq xy$ where x and y are real numbers. **2**

(ii) If $x + y = 3z$ show that $x^2 + y^2 \geq 3z^2$. **2**

(b) The complex numbers z and w each have a modulus of 2. The arguments of z and w are $\frac{4\pi}{9}$ and $\frac{7\pi}{9}$ respectively.

(i) Sketch vectors representing z, w and $z+w$ on the Argand diagram, showing any geometrical relationships between the three vectors. **2**

(ii) Find $\arg(z+w)$. **1**

(iii) Evaluate $|z+w|$. **1**

(c) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$. **3**

(ii) Show that $\int_0^{2a} f(x)\, dx = \int_0^{a} \left[f(x) + f(2a-x) \right] dx$. **2**

(iii) Hence, or otherwise, evaluate $\int_0^{\pi} \frac{x}{2+\sin x} dx$. **2**

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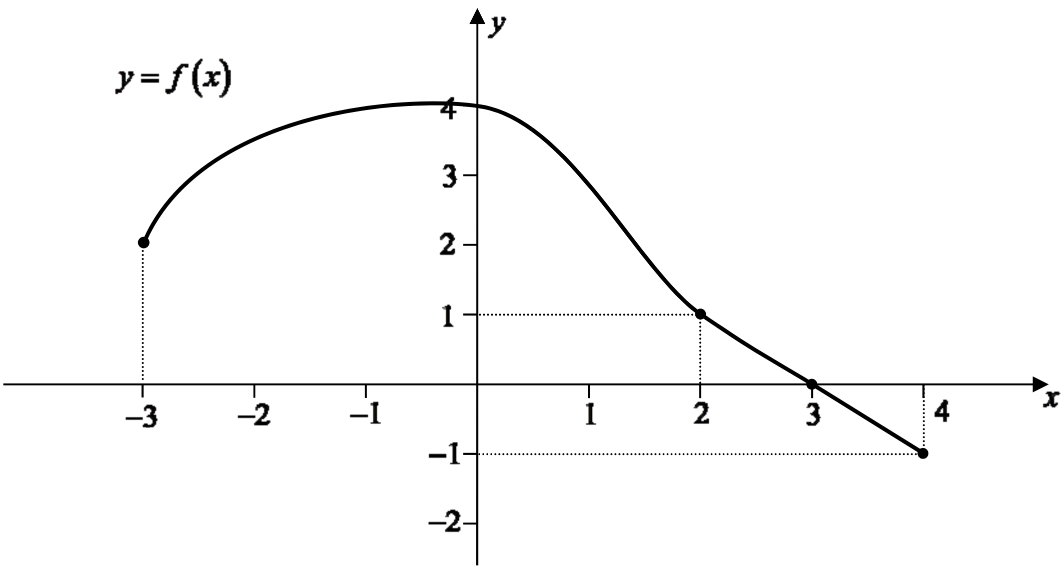
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y = f(x)$ which is only defined over the domain $-3 \le x \le 4$.

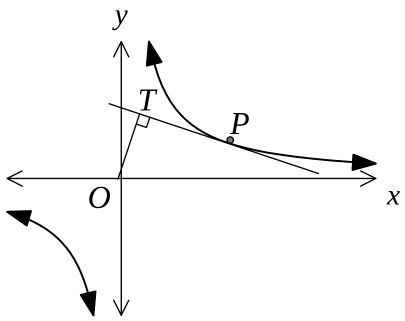


Draw separate one-third page sketches of the graphs of the following:

(i) $y = f\left(|x|\right)$ **1**

(ii) $y = \ln\left(f\left(x\right)\right)$ **2**

(b)



The point $P\left(ct, \dfrac{c}{t}\right)$ lies on the hyperbola $xy = c^2$. The point T lies at the foot of the perpendicular drawn from the origin O to the tangent at P.

(i) Show that the tangent at P has equation $x + t^2 y = 2ct$. **2**

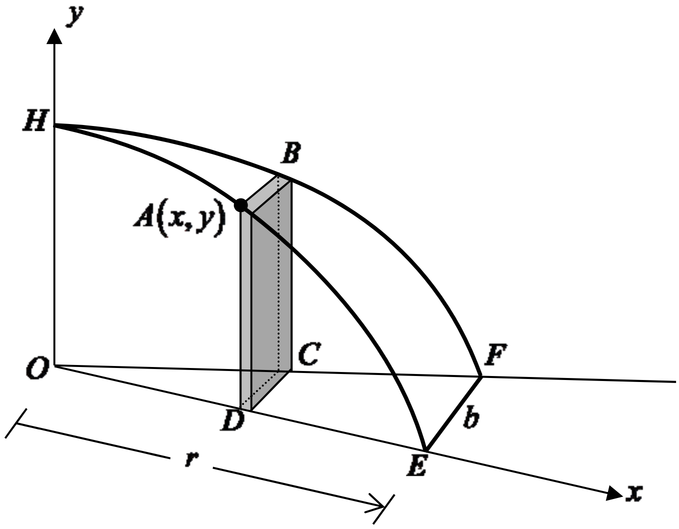
(ii) If the coordinates of T are $\left(x_1, y_1\right)$ show that $y_1 = t^2 x_1$. **1**

(iii) Show that the locus of T is given by $\left(x^2 + y^2\right)^2 = 4c^2 xy$. **2**

Question 14 continues on page 11

Question 14 (continued)

(c)



The horizontal base of a solid is an isosceles triangle OEF where $OE = OF = r$ and $EF = b$. HAE is the parabolic arc with equation $y = r^2 - x^2$ where E lies on the x-axis. HBF is another parabolic arc, congruent to HAE, so that the plane $OHBF$ is vertical. A rectangular slice $ABCD$ of width δx is taken perpendicular to the base, such that CD lies in the base and $CD \parallel EF$.

(i) Show that the volume of the slice $ABCD$ is $\dfrac{bx}{r}\left(r^2 - x^2\right)\delta x$. **2**

(ii) Hence show that the solid $HOEF$ has volume $\dfrac{br^3}{4}$. **2**

(iii) Suppose now that $\angle EOF = \dfrac{2\pi}{n}$ and that n identical solids $HOEF$ are arranged about O as centre with common vertical axis OH to form a solid ***S***. Show that the volume V_n of ***S*** is given by $V_n = \dfrac{1}{2} r^4 n \sin\dfrac{\pi}{n}$. **2**

(iv) When n is large, the solid ***S*** approximates the volume of the solid of revolution formed by rotating the region bound by the x axis and the curve $y = r^2 - x^2$ about the y axis. **1**

Using the fact that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$ find $\lim\limits_{n\to\infty} V_n$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $2x^3 - 5x + 1 = 0$ has roots α, β, γ . Find the equation whose roots are $-2\alpha, -2\beta,$ and -2γ . **2**

(b) (i) For $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **2**

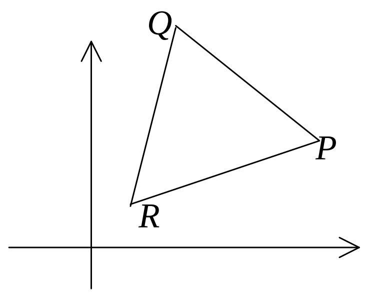
(ii) If $z + \frac{1}{z} = u$, find an expression for $z^3 + \frac{1}{z^3}$ in terms of *u*. **2**

(iii) It can be shown that $z^5 + \frac{1}{z^5} = u^5 - 5u^3 + 5u$. (Do not prove this). **3**

Show that

$$1 + \cos 10\theta = 2\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)^2$$

(c)



In the Argand diagram, the points *P*, *Q* and *R* represent the complex numbers *p*, *q* and *r*.

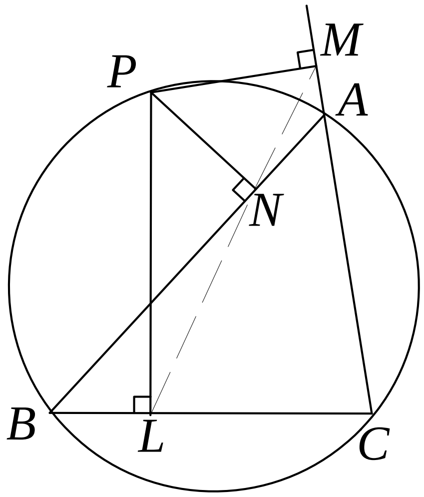
(i) Given that the triangle *PQR* is equilateral, explain why **1**

$$r - q = \text{cis}\frac{2\pi}{3}(q - p)$$

(ii) Hence, or otherwise show $2r = (p + q) + i\sqrt{3}(q - p)$ **1**

Question 15 continues on page 13

(d)



In the diagram, P is any point on the circle ABC. The point N lies on AB such that PN is perpendicular to AB. Similarly, points M and L lie at the foot of the perpendiculars drawn from P to CA (produced) and BC respectively.

(i) State why $BLNP$ is a cyclic quadrilateral. **1**

(ii) Prove that the points L, M and N are collinear. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P\left(a\cos\theta, b\sin\theta\right)$ be a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The focus of the ellipse Is $S\left(ae,0\right)$ where e is the eccentricity and O is the origin.

(i) Find the coordinates of the centre C and the radius of the circle of which SP is a diameter. **2**

(ii) Show that $OC=\dfrac{a}{2}\left(e\cos\theta+1\right)$ **2**

(b) (i) Show that the polynomial $P(x)=4x^3+10x^2+8x+3$ is divisible by $\left(2x+3\right)$. **1**

(ii) Hence express the polynomial in the form $P\left(x\right)=A\left(x\right)Q\left(x\right)$ where $Q\left(x\right)$ is a real quadratic polynomial. **2**

(c) Using the fact that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$ for $0\le x<1$ and $0\le y<1$ **3**

prove by mathematical induction that for all positive integers n,

$$\tan^{-1}\frac{1}{2\times1^2}+\tan^{-1}\frac{1}{2\times2^2}+\tan^{-1}\frac{1}{2\times3^2}+\ldots+\tan^{-1}\frac{1}{2\times n^2}=\frac{\pi}{4}-\tan^{-1}\frac{1}{2n+1}$$

Question 16 continues on page 15

Question 16 (continued)

(d) Consider $f\left(x\right)=\log x-x+1$.

(i) Show that $f\left(x\right)\leq 0$ for all $x>0$. **2**

(ii) Consider the set of n positive numbers $p_1\ ,\ p_2\ ,\ p_3\ ,\ldots p_n$ such that **1**

$$p_1+\ p_2+p_3+\ldots\ldots..+p_n=1.$$

By using the result in part (i), deduce that

$$\sum_{r=1}^{n}\log\left(np_r\right)\leq np_1+np_2+np_3\ldots+np_n-n$$

(iii) Show that $\sum_{r=1}^{n}\log np_r\leq 0$. **1**

(iv) Hence deduce that $0<n^n p_1\ p_2\ p_3\ldots..p_n\leq 1$ **1**

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2013 Extension 2 Trial HSC Solutions

1. (A) $z+\overline{z}=(a+ib)+(a-ib)$

$=2a$ (which is real)

(But you should know that $z+\overline{z}=2\operatorname{Re}z$)

(B) $\dfrac{z}{\overline{z}}=\dfrac{a+ib}{a-ib}\times\dfrac{a+ib}{a+ib}$

$=\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{2ab}{a^2+b^2}i$ (which is not real since $a,b\neq 0$)

Alternatively

$$\frac{z}{\overline{z}}=\frac{r\operatorname{cis}\theta}{r\operatorname{cis}(-\theta)}$$

$$=\operatorname{cis}2\theta$$

which is real if $2\theta=k\pi$ (where k is an integer)

$\theta=\dfrac{k}{2}\pi$ (ie. z is either real or pure imaginary)

But this is not the case, since neither a nor b is zero

(C) $z^2-(\overline{z})^2=(a+ib)^2-(a-ib)^2$

$=a^2+2abi-b^2-a^2+2abi+b^2$

$=4abi$ (which is never real since $a,b\neq 0$)

(D) $z\overline{z}=|z|^2$ which by definition of modulus is real and positive **(C)**

2. Without doing any algebra, this is the set of point which are equidistant from $(0,-2)$ and $(0,0)$.
ie. $y=-1$ **(B)**

3. $\dfrac{1}{\beta\gamma}+\dfrac{1}{\alpha\gamma}+\dfrac{1}{\alpha\beta}=\dfrac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$ **(A)**

$=\dfrac{\frac{27}{9}}{\frac{7}{9}}$

$=\dfrac{27}{7}$

4. Since $P(x)=0$ has real coefficients, the conjugate of the root $x=-2+i$ must also be a root. **(C)**
So the polynomial must have at least 3 roots (and there is not enough information to conclude more.)

5. $\displaystyle\int_0^{\frac{\pi}{3}}\sin^3 x\cos^4 x\,dx=\int_0^{\frac{\pi}{3}}\cos^4 x\left(1-\cos^2 x\right)\cdot\sin x\,dx$ Let $u=\cos x$, $du=-\sin x\,dx$; $x=0,\ u=1$; $x=\dfrac{\pi}{3},\ u=\dfrac{1}{2}$ **(B)**

$\displaystyle=\int_1^{\frac{1}{2}}u^4\left(1-u^2\right)\cdot(-du)$

$\displaystyle=\int_1^{\frac{1}{2}}\left(u^6-u^4\right)du$

6. 1st sock can be any of them.
2nd sock cannot be the only matching sock – 8 possibilities of 9 socks remaining
3rd sock cannot be either of the two matching socks – 6 possibilities of the 8 socks remaining
4th sock cannot be either of the three matching socks – 4 possibilities of the 7 socks remaining **(D)**

7.
$$x^3 + y^3 = 3xy$$
$$\cancel{3}x^2 + \cancel{3}y^2 \cdot \frac{dy}{dx} = \cancel{3}x \cdot \frac{dy}{dx} + y \cdot \cancel{3} \quad \text{(product rule)}$$
$$\left(y^2 - x\right)\frac{dy}{dx} = y - x^2$$
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Alternative setting out:
$$x^3 + y^3 = 3xy$$
(A)
$$\cancel{3}x^2 \cdot dx + \cancel{3}y^2 \cdot dy = \cancel{3}\left(x \cdot dy + y \cdot dx\right)$$
$$\left(y^2 - x\right)dy = \left(y - x^2\right)dx$$
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

8. Negative root (single root) must become a vertical point.
To the left of the negative root, $f(x)$ is –ve, so can’t be square rooted.
Positive root is a multiple root, so we can’t determine the nature of the corresponding point on the new graph. (But we are only asked for the *best* answer.)
$y^2 = f(x)$ becomes $y = \pm\sqrt{f(x)}$, hence symmetry in the x-axis. **(D)**

9. Easiest method – check the options by differentiating to get v.
The only options whose derivatives are the function itself (ie. $v = x$) are (A) and (B).
But (B) is the only option that also allows x to equal -1.

Alternative method:
$$\frac{dx}{dt} = x$$
$$\frac{dx}{x} = dt$$
$$\int \frac{dx}{x} = \int dt$$
$$\ln|x| = t + c$$

$$(t = 3,\ x = -1) \quad 0 = 3 + c$$
$$c = -3$$

$$\ln|x| = t - 3$$
$$|x| = e^{t-3}$$
$$x = \pm e^{t-3}$$

But for x to equal -1, we need the –ve case:
$$x = -e^{t-3}$$
(B)

10. $\delta V = \pi\left(R^2 - r^2\right)h$

$= \pi\left(2^2 - x^2\right)\delta y$

But $y = \sqrt{x^2 - 1}$

$y^2 = x^2 - 1$

$x^2 = y^2 + 1$

$\therefore\ \delta V = \pi\left[4 - \left(y^2 + 1\right)\right]\delta y$ **(A)**

$= \pi\left(3 - y^2\right)\delta y$

Question 11

(a) (i) $\left(1+i\right)\overline{\omega} = \left(1+i\right)\left(2+i\right)$

$= 2 + i + 2i - 1$

$= 1 + 3i$

(ii) $\dfrac{z}{2-3i} = \dfrac{-5-12i}{2-3i} \times \dfrac{2+3i}{2+3i}$

$= \dfrac{-10-15i-24i+36}{4+9}$

$= \dfrac{26-39i}{13}$

$= 2-3i$

(b) $w = 2\,\mathrm{cis}\dfrac{5\pi}{6}$

$w^3 - 8i = \left(2\,\mathrm{cis}\dfrac{5\pi}{6}\right)^3 - 8i$

$= 8\,\mathrm{cis}\dfrac{5\pi}{2} - 8i$

$= 8i - 8i$ $\left(\text{hopefully you don't need to write } \mathrm{cis}\dfrac{5\pi}{2} \text{ as } \cos\dfrac{5\pi}{2} + i\sin\dfrac{5\pi}{2} \text{ to see this}\right)$

$= 0$

(c) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{-\left(x^2+2x+1\right)+3+1}}$

$\displaystyle = \int \frac{dx}{\sqrt{4-\left(x+1\right)^2}}$

$= \sin^{-1}\dfrac{x+1}{2} + c$

(d) $\displaystyle\int_0^{\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+1}} = \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{\sqrt{x^2+1}}\cdot 2xdx$

Let $u = x^2+1$

$du = 2x\,dx$

$\begin{pmatrix} x=0, & u=1 \\ x=\sqrt{3}, & u=4 \end{pmatrix}$

$$= \frac{1}{2}\int_1^4 \frac{u-1}{\sqrt{u}}\cdot du$$

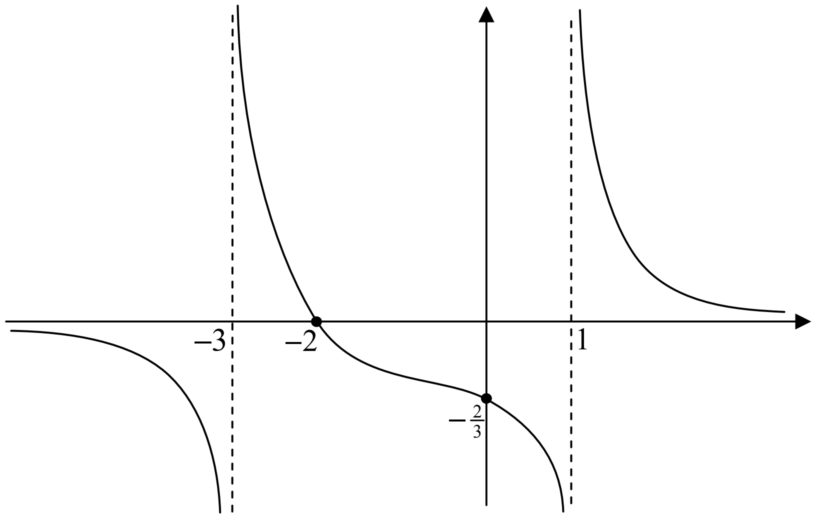
$$= \frac{1}{2}\int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$$

$$= \left[\frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}}\right]_1^4$$

$$= \frac{8}{3} - 2 - \frac{1}{3} + 1$$

$$= \frac{4}{3}$$

(e) (i)



(ii) $\displaystyle A = \int_2^5 \frac{(x+2)\,dx}{(x-1)(x+3)}$

Let $\displaystyle \frac{x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$

$$x+2 = A(x+3) + B(x-1)$$

$(x=1)\quad 3 = 4A \quad \Rightarrow \quad A = \frac{3}{4}$

$(x=-3)\quad -1 = -4B \quad \Rightarrow \quad B = \frac{1}{4}$

$$A = \frac{1}{4}\int_2^5 \left(\frac{3}{x-1} + \frac{1}{x+3}\right) dx$$

$$= \frac{1}{4}\Big[3\ln|x-1| + \ln|x+3|\Big]_2^5$$

$$= \frac{1}{4}\left(3\ln 4 + \ln 8 - 0 - \ln 8\right)$$

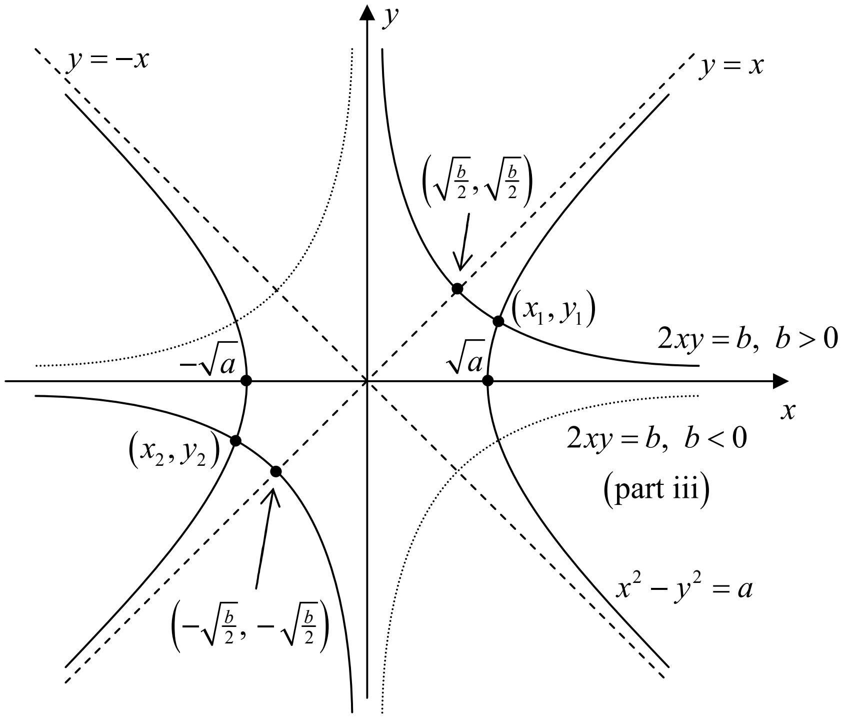
$$= \frac{1}{4}\ln\frac{4^3\times 8}{5}$$

$$= \frac{1}{4}\ln\frac{512}{5}$$

Question 12

(a) (i) When $y = x$, $2x^2 = b \quad (b > 0)$

$$x = \pm\sqrt{\frac{b}{2}}$$



(ii) Let $z = x + iy$

$$z^2 = a + ib$$
$$(x + iy)^2 = a + ib$$
$$(x^2 - y^2) + 2xyi = a + ib$$

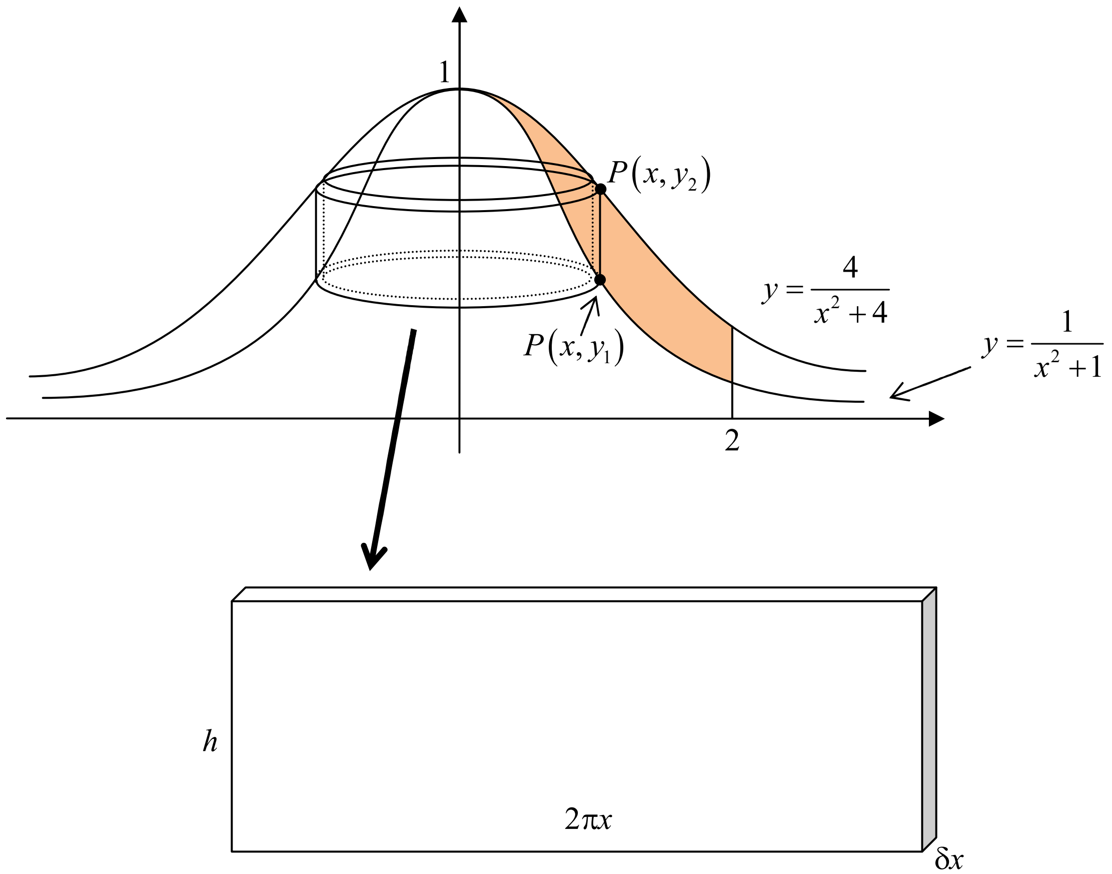
Equating real and imaginary parts: $x^2 - y^2 = a$

$$2xy = b$$

Solving simultaneously for x and y, we get the graphs of part (i).
The graphs show that there are two distinct points of intersection (x_1, y_1) and (x_2, y_2) corresponding to two distinct complex roots $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ of the complex number $a + ib$.

(iii) When b is negative, the graph of $2xy = b$ lies in the 2nd and 4th quadrants.
So the points of intersection with $x^2 - y^2 = a$ are $(x_1, -y_1)$ and $(x_2, -y_2)$.
ie. the new square roots are the conjugates of the roots found in part (ii).

(b)



$$h = y_2 - y_1$$
$$= \frac{4}{x^2+4} - \frac{1}{x^2+1}$$

Volume of shell $\delta V \approx 2\pi x h \cdot \delta x$

$$= 2\pi x\left(\frac{4}{x^2+4} - \frac{1}{x^2+1}\right)\delta x$$

Volume $V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x\left(\frac{4}{x^2+4} - \frac{1}{x^2+1}\right)\delta x$

$$= \pi \int_0^2 \left(\frac{8x}{x^2+4} - \frac{2x}{x^2+1}\right)dx$$
$$= \pi\left[4\ln\left(x^2+4\right) - \ln\left(x^2+1\right)\right]_0^2$$
$$= \pi\left(4\ln 8 - \ln 5 - 4\ln 4 + 0\right)$$
$$= \pi \ln \frac{8^4}{5\times 4^4}$$
$$= \pi \ln \frac{16}{5} \text{ units}^3$$

(c) (i) $\ddot{x} = 3(1-x)(1+x)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 3x^2$$

$$\frac{1}{2}v^2 = 3x - x^3 + c$$

$(x=0,\ v=2)$ $\quad 2 = c$

$$\frac{1}{2}v^2 = 3x - x^3 + 2$$

$$v^2 = 6x - 2x^3 + 4$$

Check RHS: $\quad 2(2-x)(x+1)^2 = (4-2x)(x^2+2x+1)$

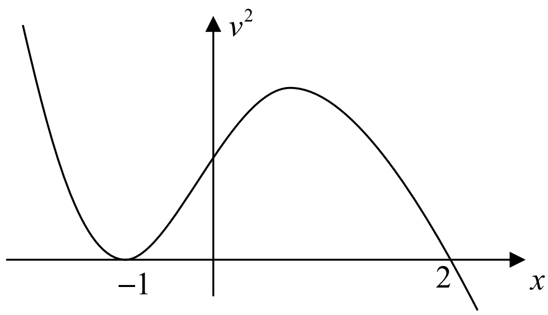
$$= 4x^2 + 8x + 4 - 2x^3 - 4x^2 - 2x$$
$$= 6x - 2x^3 + 4$$
$$= \text{LHS}$$

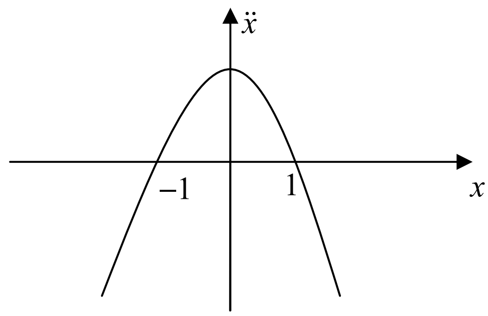
$\therefore\ v^2 = 2(2-x)(x+1)^2$

(ii) $x = 2$: $\quad v = 0$

$$\ddot{x} = 3(1-2)(1+2)$$
$$= -9 \text{ ms}^{-2}$$

(iii)





Firstly, the particle cannot ever be to the right of $x = 2$, as v^2 would be –ve.

Secondly, the particle can *possibly* change direction only when $v = 0$, ie. at $x = -1$ and $x = 2$.

Initially, the velocity is +2, so the particle moves to the right, speeding up until it reaches $x = 1$, then slowing to a stop at $x = 2$.

Since the acceleration at $x = 2$ is –ve, it then changes direction, speeds up until it again reaches $x = 1$, then slowing to a stop at $x = -1$.

At $x = -1$ the velocity and acceleration are both zero (and dependent only on position, not time), so the particle remains at $x = -1$.

(iv) From the graphs, the max speed (over the restricted domain $-1 \le x \le 2$) occurs at $x = 1$ $(\ddot{x} = 0)$.

$$v_{\text{max}}^2 = 2(2-1)(1+1)^2$$
$$= 8$$
$$v_{\text{max}} = 2\sqrt{2} \text{ m/s}$$

Question 13

(a) (i)

$$(x-y)^2 \ge 0 \quad \forall \text{ real } x, y$$

$$x^2 - 2xy + y^2 \ge 0$$

$$x^2 + y^2 \ge 2xy$$

If x, y have the same sign, $xy > 0$, so $2xy > xy$, so $x^2 + y^2 \ge xy$.
If x, y have opposite sign, $xy < 0$, so $x^2 + y^2 \ge xy$ as $x^2 + y^2 \ge 0$.

OR

$$x^2 + y^2 - xy = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2$$

$$\ge 0 \qquad \text{(sum of 2 perfect squares)}$$

$$x^2 + y^2 \ge xy$$

(ii)

$$x^2 + y^2 - 3z^2 = x^2 + y^2 - 3\left(\frac{x+y}{3}\right)^2$$

$$= x^2 + y^2 - \frac{x^2 + 2xy + y^2}{3}$$

$$= \frac{3x^2 + 3y^2 - x^2 - 2xy - y^2}{3}$$

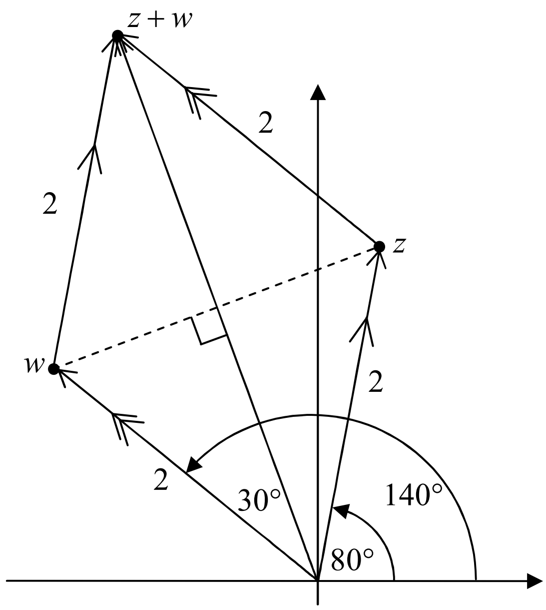
$$= \frac{2x^2 + 2y^2 - 2xy}{3}$$

$$= \frac{2}{3}\left(x^2 + y^2 - xy\right)$$

$$\ge 0 \qquad \left(\text{since } x^2 + y^2 \ge xy \text{ from i}\right)$$

$$x^2 + y^2 \ge 3z^2$$

(b) (i)



(ii) Since this shape is a rhombus, the vertex angles are bisected by the diagonals.

$\therefore\ \arg(z+w)$ is the average of $\frac{4\pi}{9}$ and $\frac{7\pi}{9}$

$$\arg(z+w) = \frac{1}{2}\left(\frac{4\pi}{9}+\frac{7\pi}{9}\right) \qquad \left[\text{OR}\ \ \arg(z+w) = \frac{4\pi}{9}+\frac{1}{2}\left(\frac{7\pi}{9}-\frac{4\pi}{9}\right)\right]$$

$$= \frac{11\pi}{18}$$

(iii) Since diagonals bisect each other at right angles, $\frac{1}{2}|z+w| = 2\cos 30°$

$$|z+w| = 2\sqrt{3}$$

(c) (i)

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}} \times \frac{1+t^2}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{2\left(1+t^2\right)+2t}$$

$$= \int_0^1 \frac{dt}{t^2+t+1}$$

$$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2\left(t+\frac{1}{2}\right)}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

Let $t = \tan\frac{x}{2}$

$\tan^{-1} t = \frac{x}{2}$

$x = 2\tan^{-1} t$

$dx = \frac{2}{1+t^2}\,dt$

$x = 0,\ \ t = 0$

$x = \frac{\pi}{2},\ t = 1$

(ii) $$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Let $u = 2a - x$ (in 2nd integral)

(so $x = u - 2a$)

$du = -dx$

$x = 0, \quad u = a$

$x = 2a, \; u = 0$

$$= \int_0^{a} f(x)\,dx + \int_a^{0} f(2a-u)\cdot(-du)$$

$$= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-u)\,du$$

$$= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx \quad \text{(since choice of var does not affect def int)}$$

$$= \int_0^{a} \left[f(x) + f(2a-x)\right]dx$$

(iii) $$\int_0^{\pi} \frac{x}{2+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{x}{2+\sin x} + \frac{\pi - x}{2+\sin(\pi - x)}\right)dx \quad \text{(by part ii)}$$

$$= \int_0^{\frac{\pi}{2}} \left(\cancel{\frac{x}{2+\sin x}} + \frac{\pi}{2+\sin x} - \cancel{\frac{x}{2+\sin x}}\right)dx$$

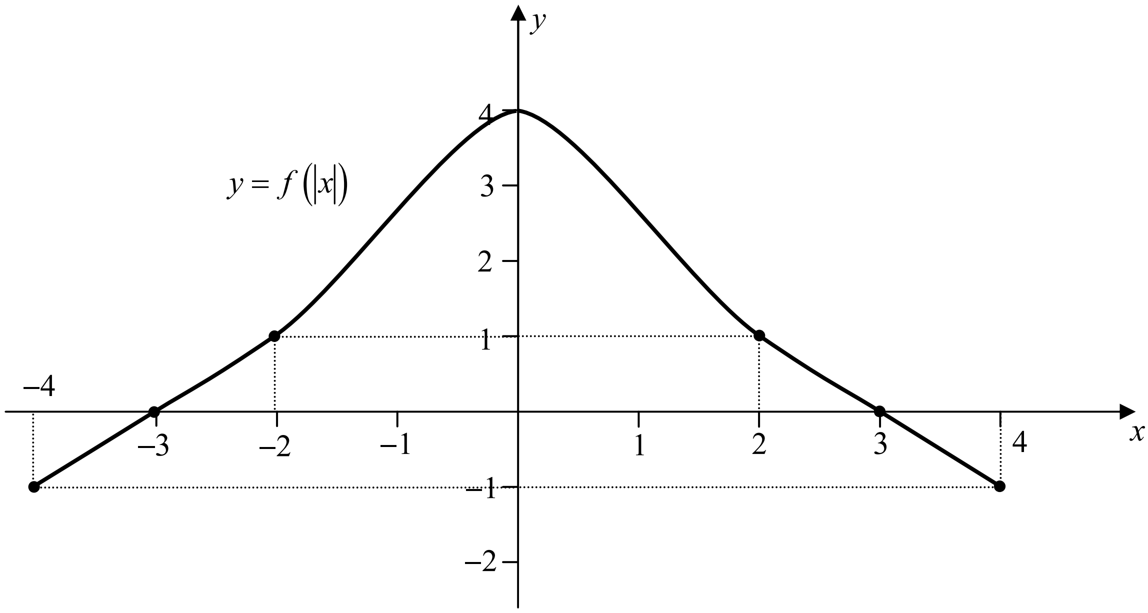
$$= \pi\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$$

$$= \pi\cdot\frac{\pi}{3\sqrt{3}} \quad \text{(by part i)}$$

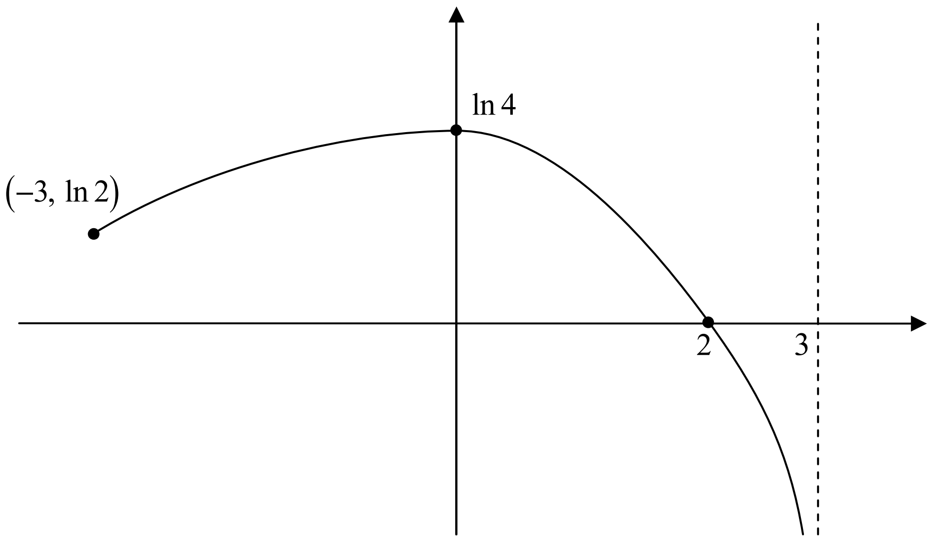
$$= \frac{\pi^2}{3\sqrt{3}}$$

Question 14

(a) (i)



(ii)



(b) (i)

$$xy = c^2$$

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

at $P\left(ct, \frac{c}{t}\right)$, $m_{\text{T}} = -\frac{\frac{c}{t}}{ct}$

$$= -\frac{1}{t^2}$$

Tangent: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$$t^2 y - ct = -x + ct$$

$$x + t^2 y = 2ct$$

(ii) OT has gradient $\dfrac{y_1}{x_1}$, and $OT \perp PT$.

So $m_{OT} \cdot m_{PT} = -1$

$$\frac{y_1}{x_1} \cdot \left(-\frac{1}{t^2}\right) = -1$$

$$y_1 = t^2 x_1$$

(iii) Since T satisfies equation of tangent:

$$x_1 + t^2 y_1 = 2ct$$

$$x_1 + \frac{y_1}{x_1} \cdot y_1 = 2c \cdot \sqrt{\frac{y_1}{x_1}} \quad \text{(from part ii)}$$

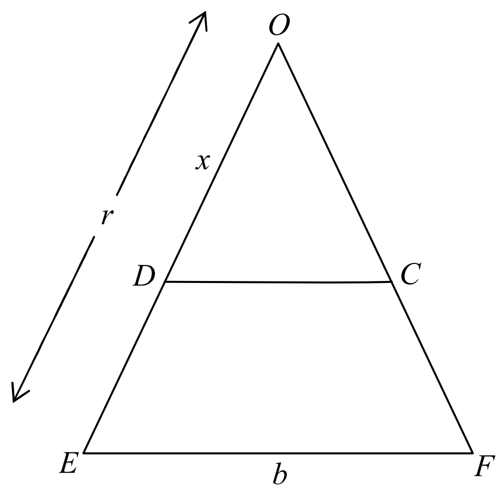
$$(\times x_1) \quad x_1^2 + y_1^2 = 2cx_1\sqrt{\frac{y_1}{x_1}}$$

$$= 2c\sqrt{x_1 y_1}$$

$$\text{(squaring)} \quad \left(x_1^2 + y_1^2\right)^2 = 4c^2 x_1 y_1$$

ie. locus of T is $\left(x^2 + y^2\right)^2 = 4c^2 xy$

(c) (i) By similar triangles OCD and OFE in the base:



$$\frac{DC}{EF} = \frac{OD}{OE}$$

$$\frac{DC}{b} = \frac{x}{r}$$

$$DC = \frac{bx}{r} \quad \text{(base of rectangular slice)}$$

Height of slice $h = y$

$$= r^2 - x^2$$

Thickness of Slice = δx

$\therefore$ Volume of slice $\delta V = \dfrac{bx}{r} \cdot \left(r^2 - x^2\right) \cdot \delta x$

(ii) Volume $V = \lim_{\delta x \to 0} \sum_{x=0}^{r} \frac{bx}{r}\left(r^2 - x^2\right)\delta x$

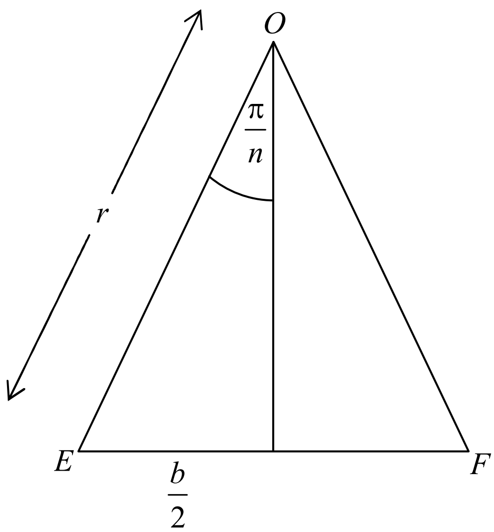
$$= \frac{b}{r}\int_0^r \left(r^2 x - x^3\right) dx$$

$$= \frac{b}{4r}\left[2r^2x^2 - x^4\right]_0^r$$

$$= \frac{b}{4r}\left(2r^4 - r^4\right)$$

$$= \frac{br^3}{4}$$

(iii)



$$\sin\frac{\pi}{n} = \frac{b}{2r}$$

$$b = 2r\sin\frac{\pi}{n}$$

$$V_n = \frac{1}{4}\cdot b \cdot r^3$$

$$= n \cdot \frac{1}{4}\cdot 2r\sin\frac{\pi}{n}\cdot r^3$$

$$= \frac{1}{2}r^4 n \sin\frac{\pi}{n}$$

(iv) As $n \to \infty$, $\frac{\pi}{n} \to 0$, so $\sin\frac{\pi}{n} \to \frac{\pi}{n}$

So $\lim_{n\to\infty} V_n = \frac{1}{2}r^4 n \cdot \frac{\pi}{n}$

$$= \frac{1}{2}\pi r^4$$

Question 15

(a) Let $P(x) = 2x^3 - 5x + 1$

$P\left(-\frac{x}{2}\right) = 0$ has roots $-2\alpha,\ -2\beta,\ -2\gamma$

$$2\left(-\frac{x}{2}\right)^3 - 5\left(-\frac{x}{2}\right) + 1 = 0$$

$$-\frac{x^3}{4} + \frac{5x}{2} + 1 = 0$$

$$x^3 - 10x - 4 = 0$$

(b) (i)

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta \end{aligned}$$

(ii)

$$\left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$\begin{aligned} z^3 + \frac{1}{z^3} &= \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right) \\ &= u^3 - 3u \end{aligned}$$

(iii) IF you HAD to show this result:

$$\left(z + \frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$\begin{aligned} z^5 + \frac{1}{z^5} &= \left(z + \frac{1}{z}\right)^5 - 5\left(z^3 + \frac{1}{z^3}\right) - 10\left(z + \frac{1}{z}\right) \\ &= u^5 - 5(u^3 - 3u) - 10u \\ &= u^5 - 5u^3 + 5u \end{aligned}$$

$$\begin{aligned} 1 + \cos 10\theta &= 1 + (2\cos^2 5\theta - 1) \\ &= 2\cos^2 5\theta \\ &= \frac{1}{2}(2\cos 5\theta)^2 \\ &= \frac{1}{2}(z^5 + z^{-5})^2 \qquad \text{(from part i)} \\ &= \frac{1}{2}(u^5 - 5u^3 + 5u)^2 \qquad \text{(given)} \\ &= \frac{1}{2}\left[(2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)\right]^2 \\ &= \frac{1}{2}(32\cos^5\theta - 40\cos^3\theta + 10\cos\theta)^2 \\ &= 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2 \end{aligned}$$

(c) (i) $\overrightarrow{QP} = \overrightarrow{QR} \cdot \text{cis}\frac{\pi}{3}$ $\left(\text{since angle in equilateral triangle is } \frac{\pi}{3}\right)$

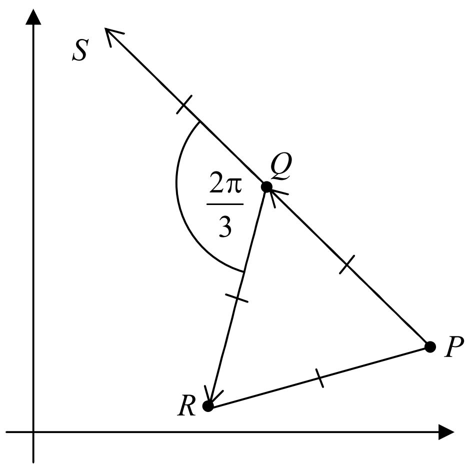
$$p - q = (r - q) \cdot \text{cis}\frac{\pi}{3}$$

$$r - q = (p - q) \cdot \text{cis}\left(-\frac{\pi}{3}\right) \qquad \text{(to divide by a complex number, multiply by its conjugate)}$$

$$r - q = (q - p) \cdot \text{cis}\left(\pi - \frac{\pi}{3}\right) \qquad \text{(to multiply by } -1\text{, add } \pi \text{ to the argument)}$$

$$r - q = \text{cis}\frac{2\pi}{3}(q - p)$$

OR



$$\angle SQR = \frac{2\pi}{3} \qquad \text{(exterior angle of triangle = opposite interior angle)}$$

$$\begin{aligned} r - q &= \overrightarrow{QR} \\ &= \text{cis}\frac{2\pi}{3} \cdot \overrightarrow{QS} \qquad \left(\text{anticlockwise rotation by } \frac{2\pi}{3}\right) \\ &= \text{cis}\frac{2\pi}{3} \cdot \overrightarrow{PQ} \\ &= \text{cis}\frac{2\pi}{3}(q - p) \end{aligned}$$

(ii)

$$\begin{aligned} r - q &= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)(q - p) \\ &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(q - p) \end{aligned}$$

$$2r - 2q = \left(-1 + i\sqrt{3}\right)(q - p)$$

$$2r - 2q = -q + p + iq\sqrt{3} - ip\sqrt{3}$$

$$2r = (p + q) + i\sqrt{3}(q - p)$$

(d) (i) PB subtends equal angles at N and L on the same side of PB.

OR

$\angle BLP = \angle BNP$ (given)

$\therefore$ $BLNP$ is cyclic (converse of angles in same segment [or angles standing on same arc])

(ii)

NOTE: You may NOT say $\angle BNL = \angle MNA$ (vertically opposite) OR $\angle MNP = \angle PBL$ (ext angle of cyclic quad = opposite interior angle) as these assume that LNM is a straight line, WHICH IS WHAT YOU ARE TRYING TO PROVE.

$\angle PMA = \angle PNA = 90°$ (given)

$\therefore$ $PNAM$ is cyclic (exterior angle of cyclic quad equals opposite interior angle)

$\angle MNP = \angle MAP$ (both angles stand on chord PM of cyclic quad $PNAM$)

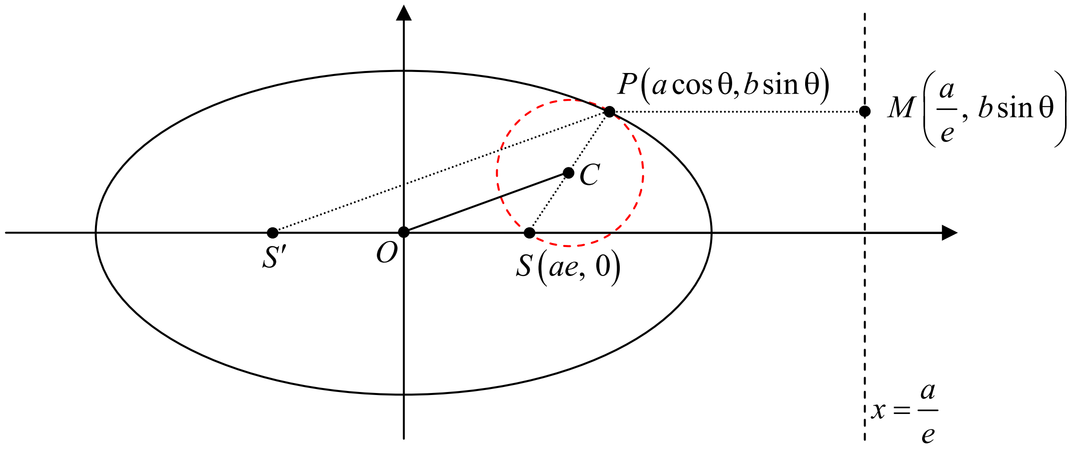
$= \angle PBC$ (exterior angle of cyclic quad $APBC$ = opposite interior angle)

$\therefore$ $\angle MNP$ is the exterior angle of cyclic quad $BLNP$ (it equals the opposite interior angle)

ie. LNM is straight (ie. L, M and N are collinear)

Question 16

(a) (i)



Centre: $C\left(\dfrac{ae + a\cos\theta}{2}, \dfrac{b\sin\theta}{2}\right) = C\left(\dfrac{a(e+\cos\theta)}{2}, \dfrac{b\sin\theta}{2}\right)$

Diameter: $PS = ePM$

$$= e\left(\frac{a}{e} - a\cos\theta\right)$$

$$= a(1 - e\cos\theta)$$

$\therefore$ radius $= \dfrac{a}{2}(1 - e\cos\theta)$

(ii) [The sneaky way]

Since $OS = \frac{1}{2}S'S$, $CS = \frac{1}{2}PS$ and $\angle OSC = \angle S'SC$, then ΔOSC and $\Delta S'SP$ are similar

$\therefore \; OC = \frac{1}{2}PS'$

But $PS + PS' = 2a$ (sum of focal lengths = length of major axis)

$\therefore \quad CS + CO = a$

$$\frac{a}{2}(1 - e\cos\theta) + OC = a$$

$$OC = a - \frac{a}{2} + \frac{a}{2}e\cos\theta$$

$$OC = \frac{a}{2}(1 + e\cos\theta)$$

[The hard slog]

$$OC^2 = \frac{a^2}{4}(e + \cos\theta)^2 + \frac{b^2}{4}\sin^2\theta$$

$$= \frac{1}{4}\left(a^2e^2 + 2a^2e\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta\right)$$

$$= \frac{1}{4}\left(a^2e^2 + 2a^2e\cos\theta + a^2\cos^2\theta + a^2\left(1 - e^2\right)\sin^2\theta\right)$$

$$= \frac{a^2}{4}\left(e^2\left[1 - \sin^2\theta\right] + 2e\cos\theta + \left[\cos^2\theta + \sin^2\theta\right]\right)$$

$$= \frac{a^2}{4}\left(e^2 \cos^2 \theta + 2e\cos\theta + 1\right)$$

$$= \frac{a^2}{4}\left(e\cos\theta + 1\right)^2$$

Since $e < 1$ for ellipse
and $|\cos\theta| \le 1$
then $|e\cos\theta| < 1$
So $1 + e\cos\theta > 0$

$$\therefore \ OC = \frac{a}{2}\left(e\cos\theta + 1\right)$$

(b) (i) $P\left(-\frac{3}{2}\right) = 4\left(-\frac{3}{2}\right)^3 + 10\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{3}{2}\right) + 3$

$$= -\frac{27}{2} + \frac{45}{2} - 12 + 3$$

$$= 0$$

$\therefore$ $P(x)$ is divisible by $(2x+3)$

(ii) Let zeros be $-\frac{3}{2}$, α, β

Sum: $-\frac{3}{2} + \alpha + \beta = -\frac{5}{2}$

$$\alpha + \beta = -1$$

Product: $-\frac{3}{2}\alpha\beta = -\frac{3}{4}$

$$\alpha\beta = \frac{1}{2}$$

$\therefore$ A polynomial with zeros α and β is $x^2 + x + \frac{1}{2}$.

But to get equal leading coefficients: $P(x) = (2x+3)\left(2x^2 + 2x + 1\right)$

[Alternatively: divide]

(c) In case you had to prove the given result:

$$\tan\left(\tan^{-1} x + \tan^{-1} y\right) = \frac{\tan\left(\tan^{-1} x\right) + \tan\left(\tan^{-1} y\right)}{1 - \tan\left(\tan^{-1} x\right) \cdot \tan\left(\tan^{-1} y\right)}$$

$$= \frac{x+y}{1-xy}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

The Induction Proof:

$$\text{RTP} \quad \tan^{-1}\frac{1}{2\times 1^2} + \tan^{-1}\frac{1}{2\times 2^2} + \tan^{-1}\frac{1}{2\times 3^2} + \ldots + \tan^{-1}\frac{1}{2n^2} = \frac{\pi}{4} - \tan^{-1}\frac{1}{2n+1}$$

Test $n=1$:

$$\text{LHS} = \tan^{-1}\frac{1}{2}$$

$$\text{RHS} = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$$

$$\begin{aligned}
\text{LHS} - \text{RHS} &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} - \frac{\pi}{4} \\
&= \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} - \frac{\pi}{4} \qquad \text{(using given rule)} \\
&= \tan^{-1}1 - \frac{\pi}{4} \\
&= \frac{\pi}{4} - \frac{\pi}{4} \\
&= 0
\end{aligned}$$

$$\text{LHS} = \text{RHS}$$

$\therefore$ true for $n=1$

Assume true for $n=k$:

$$\text{ie.} \quad \tan^{-1}\frac{1}{2\times 1^2} + \tan^{-1}\frac{1}{2\times 2^2} + \tan^{-1}\frac{1}{2\times 3^2} + \ldots + \tan^{-1}\frac{1}{2k^2} = \frac{\pi}{4} - \tan^{-1}\frac{1}{2k+1}$$

Prove true for $n=k+1$:

$$\text{ie. RTP} \quad \tan^{-1}\frac{1}{2\times 1^2} + \tan^{-1}\frac{1}{2\times 2^2} + \tan^{-1}\frac{1}{2\times 3^2} + \ldots + \tan^{-1}\frac{1}{2k^2} + \tan^{-1}\frac{1}{2(k+1)^2} = \frac{\pi}{4} - \tan^{-1}\frac{1}{2k+3}$$

$$\begin{aligned}
\text{LHS} - \text{RHS} &= \left(\frac{\pi}{4} - \tan^{-1}\frac{1}{2k+1}\right) + \tan^{-1}\frac{1}{2(k+1)^2} - \left(\frac{\pi}{4} - \tan^{-1}\frac{1}{2k+3}\right) \qquad \text{(by assumption)} \\
&= \tan^{-1}\frac{1}{2(k+1)^2} + \tan^{-1}\frac{1}{2k+3} - \tan^{-1}\frac{1}{2k+1} \\
&= \tan^{-1}\frac{\frac{1}{2(k+1)^2}+\frac{1}{2k+3}}{1-\frac{1}{2(k+1)^2}\cdot\frac{1}{2k+3}} \times \frac{2(k+1)^2(2k+3)}{2(k+1)^2(2k+3)} - \tan^{-1}\frac{1}{2k+1} \\
&= \tan^{-1}\frac{(2k+3)+2(k+1)^2}{2(k+1)^2(2k+3)-1} - \tan^{-1}\frac{1}{2k+1} \\
&= \tan^{-1}\frac{2k^2+6k+5}{4k^3+14k^2+16k+5} - \tan^{-1}\frac{1}{2k+1} \\
&= \tan^{-1}\frac{2k^2+6k+5}{(2k+1)(2k^2+6k+5)} - \tan^{-1}\frac{1}{2k+1} \\
&= \tan^{-1}\frac{1}{2k+1} - \tan^{-1}\frac{1}{2k+1} \\
&= 0
\end{aligned}$$

$$\text{LHS} = \text{RHS}$$

OR

Using the fact that $\tan^{-1}(-x) = -\tan^{-1} x$:

$$\text{LHS} = \frac{\pi}{4} - \tan^{-1}\frac{1}{2k+1} + \tan^{-1}\frac{1}{2(k+1)^2}$$

$$= \frac{\pi}{4} - \left(\tan^{-1}\frac{1}{2k+1} - \tan^{-1}\frac{1}{2(k+1)^2}\right)$$

$$= \frac{\pi}{4} - \tan^{-1}\frac{\frac{1}{2k+1} - \frac{1}{2(k+1)^2}}{1 + \frac{1}{2k+1}\cdot\frac{1}{2(k+1)^2}} \times \frac{2(2k+1)(k+1)^2}{2(2k+1)(k+1)^2}$$

$$= \frac{\pi}{4} - \tan^{-1}\frac{2(k+1)^2 - (2k+1)}{2(2k+1)(k+1)^2 + 1}$$

$$= \frac{\pi}{4} - \tan^{-1}\frac{2k^2 + 2k + 1}{4k^3 + 10k^2 + 8k + 3}$$

$$= \frac{\pi}{4} - \tan^{-1}\frac{1}{2k+3} \qquad \text{(by part b)}$$

$$= \text{RHS}$$

$\therefore$ true for $n = k+1$ when true for $n = k$

$\therefore$ by Mathematical Induction, true for all positive integers n.

(d) (i) $f(x) = \log x - x + 1$

$$f'(x) = \frac{1}{x} - 1$$

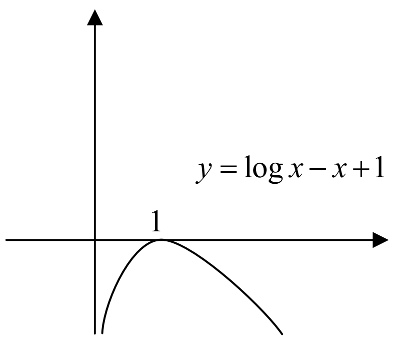
$$= \frac{1-x}{x}$$

$\therefore$ stationary point at $x = 1$

$$f''(x) = -\frac{1}{x^2} < 0 \ \forall x$$

$\therefore$ minimum turning point at $(1, 0)$

Also domain $x > 0$ (and continuous for all x in the domain).



$\therefore \ f(x) \le 0 \ \ \forall x > 0$

(ii) $$\sum_{r=1}^{n} \log(np_r) \le \sum_{r=1}^{n} (np_r - 1) \qquad \left(\text{from part i - } \log x \le x - 1 \ \ \forall x > 0\right)$$

$$= \sum_{r=1}^{n} np_r - \sum_{r=1}^{n} 1$$

$$\sum_{r=1}^{n} \log(np_r) \le \sum_{r=1}^{n} np_r - n$$

(iii) Continuing from part ii:

$$\sum_{r=1}^{n} \log np_r \le n\sum_{r=1}^{n} p_r - n \qquad (\text{since } n \text{ is a constant})$$

$$= n\cdot 1 - n$$

$$\sum_{r=1}^{n} \log np_r \le 0$$

(iv) Continuiing from part iii:

$$\log np_1 + \log np_2 + ... + \log np_n \le 0$$

$$\log\left(np_1 \cdot np_2 \cdot ... \cdot np_n\right) \le 0$$

$$\log\left(n^n \cdot p_1 p_2 ... p_n\right) \le 0$$

$$n^n \cdot p_1 p_2 ... p_n \le 1$$

Also, since $p_1,\ p_2, ...,\ p_n$ and n are all positive, then $n^n \cdot p_1 p_2 ... p_n > 0$

$\therefore\ 0 < n^n \cdot p_1 p_2 ... p_n \le 1$