NORTH SYDNEY GIRLS HIGH SCHOOL

2014
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- Show all necessary working in questions 11–16

Total Marks – 100

Section I Pages 2–6

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 7–13

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

NAME: ______________________ **TEACHER** ______________

NUMBER: ______________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is true for all complex numbers z?

(A) $\operatorname{Im} z = \dfrac{z - \overline{z}}{2}$

(B) $\operatorname{Im} z = \dfrac{z + \overline{z}}{2}$

(C) $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(D) $\operatorname{Im} z = \dfrac{z + \overline{z}}{2i}$

2 What is the eccentricity of the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{11} = 1$?

(A) $\dfrac{6}{5}$

(B) $\dfrac{\sqrt{14}}{5}$

(C) $\dfrac{6}{\sqrt{11}}$

(D) $\dfrac{5}{6}$

3 If $\displaystyle\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2 \cos x\,dx$, which of the following could be $f(x)$?

(A) $3x^2$

(B) x^3

(C) $-x^3$

(D) $-3x^2$

4 Which of the following is $\dfrac{3x+11}{(x-3)(x+1)}$ expressed in partial fractions?

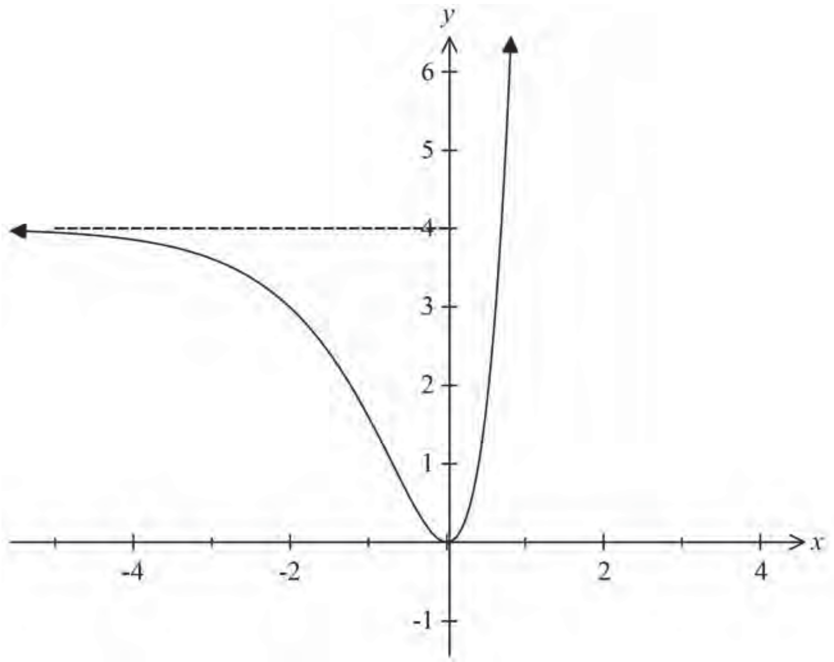
(A) $-\dfrac{1}{x-3}-\dfrac{4}{x+1}$

(B) $\dfrac{5}{x-3}-\dfrac{2}{x+1}$

(C) $\dfrac{5}{x-3}+\dfrac{2}{x+1}$

(D) $-\dfrac{1}{x-3}+\dfrac{4}{x+1}$

5 The graph below is that of $y=\left[f(x)\right]^2$.
The line $y = 4$ is a horizontal asymptote.



What is a possible equation for $f(x)$?

(A) $f(x)=4+2e^x$

(B) $f(x)=4-2e^x$

(C) $f(x)=2+2e^x$

(D) $f(x)=2-2e^x$

6 The polynomial equation $x^3 + x^2 - x - 4 = 0$ has roots α, β and γ.
Which of the following polynomial equations has roots α^2, β^2 and γ^2?

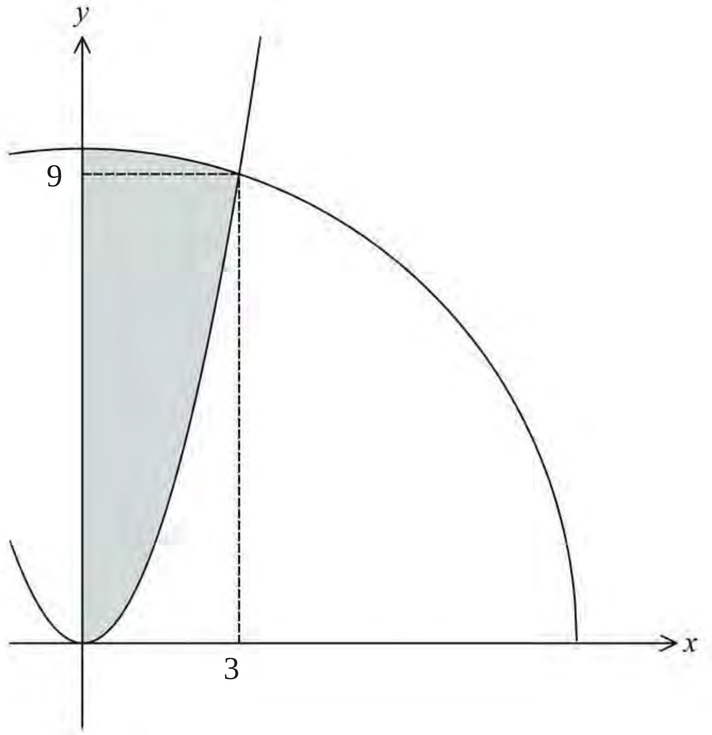
(A) $x^3 - 3x^2 - 8x - 16 = 0$

(B) $x^3 - 3x^2 + 9x - 16 = 0$

(C) $x^3 - 3x^2 - 8x - 1 = 0$

(D) $x^3 - x^2 + 9x - 1 = 0$

7 The shaded region, bounded by the curves $y = x^2$, $y = \sqrt{90 - x^2}$ and the y-axis is rotated about the y-axis.



What is the correct expression for the volume of the solid of revolution using the method of cylindrical shells?

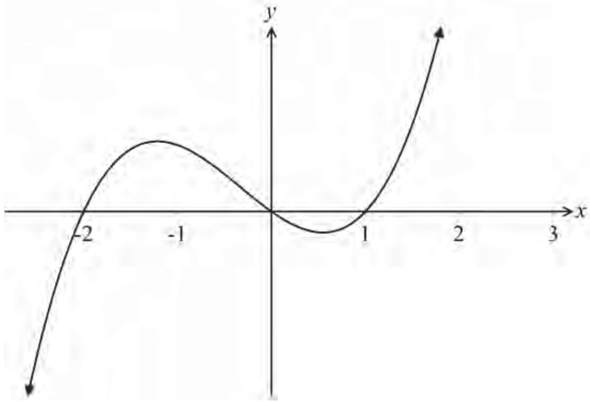
(A) $V = 2\pi \displaystyle\int_0^3 x\left(\sqrt{90 - x^2} - x^2\right) dx$

(B) $V = 2\pi \displaystyle\int_0^3 \left(\sqrt{90 - x^2} - x^2\right) dx$

(C) $V = 2\pi \displaystyle\int_0^9 \left(\sqrt{90 - y^2} - y^2\right) dy$

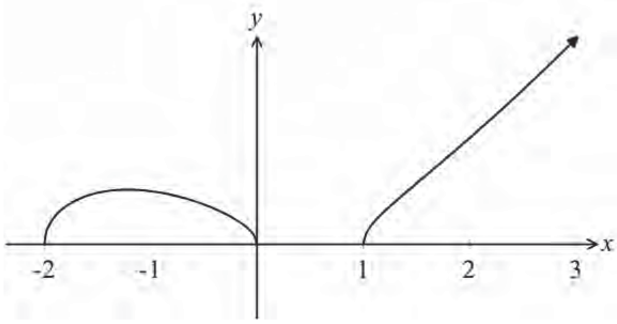
(D) $V = 2\pi \displaystyle\int_0^9 y\left(y^2 - \sqrt{90 - y^2}\right) dy$

8 The diagram shows the graph of $y = f(x)$.

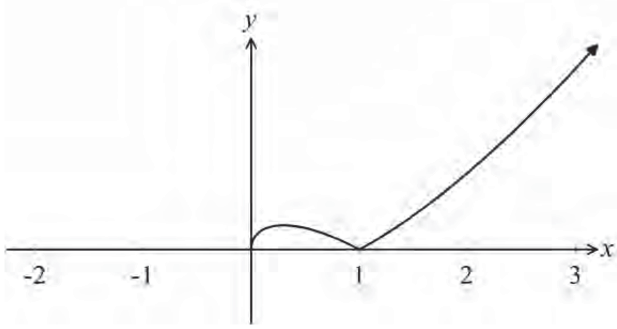


Which diagram best represents the graph of $y = f\left(\sqrt{x}\right)$?

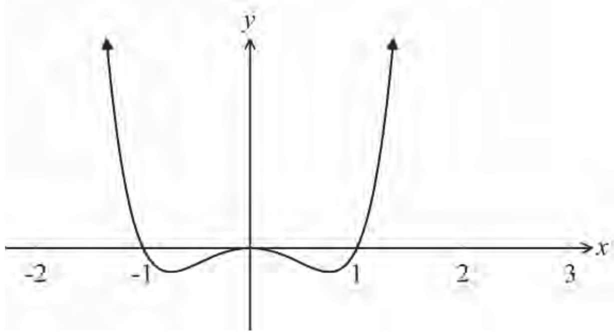
(A)



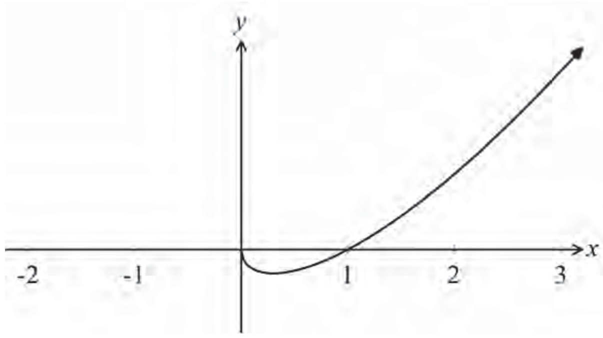
(B)



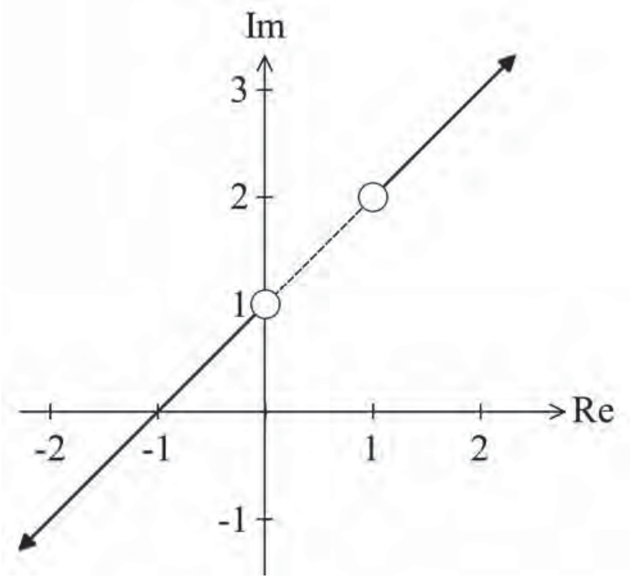
(C)



(D)



9 Which of the following defines the locus of the complex number z sketched in the diagram below?



(A) $\arg\left(\dfrac{z-i}{z-1-2i}\right)=\pi$

(B) $\arg(z+i)=\arg(z-1-2i)$

(C) $\arg(z-i)=\arg(z-1-2i)$

(D) $\arg\left(\dfrac{z+i}{z-1-2i}\right)=\pi$

10 If the eccentricity of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is e, what are the equations of the directrices of the ellipse $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$?

(A) $x=\pm e^2\sqrt{a^2+b^2}$

(B) $x=\pm e\sqrt{a^2+b^2}$

(C) $x=\pm\dfrac{\sqrt{a^2+b^2}}{e^2}$

(D) $x=\pm\dfrac{\sqrt{a^2+b^2}}{e}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW Writing Booklet

(a) Find $\displaystyle\int \frac{dx}{1+9x^2}$. **1**

(b) Find $\displaystyle\int \sqrt{x}\log_e x\,dx$. **2**

(c) (i) Find the real numbers a, b and c such that **2**

$$\frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)} \equiv \frac{x+a}{x^2}+\frac{bx+c}{x^2+1}.$$

(ii) Find $\displaystyle\int \frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)}\,dx$. **2**

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4+2\sin^2 x}}\,dx$. **3**

(e) Solve the equation $w^2 + 6w + 34 = 0$, giving your answers in the form $p + qi$, where p and q are integers. **2**

(f) It is given that $z = i(1 + i)(2 + i)$.

(i) Express z in the form $a + ib$, where a and b are integers. **1**

(ii) Find integers m and n such that $z + m\overline{z} = ni$. **2**

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) Two loci, L_1 and L_2, in an Argand diagram are given by

L_1: $|z + 6 - 5i| = 4\sqrt{2}$

L_2: $\arg(z + i) = \dfrac{3\pi}{4}$

The point P represents the complex number $-2 + i$.

(i) Show that the point P is a point of intersection of L_1 and L_2. **2**

(ii) Sketch L_1 and L_2 on one Argand diagram. **3**
Include the above information.

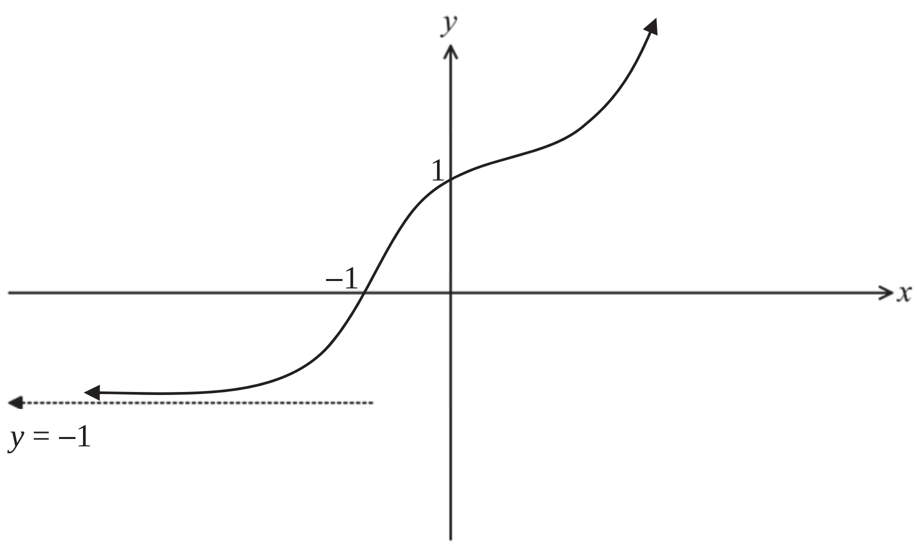
(iii) The point Q is also a point of intersection of L_1 and L_2. **1**
Write down the complex number that is represented by Q.

(b) (i) By expressing in modulus-argument form, or otherwise, **2**
show that $2\cos\dfrac{2\pi}{9} + 2i\sin\dfrac{2\pi}{9}$ is a solution of $z^3 = -4 + 4\sqrt{3}i$.

(ii) The roots of $z^3 = -4 + 4\sqrt{3}i$ are represented by the points P, Q and R **3**
on an Argand diagram.

Find the area of the $\Delta\, PQR$, giving your answer in the form $k\sqrt{3}$,
where k is an integer.

(c) The diagram below shows the graph of $y = f(x)$ which has a horizontal asymptote at $y = -1$. It also has intercepts at (0, 1) and (–1, 0).



Sketch the following curves on separate half-page diagrams.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \tan^{-1} f(x)$ **2**

Question 13 (15 Marks) Start a NEW Writing Booklet

(a) (i) If $T(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, by considering $\lim\limits_{x\to\infty}T(x)$, or otherwise, show that the domain of $T^{-1}(x)$ is $|x|<1$. **1**

(ii) Show that $T^{-1}(x)=\dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$. **2**

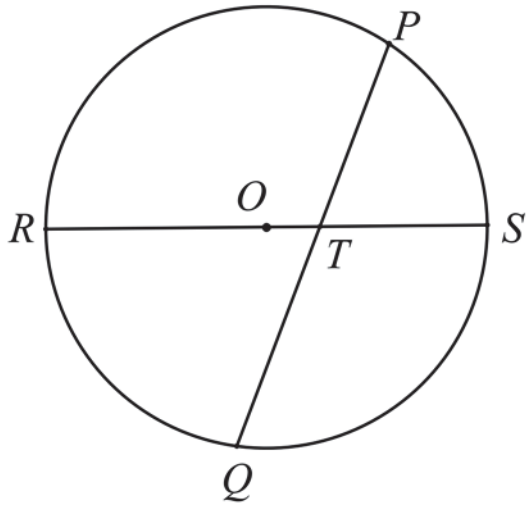
(iii) Hence, or otherwise, show that $\dfrac{d}{dx}\left[T^{-1}(x)\right]=\dfrac{1}{1-x^2}$. **2**

(iv) Use integration by parts to show that **3**

$$\int_0^{\frac{1}{2}} 4T^{-1}(x)\,dx=\log_e\left(\frac{3^m}{2^n}\right)$$

where m and n are positive integers.

(b) The diagram below shows a circle with center O and a diameter RS. **2**
A chord, PQ, intersects RS at T which is a point within the circle.



Prove that $RT^2+TS^2\geq PT^2+TQ^2$.

(c) Let α, β, and γ be the roots of the cubic equation $x^3+Ax^2+Bx+8=0$, where A and B are real.
Furthermore, $\alpha^2+\beta^2=0$ and $\beta^2+\gamma^2=0$.

(i) Explain why β is real and α and γ are not real. **2**

(ii) Show that α and γ are purely imaginary. **1**

(iii) Find A and B. **2**

Question 14 (15 Marks) Start a NEW Writing Booklet

(a) (i) By using de Moivre's theorem, or otherwise, show that **3**

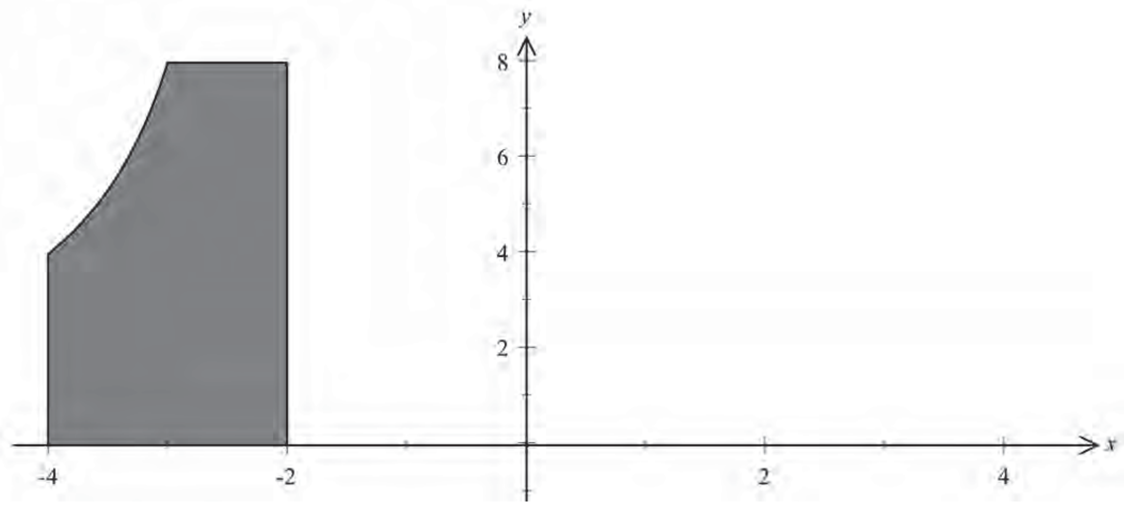
$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(ii) Hence, explain why $t = \tan\dfrac{\pi}{16}$ is a root of the equation **1**

$$t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$$

(iii) Hence show that $\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{3\pi}{16} + \tan^2\dfrac{5\pi}{16} + \tan^2\dfrac{7\pi}{16} = 28$ **3**

(b) In the diagram below, the shaded region is bounded by the lines **3**
$x = -4$, $x = -2$, $y = 8$, the x-axis and by the curve $y = -\dfrac{8}{x+2}$.



The region is rotated about the line $x = 3$ thus forming a solid.

Find the volume of the solid.

(c) Let $I_n = \displaystyle\int_0^1 x\left(x^2 - 1\right)^n \, dx$ for $n \geq 0$

(i) Use integration by parts to show that **3**

$$I_n = -\frac{n}{n+1} I_{n-1} \text{ for } n \geq 1$$

(ii) Hence find I_3. **2**

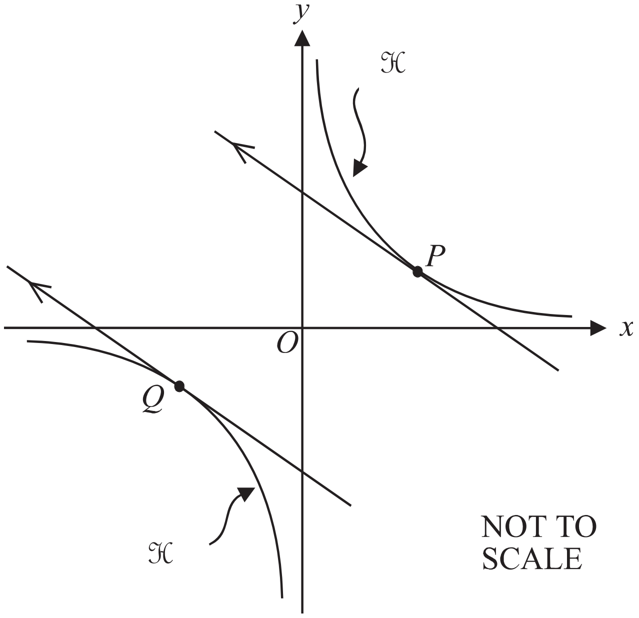
Question 15 (15 Marks) Start a NEW Writing Booklet

(a) (i) Prove for positive numbers x and y that $x + y \geq 2\sqrt{xy}$. **1**

(ii) Hence, or otherwise, prove $x^2 + \dfrac{1}{x^2} \geq 2$, for all real x, $x \neq 0$. **2**

Also, state for what value(s) of x there is equality.

(b) In the diagram below, the points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ with $c, p > 0$, lie on different branches of the hyperbola, $\mathcal{H}$, with equation $xy = c^2$.
The tangent to $\mathcal{H}$ at P and the tangent to $\mathcal{H}$ at Q are parallel.



(i) Show that the equation of the tangent at P is $y = \dfrac{2c}{p} - \dfrac{x}{p^2}$. **2**

(ii) Show that $p = -q$. **1**

(iii) Show that the perpendicular distance from P to the tangent through Q is given by $\dfrac{4cp}{\sqrt{p^4+1}}$. **2**

(iv) Using part (a), or otherwise, find the coordinates of the points P and Q when the perpendicular distance from P to the tangent through Q is a maximum. **3**

(c) (i) Using part (a), or otherwise, prove for positive numbers x, y and z that **2**

$$(x+y)(y+z)(z+x) \geq 8xyz.$$

(ii) By using part (i) above, or otherwise, if a, b and c are the sides of a triangle, prove that **2**

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Question 16 (15 Marks) Start a NEW Writing Booklet

(a) The horizontal base of an igloo is the major segment of an ellipse, as shown in Figure I below, with semi-major and semi-minor axes of 2 m and 1 m respectively.

In Figure 1, PQ is the base of a typical vertical cross section in the shape of a parabolic arc, shown in Figure 2.

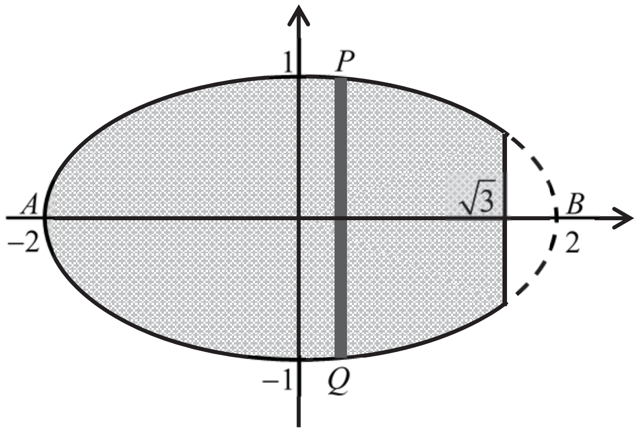


Figure 1

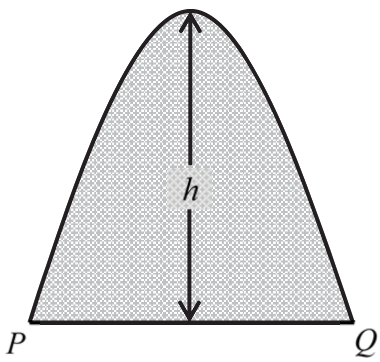


Figure 2

The height of this parabola is determined by another parabola whose base is AB, as shown in Figure 3.

The maximum height of the interior is to be 2 metres as shown, and the internal dimensions are indicated on the diagram.
The entrance to the igloo is formed by slicing the structure vertically at right angles to the major axis of the ellipse $\sqrt{3}$ metres from the centre and removing the material from this point outwards to the end of the major axis as shown in the diagram.

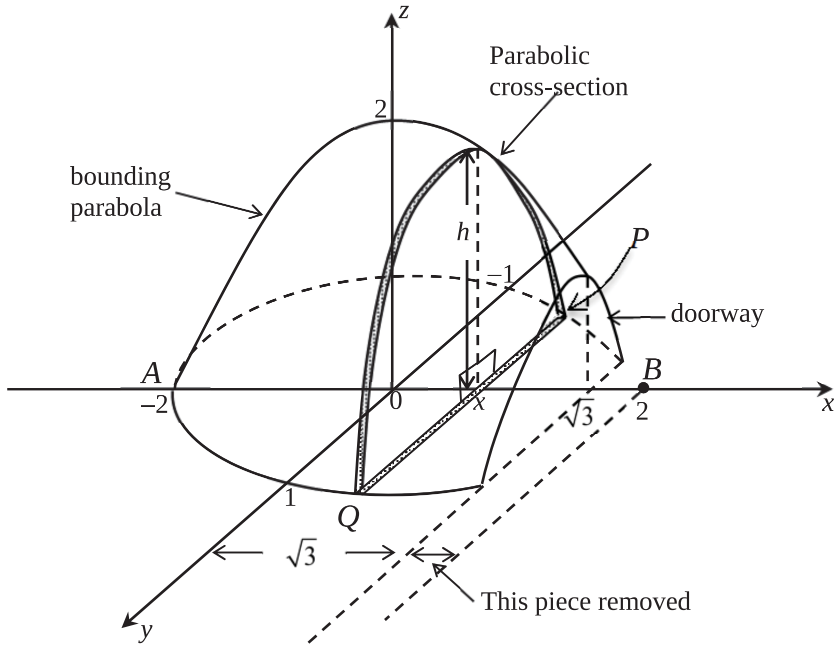


Figure 3

Question 16 continues on page 13

Question 16 (continued)

(a) (i) Find the equation of the ellipse that bounds the floor of the interior of the igloo. **1**

(ii) By finding the equation of the bounding parabola indicated in the diagram, show that the height h of the vertical cross-section, drawn at a distance x from the centre of the ellipse is given by **1**

$$h = \tfrac{1}{2}\left(4 - x^2\right).$$

(iii) Use Simpson's rule, or otherwise, to find the area of the indicated cross-section of height h and show that the volume of air in the igloo is given by **3**

$$V = \frac{1}{3}\int_{-2}^{\sqrt{3}} \left(4 - x^2\right)^{\frac{3}{2}}\, dx$$

(iv) Using $x = 2\sin\theta$, or otherwise, calculate the volume of air in the igloo. **4**

(b) A sequence of numbers U_n is such that $U_1 = 12$, $U_2 = 30$ and **3**

$$U_n = 5U_{n-1} - 6U_{n-2} \text{ for } n \geq 3$$

Prove by mathematical induction that $U_n = 2 \times 3^n + 3 \times 2^n$ for $n \geq 1$.

(c) Let $z = \left(a + \cos\theta\right) + i\left(2a + \sin\theta\right)$, where a is a real number and $0 \leq \theta \leq 2\pi$. **3**

If $|\, z \,| \leq 2$, then $|\, a \,| \leq k$ for some real number k.

Find the value of k.

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

NORTH SYDNEY GIRLS HIGH SCHOOL

2014

TRIAL HSC EXAMINATION

Mathematics Extension 2

SUGGESTED SOLUTIONS

Multiple Choice Answers

1. C
2. A
3. B
4. B
5. D
6. B
7. A
8. D
9. C
10. B

1 Which of the following is true for all complex numbers z?

(A) $\operatorname{Im} z = \dfrac{z - \bar{z}}{2}$

(B) $\operatorname{Im} z = \dfrac{z + \bar{z}}{2}$

(C) $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$

(D) $\operatorname{Im} z = \dfrac{z + \bar{z}}{2i}$

$z = x + iy \Rightarrow \bar{z} = x - iy$

$\therefore z - \bar{z} = 2iy \Rightarrow \operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$

2 What is the eccentricity of the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{11} = 1$?

(A) $\dfrac{6}{5}$

(B) $\dfrac{\sqrt{14}}{5}$

(C) $\dfrac{6}{\sqrt{11}}$

(D) $\dfrac{5}{6}$

$a^2 = 25,\ b^2 = 11$

$e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \dfrac{11}{25} = \dfrac{36}{25}$

$\therefore e = \dfrac{6}{5}$

3 If $\displaystyle\int f(x)\sin x\, dx = -f(x)\cos x + \int 3x^2 \cos x\, dx$, which of the following could be $f(x)$?

(A) $3x^2$

(B) x^3

(C) $-x^3$

(D) $\cos x$

$$\int f(x)\sin x\, dx = \int f(x)\, d(-\cos x)$$

$$= -\cos x f(x) - \int -\cos x f'(x)\, dx$$

$$= -\cos x f(x) + \int \cos x f'(x)\, dx$$

4 Which of the following is $\dfrac{3x + 11}{(x-3)(x+1)}$ expressed in partial fractions?

(A) $-\dfrac{1}{x-3} - \dfrac{4}{x+1}$

(B) $\dfrac{5}{x-3} - \dfrac{2}{x+1}$

(C) $\dfrac{5}{x-3} + \dfrac{2}{x+1}$

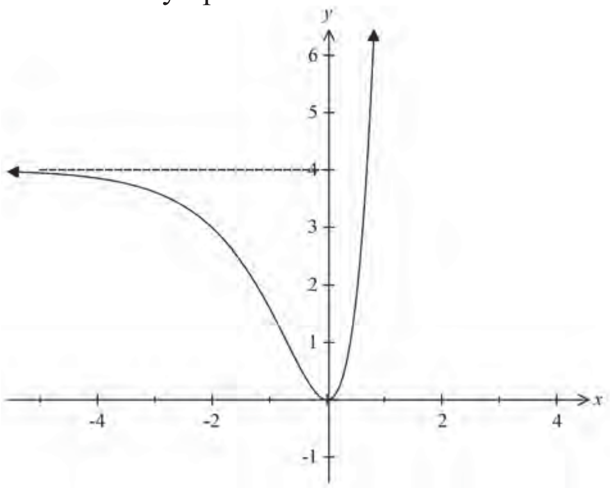
(D) $-\dfrac{1}{x-3} + \dfrac{4}{x+1}$

$x = 3: \quad \dfrac{3 \times 3 + 11}{\cancel{(x-3)}(3+1)} = 5 \Rightarrow \dfrac{5}{x-3}$

$x = -1: \quad \dfrac{3 \times (-1) + 11}{(-1-3)\cancel{(x+1)}} = -2 \Rightarrow \dfrac{-2}{x+1}$

5 The graph below is that of $y=\left[f(x)\right]^2$.

The line $y = 4$ is a horizontal asymptote.



Which of the following could be the equation for $f(x)$?

(A) $f(x)=4+2e^x$

(B) $f(x)=4-2e^x$

(C) $f(x)=2+2e^x$

(D) $f(x)=2-2e^x$

6 The polynomial equation $x^3 + x^2 - x - 4 = 0$ has roots α, β and γ.

Which of the following polynomial equations has roots α^2, β^2 and γ^2?

(A) $x^3 - 3x^2 - 8x - 16 = 0$

(B) $x^3 - 3x^2 + 9x - 16 = 0$

(C) $x^3 - 3x^2 - 8x - 1 = 0$

(D) $x^3 - x^2 + 9x - 1 = 0$

Let $y = x^2$

$$x^3+x^2-x-4=0 \Rightarrow x^3-x=4-x^2$$

$$\therefore x\left(x^2-1\right)=4-x^2 \Rightarrow x^2\left(x^2-1\right)^2=\left(4-x^2\right)^2$$

$$\therefore y\left(y-1\right)^2=\left(4-y\right)^2 \Rightarrow y\left(y^2-2y+1\right)=16-8y+y^2$$

$$\therefore y^3-2y^2+y=16-8y+y^2$$

$$\therefore y^3-3y^2+9y-16=0$$

Alternatively: Swap x with $\sqrt{x}$

$$\therefore\left(\sqrt{x}\right)^3+\left(\sqrt{x}\right)^2-\left(\sqrt{x}\right)-4=0 \Rightarrow x\left(\sqrt{x}\right)+x-\left(\sqrt{x}\right)-4=0$$

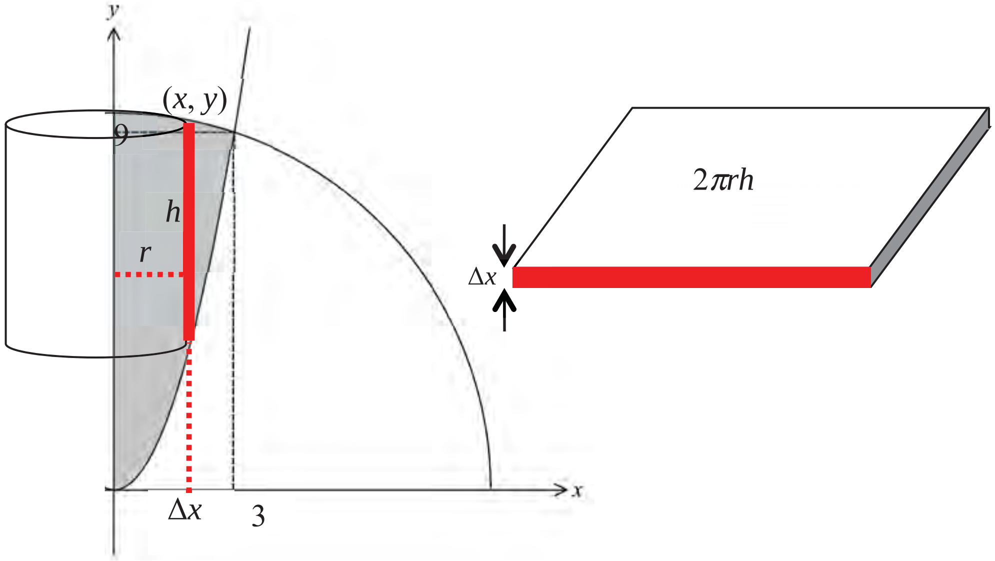
$$\therefore \sqrt{x}\left(x-1\right)=4-x \Rightarrow \left[\sqrt{x}\left(x-1\right)\right]^2=\left(4-x\right)^2$$

$$\therefore x\left(x^2-2x+1\right)=16-8x+x^2$$

$$\therefore x^3-2x^2+x=16-8x+x^2$$

$$\therefore x^3-3x^2+9x-16=0$$

7 The shaded region, bounded by the curves $y = x^2$, $y = \sqrt{90 - x^2}$ and the y-axis is rotated about the y-axis.



What is the correct expression for the volume using the method of cylindrical shells?

(A) $V = 2\pi \displaystyle\int_0^3 x\left(\sqrt{90 - x^2} - x^2\right) dx$

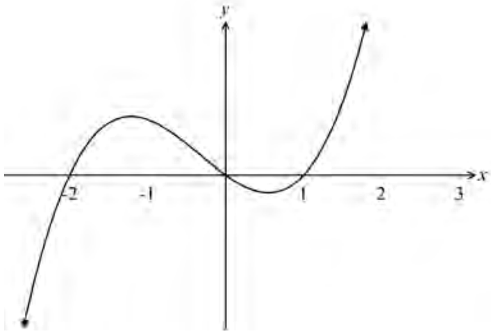
(B) $V = 2\pi \displaystyle\int_0^3 \left(\sqrt{90 - x^2} - x^2\right) dx$

(C) $V = 2\pi \displaystyle\int_0^9 \left(\sqrt{90 - y^2} - y^2\right) dy$

(D) $V = 2\pi \displaystyle\int_0^9 y\left(y^2 - \sqrt{90 - y^2}\right) dy$

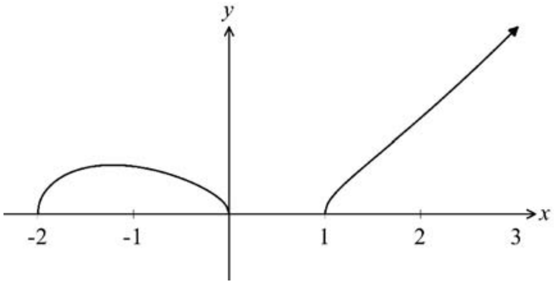
$h = \sqrt{90 - x^2} - x^2$
$r = x$
$\Delta V \doteqdot 2\pi r h \Delta x$
$= 2\pi x\left(\sqrt{90 - x^2} - x^2\right)\Delta x$

8 The graph below is that of $y = f(x)$.

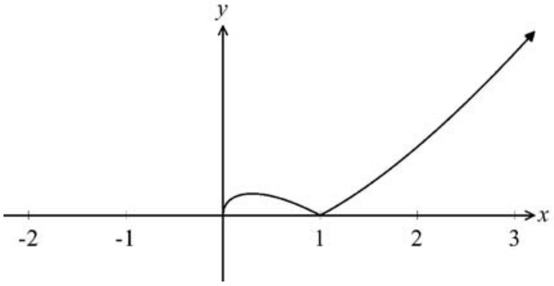


Which of the graphs below would best represent that of $y = f\left(\sqrt{x}\right)$?

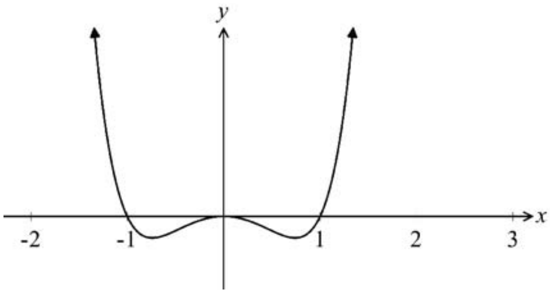
(A)



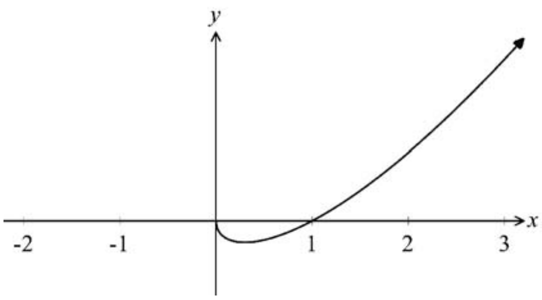
(B)



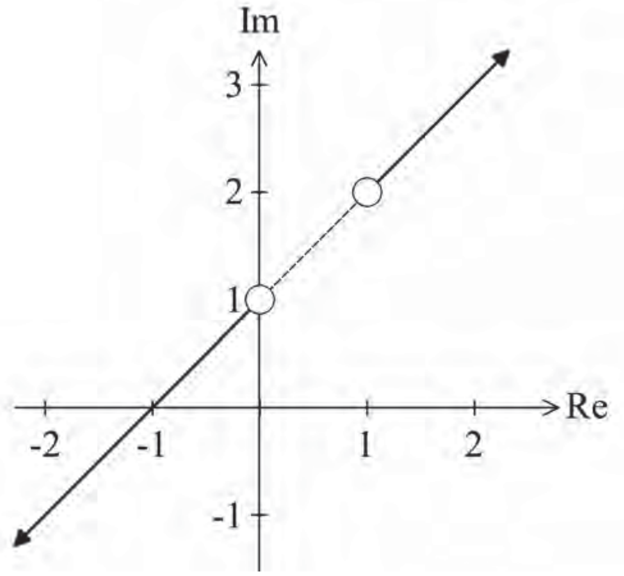
(C)



(D)



9 Which of the following defines the locus of the complex number z sketched in the diagram below?



(A) $\arg\left(\dfrac{z-i}{z-1-2i}\right)=\pi$

(B) $\arg(z+i)=\arg(z-1-2i)$

(C) $\arg(z-i)=\arg(z-1-2i)$

(D) $\arg\left(\dfrac{z+i}{z-1-2i}\right)=\pi$

10 If the eccentricity of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is e, what are the equations of the directrices of the ellipse $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$?

(A) $x=\pm e^2\sqrt{a^2+b^2}$

(B) $x=\pm e\sqrt{a^2+b^2}$

(C) $x=\pm\dfrac{\sqrt{a^2+b^2}}{e^2}$

(D) $x=\pm\dfrac{\sqrt{a^2+b^2}}{e}$

Let E be the eccentricity of the ellipse.

$$E^2=1-\frac{b^2}{a^2+b^2}=\frac{a^2+b^2-b^2}{a^2+b^2}$$
$$=\frac{a^2}{a^2+b^2}$$
$$e^2=1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2}$$
$$E^2=\frac{1}{e^2}\Rightarrow E=\frac{1}{e}$$
$$x=\pm\frac{\sqrt{a^2+b^2}}{E}=\pm e\sqrt{a^2+b^2}$$

Section II

Question 11

(a) Find $\int \frac{dx}{1+9x^2}$ **1**

$$\int \frac{dx}{1+9x^2} = \frac{1}{9}\int \frac{dx}{\frac{1}{9}+x^2}$$
$$= \frac{1}{9}\int \frac{dx}{\left(\frac{1}{3}\right)^2+x^2}$$
$$= \frac{1}{9}\times\frac{1}{\frac{1}{3}}\tan^{-1}\left(\frac{x}{\frac{1}{3}}\right)+C \qquad \left[\text{standard integral}\right]$$
$$= \frac{1}{3}\tan^{-1}(3x)+C$$

ALTERNATIVE:

$$\int \frac{dx}{1+9x^2} = \int \frac{dx}{1+(3x)^2} = \frac{1}{3}\tan^{-1}(3x)+C$$

(b) Find $\int \sqrt{x}\log_e x\,dx$ **2**

$$\int \sqrt{x}\log_e x\,dx = \int \ln x\, d\left(\frac{2}{3}x^{\frac{3}{2}}\right)$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}d\ln x$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}\times\frac{1}{x}dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}+C$$
$$= \frac{2}{9}x^{\frac{3}{2}}(3\ln x-2)+C$$

(c) (i) Find the real numbers a, b and c such that **2**

$$\frac{x^3+5x^2+x+2}{x^2(x^2+1)} \equiv \frac{x+a}{x^2}+\frac{bx+c}{x^2+1}$$

$\therefore x^3+5x^2+x+2 \equiv (x+a)(x^2+1)+x^2(bx+c)$

Let $x=0$: $\quad 2 = a(1)+0$

$\therefore\ a=2$

$1+b=1$ (comparing coefficients of x^3)

$\therefore\ b=0$

$a+c=5$ (comparing coefficients of x^2)

$\therefore\ c=3$

$\therefore\ a=2,\ b=0,\ c=3$

(c) (ii) Find $\int \frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)}\,dx$ **2**

$$\int \frac{x^3+5x^2+x+2}{x^2\left(x^2+1\right)}\,dx = \int \left(\frac{x+2}{x^2}+\frac{3}{x^2+1}\right)dx$$
$$= \int \left(x^{-1}+2x^{-2}+\frac{3}{x^2+1}\right)dx$$
$$= \ln|\,x\,| - \frac{2}{x} + 3\tan^{-1}x + C$$

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4+2\sin^2 x}}\,dx$ **3**

Let $u = \sin x \Rightarrow du = \cos x\,dx$

$x = 0$, $u = 0$ and $x = \frac{\pi}{2}, u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4+2\sin^2 x}}\,dx = \int_0^1 \frac{du}{\sqrt{4+2u^2}}$$
$$= \int_0^1 \frac{du}{\sqrt{2\left(2+u^2\right)}}$$
$$= \frac{1}{\sqrt{2}}\int_0^1 \frac{du}{\sqrt{2+u^2}}$$
$$= \frac{1}{\sqrt{2}}\left[\ln\left(u+\sqrt{2+u^2}\right)\right]_0^1$$
$$= \frac{1}{2}\left[\ln\left(1+\sqrt{3}\right) - \ln\sqrt{2}\right]$$
$$= \frac{1}{2}\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

Question 11 (continued)

(e) Solve the equation $w^2 + 6w + 34 = 0$, giving your answers in the form $p + qi$, where p and q are integers. **2**

$\therefore\ w^2 + 6w + 9 = -25$
$\therefore\ (w + 3)^2 = (5i)^2$
$\therefore\ w + 3 = \pm\, 5i$
$\therefore\ w = -\,3 \pm 5i$

(f) It is given that $z = i(1 + i)(2 + i)$.

(i) Express z in the form $a + ib$, where a and b are integers. **1**

$z = i(1 + i)(2 + i)$
$\ \ = i(2 + 3i + i^2)$
$\ \ = i(1 + 3i)$
$\ \ = -3 + i$

(ii) Find integers m and n such that $z + m\bar{z} = ni$. **2**

$\therefore -3 + i + m(-3 - i) = ni$
$\therefore -3 - 3m + i(1 - m) = ni$
$\therefore -3 - 3m = 0$ and $1 - m = n$ (Equating real and imaginary parts)
$\therefore\ m = -1$ and $n = 1 - (-1) = 2$

$\therefore\ m = -1,\ n = 2$

Question 12

(a) Two loci, L_1 and L_2, on an Argand diagram are given by

L_1: $|z + 6 - 5i| = 4\sqrt{2}$

L_2: $\arg(z + i) = \dfrac{3\pi}{4}$

The point P represents the complex number $-2 + i$.

(i) Show that the point P is a point of intersection of L_1 and L_2. **2**

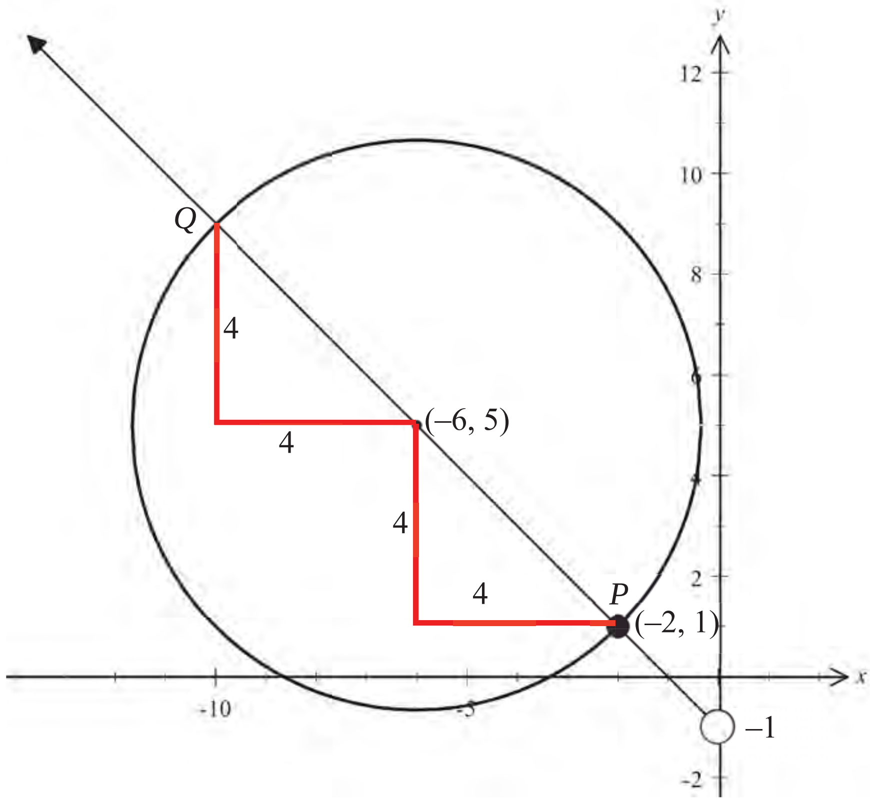
L_1:
$$\begin{aligned} \text{LHS} &= |-2 + i + 6 - 5i| \\ &= |4 - 4i| \\ &= \sqrt{4^2 + (-4)^2} = \sqrt{32} \\ &= 4\sqrt{2} \\ &= \text{RHS} \end{aligned}$$

$\therefore$ P lies on L_1.

L_2:
$$\begin{aligned} \text{LHS} &= \arg(-2 + i + i) \\ &= \arg(-2 + 2i) \\ &= \pi - \tan^{-1} 1 \\ &= \frac{3\pi}{4} \\ &= \text{RHS} \end{aligned}$$

$\therefore$ P lies on L_2.

(ii) Sketch L_1 and L_2 on one Argand diagram. **3**



(iii) The point Q is also a point of intersection of L_1 and L_2. Find the complex number that is represented by Q. **1**

L_2 is a ray of gradient -1.
$\therefore$ the centre $(-6, 5)$ lies on L_2.
$\therefore$ $Q(-6 - 4, 5 + 4) = (-10, 9)$.
$\therefore -10 + 9i$

Question 12 (continued)

(b) (i) By expressing in modulus-argument form, or otherwise, **2**

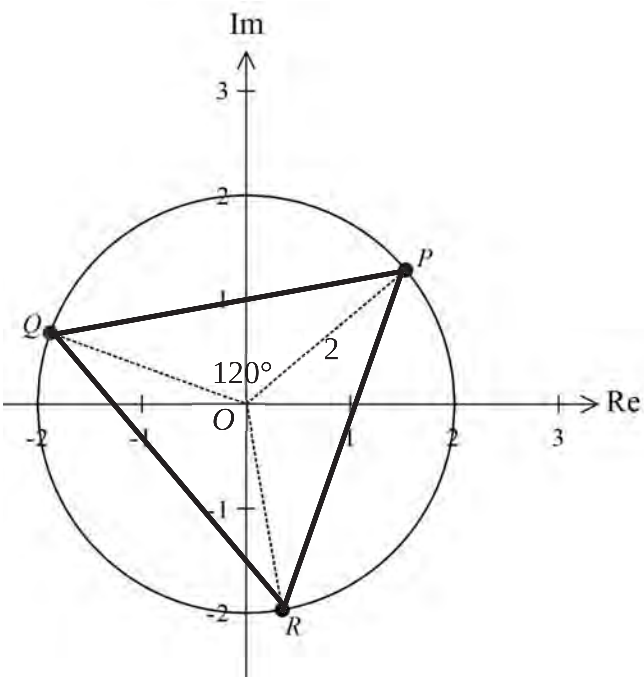
show that $2\cos\frac{2\pi}{9}+2i\sin\frac{2\pi}{9}$ is a solution of $z^3=-4+4\sqrt{3}i$.

$$\begin{aligned}\text{LHS} &= z^3\\ &= \left(2\text{cis}\frac{2\pi}{9}\right)^3\\ &= 8\text{cis}\frac{6\pi}{9}=8\text{cis}\frac{2\pi}{3}\\ &= 8\cos\frac{2\pi}{3}+8i\sin\frac{2\pi}{3}\\ &= 8\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\ &= -4+4i\sqrt{3}\end{aligned}$$

(ii) The roots of $z^3=-4+4\sqrt{3}i$ are represented by the points P, Q and R on an Argand diagram. **3**

Find the area of the $\Delta\, PQR$, giving your answer in the form $k\sqrt{3}$, where k is an integer.

The roots of $z^3=-4+4\sqrt{3}i$ will lie equally spaced around a circle of radius 2.
The angular difference of the roots is $\frac{2\pi}{3}$.



Area $\Delta\, POQ=\frac{1}{2}\times2\times2\times\sin120°=\sqrt{3}$

$\therefore$ Area $\Delta\, PQR=3\times\sqrt{3}=3\sqrt{3}$

Question 12 (continued)

(c) The diagram below shows the graph of $y = f(x)$ which has a horizontal asymptote at $y = -1$

The point of inflexion near the *y*-axis was not used in the marking scheme.

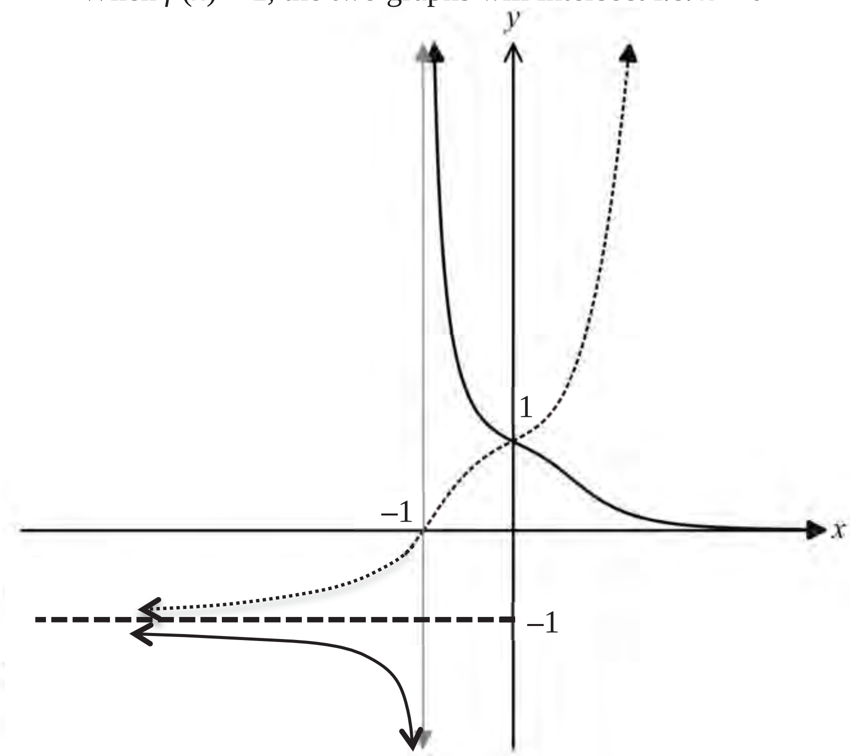
(i) $y = \dfrac{1}{f(x)}$ **2**

As $f(-1) = 0$, then $x = -1$ is a vertical asymptote.

As $x \to -\infty$, $f(x) \to -1^{+}$ and so $\dfrac{1}{f(x)} \to (-1)^{-}$.

Similarly, $x \to \infty$, $f(x) \to \infty$ and so $\dfrac{1}{f(x)} \to 0^{+}$.

When $f(x) = 1$, the two graphs will intersect i.e. $x = 0$

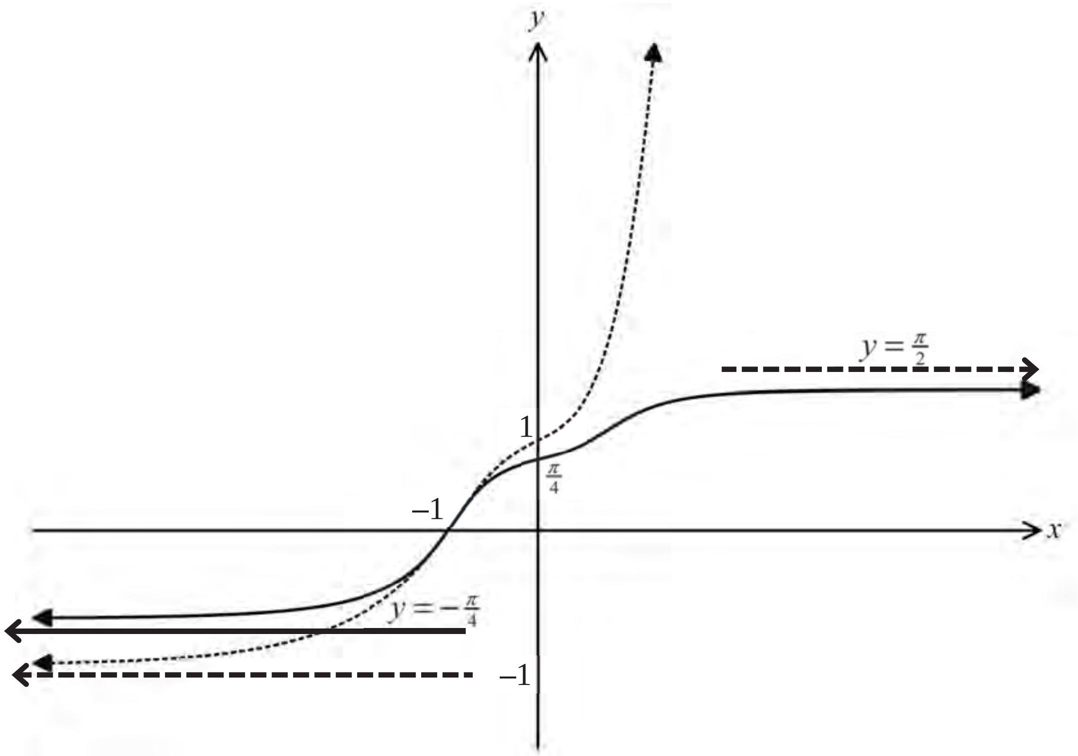


Question 12 (continued)

(c) (ii) $y = \tan^{-1} f(x)$ **2**

As $x \to -\infty$, $f(x) \to -1$ and so $\tan^{-1} f(x) \to -\frac{\pi}{4}$.

Similarly, $x \to \infty$, $f(x) \to \infty$ and so $\tan^{-1} f(x) \to \frac{\pi}{2}$.



Question 13

(a) (i) If $T(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, by considering $\lim\limits_{x\to\infty} T(x)$, or otherwise, show that the domain of $T^{-1}(x)$ is $|x|<1$. **1**

$T(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ is an odd function.

$\lim\limits_{x\to\infty} T(x)=1$ and $\lim\limits_{x\to-\infty} T(x)=-1$ so the range of $T(x)$ is $|y|<1$.

NB You do need that the graph is strictly increasing/decreasing though this was not used in the marking scheme.

$\therefore$ the domain of $T^{-1}(x)$ is $|x|<1$.

(ii) Show that $T^{-1}(x)=\dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$. **2**

$$y=\frac{e^x-\frac{1}{e^x}}{e^x+\frac{1}{e^x}}\Rightarrow y=\frac{e^{2x}-1}{e^{2x}+1}$$

$$\therefore \left(e^{2x}+1\right)y=e^{2x}-1$$

$$\therefore ye^{2x}+y=e^{2x}-1$$

$$\therefore y+1=e^{2x}\left(1-y\right)$$

$$\therefore e^{2x}=\frac{y+1}{1-y}$$

$$\therefore 2x=\ln\left(\frac{y+1}{1-y}\right)\Rightarrow x=\frac{1}{2}\ln\left(\frac{y+1}{1-y}\right)$$

$$\therefore T^{-1}(x)=\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

(iii) Hence, or otherwise, show that $\dfrac{d}{dx}\left[T^{-1}(x)\right]=\dfrac{1}{1-x^2}$. **2**

$$T^{-1}(x)=\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

$$=\frac{1}{2}\left[\ln(x+1)-\ln(1-x)\right]$$

$$\therefore \frac{d}{dx}T^{-1}(x)=\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{1-x}\right)$$

$$=\frac{1}{2}\left(\frac{2}{1-x^2}\right)$$

$$=\frac{1}{1-x^2}$$

Question 13 (continued)

(a) (iv) Use integration by parts to show that **3**

$$\int_0^{\frac{1}{2}} 4T^{-1}(x)\,dx = \log_e\left(\frac{3^m}{2^n}\right)$$

where m and n are positive integers.

$$\int_0^{\frac{1}{2}} 4T^{-1}(x)\,dx = \int_0^{\frac{1}{2}} T^{-1}(x)\times d(4x)$$

$$= \left[4xT^{-1}(x)\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 4x\,dT^{-1}(x)$$

$$= 2T^{-1}\left(\tfrac{1}{2}\right) - \int_0^{\frac{1}{2}} \frac{4x}{1-x^2}\,dx$$

$$= \ln 3 + 2\int_0^{\frac{1}{2}} \frac{-2x}{1-x^2}\,dx$$

$$= \ln 3 + 2\left[\ln\left|1-x^2\right|\right]_0^{\frac{1}{2}}$$

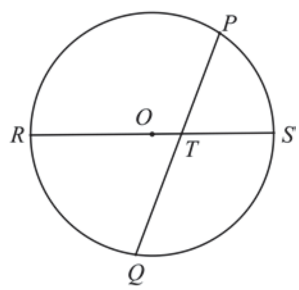
$$= \ln 3 + 2\ln\tfrac{3}{4}$$

$$= \ln\left[3\times\left(\tfrac{3}{2^2}\right)^2\right]$$

$$= \ln\left(\frac{3^3}{2^4}\right)$$

(b) The diagram below shows a circle with center O and a diameter RS. **2**
A chord, PQ, intersects RS at T which is a point within the circle.

Prove that $RT^2 + TS^2 \geq PT^2 + TQ^2$.



(1): $RT.TS = PT.TQ$ (Intersecting chords)

As RS is a diameter then $RS \geq PQ$.

$RS = RT + TS$ and $PQ = PT + TQ$

$\therefore (RT + TS)^2 \geq (PT + TQ)^2$

$\therefore RT^2 + TS^2 + 2RT.TS \geq PT^2 + TQ^2 + 2PT.TQ$

From (1): $RT^2 + TS^2 \geq PT^2 + TQ^2$

Question 13 (continued)

(c) Let α, β, and γ be the roots of the cubic equation $x^3 + Ax^2 + Bx + 8 = 0$, where A and B are real.
Furthermore, $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$.

(i) Explain why β is real and α and γ are not real. **2**

As A and B are real, then the roots are either all real or 1 pair of complex conjugates and one real root.

$\because\ \alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0 \Rightarrow \alpha^2 = -\beta^2$ and $\gamma^2 = -\beta^2$
$\therefore$ if α, β, and γ are all real then $\alpha = \beta = \gamma = 0$, but $\alpha\beta\gamma = -8$
So there is 1 pair of 1 pair of complex conjugates and one real root.

Now $\alpha^2 = \gamma^2$
$\therefore\ \alpha = \pm\gamma$
$\therefore\ \alpha$ and γ are the non-real roots, as if α is real then γ is real, and then all the roots would be real.

$\therefore\ \beta$ is real.

(ii) Show that α and γ are purely imaginary **1**

As β is real then $\beta = b$, where $b \in \mathbb{R}$.
$\alpha^2 = -\beta^2 \Rightarrow \alpha = \pm b$.
$\alpha = \pm\gamma \Rightarrow \gamma = \mp ib$

$\therefore\ \alpha$ and γ are purely imaginary

(iii) Find A and B. **2**

The roots are b, ib and $-ib$.

$b \times ib \times (-ib) = -8$ (product of roots)
$\therefore\ b^3 = -8$
$\therefore\ b = -2$

$b + ib + (-ib) = -A$ (sum of roots)
$\therefore\ A = 2$

$\Sigma\alpha\beta = B = 2(2i) + 2(-2i) + (2i)(-2i) = 4$

$\therefore\ A = 2, B = 4$

Question 14

(a) (i) By using de Moivre's theorem, or otherwise, show that **3**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} \qquad -(1)$$

By de Moivre's Theorem $\left(\cos\theta + i\sin\theta\right)^4 = \cos 4\theta + i\sin 4\theta$

Let $c = \cos\theta$, $s = \sin\theta$ and $t = \tan\theta$

$$\begin{aligned}\left(\cos\theta + i\sin\theta\right)^4 &= c^4 + 4c^3\left(is\right) + 6c^2\left(is\right)^2 + 4c\left(is\right)^3 + \left(is\right)^4 \\ &= c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4 \\ &= c^4 - 6c^2s^2 + s^4 + i\left(4c^3s - 4cs^3\right)\end{aligned}$$

Equating real and imaginary parts: $\cos 4\theta = c^4 - 6c^2s^2 + s^4$ and $\sin 4\theta = 4c^3s - 4cs^3$

$$\begin{aligned}\tan 4\theta &= \frac{\sin 4\theta}{\cos 4\theta} \\ &= \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4} \qquad \left[\div c^4\right] \\ &= \frac{4t - 4t^3}{t^4 - 6t^2 + t^4} \\ &= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}\end{aligned}$$

ALTERNATIVE:

$$\begin{aligned}\tan 4\theta &= \tan\left(2\times 2\theta\right) \\ &= \frac{2\tan 2\theta}{1-\tan^2 2\theta} \\ &= \frac{2\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2} \\ &= \frac{4\tan\theta\left(1-\tan^2\theta\right)}{\left(1-\tan^2\theta\right)^2 - \left(2\tan\theta\right)^2} \\ &= \frac{4\tan\theta - 4\tan^3\theta}{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta} \\ &= \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\end{aligned}$$

(ii) Hence, explain why $t = \tan\frac{\pi}{16}$ is a root of the equation **1**

$$t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$$

Substitute $\theta = \frac{\pi}{16}$ into equation (1) above

$$\tan\left(4 \times \frac{\pi}{16}\right) = \frac{4\tan\frac{\pi}{16} - 4\tan^3\frac{\pi}{16}}{1 - 6\tan^2\frac{\pi}{16} + \tan^4\frac{\pi}{16}}$$

$$\therefore \tan\frac{\pi}{4} = \frac{4\tan\frac{\pi}{16} - 4\tan^3\frac{\pi}{16}}{1 - 6\tan^2\frac{\pi}{16} + \tan^4\frac{\pi}{16}}$$

$$\therefore 1 = \frac{4t - 4t^3}{1 - 6t^2 + t^4} \qquad \left[\text{Let } t = \tan\frac{\pi}{16}\right]$$

$$\therefore 1 - 6t^2 + t^4 = 4t - 4t^3$$

$$\therefore t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$$

ALTERNATIVE:

Consider $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$

$$\therefore 1 - 6t^2 + t^4 = 4t - 4t^3$$

$$\therefore \frac{4t - 4t^3}{1 - 6t^2 + t^4} = 1$$

Let $t = \tan\theta$

$$\therefore \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$$

$$\therefore \tan 4\theta = 1$$

$$\therefore 4\theta = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

$$\therefore \theta = \frac{\pi}{16} + \frac{n\pi}{4}$$

For $n = 0$, $\theta = \frac{\pi}{16}$

$\therefore t = \tan\frac{\pi}{16}$ is a root of $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$

Question 14 (continued)

(a) (iii) Hence show that $\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$ **3**

The other roots of $t^4+4t^3-6t^2-4t+1=0$ come from $\tan 4\theta=1$

$\tan 4\theta = 1 \Rightarrow 4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$

$\therefore \theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots$

$\therefore t = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}, \ldots$

$\therefore$ the roots of $t^4+4t^3-6t^2-4t+1=0$ are $\tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$

NB $\tan\frac{9\pi}{16}=-\tan\frac{7\pi}{16}$ and $\tan\frac{13\pi}{16}=-\tan\frac{3\pi}{16}$

$\therefore \alpha,\beta,\gamma,\delta = \tan\frac{\pi}{16}, -\tan\frac{3\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{7\pi}{16}$

$$\begin{aligned}\sum\alpha^2 &= \left(\sum\alpha\right)^2 - 2\sum\alpha\beta \\ &= (-4)^2 - 2(-6) \\ &= 16+12 = 28\end{aligned}$$

$\therefore \tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$

ALTERNATIVE

$P(t)=t^4+4t^3-6t^2-4t+1$

$\therefore P(\sqrt{t})=(\sqrt{t})^4+4(\sqrt{t})^3-6(\sqrt{t})^2-4(\sqrt{t})+1$

$t^4+4t^3-6t^2-4t+1=0$ has roots $\alpha,\beta,\gamma,\delta = \tan\frac{\pi}{16}, -\tan\frac{3\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{7\pi}{16}$.

$\therefore (\sqrt{t})^4+4(\sqrt{t})^3-6(\sqrt{t})^2-4(\sqrt{t})+1=0$ has roots $\alpha^2,\beta^2,\gamma^2,\delta^2$.

$\therefore (\sqrt{t})^4+4(\sqrt{t})^3-6(\sqrt{t})^2-4(\sqrt{t})+1=0$

$\therefore t^2+4t\sqrt{t}-6t-4\sqrt{t}+1=0$

$\therefore t^2-6t+1=4\sqrt{t}-4t\sqrt{t}$

$\therefore (t^2-6t+1)^2=\left[4\sqrt{t}(1-t)\right]^2$

$\therefore (t^2-6t)^2+2(t^2-6t)+1=16t(1-2t+t^2)$

$\therefore t^4-12t^3+36t^2+2t^2-12t+1=16t-32t^2+16t^3$

$\therefore t^4-28t^3+70t^2-28t+1=0$

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2=-\frac{-28}{1}=28$

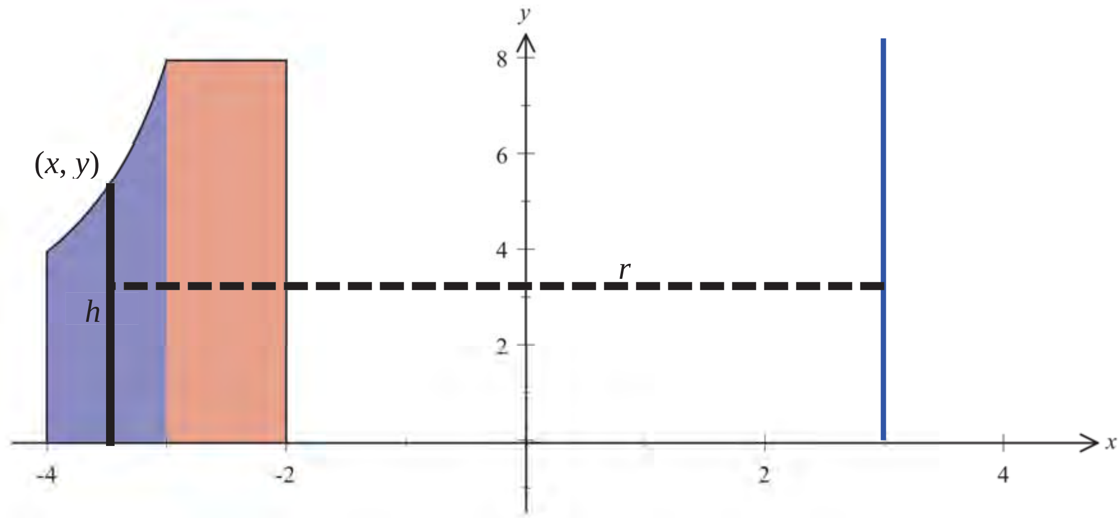
Question 14 (continued)

(b) In the diagram below, the shaded region is bounded by the lines **3**

$x = -4$, $x = -2$, $y = 8$, the x-axis and by the curve $y = -\dfrac{8}{x+2}$.

The region is rotated about the line $x = 3$ thus forming a solid.
Find the volume of the solid.

Method 1: Shells



For $-3 \le x \le -2$: Annular cylinder with inner radius 5 and outer radius 6; height 8

$\therefore$ Volume = $\pi\left(6^2 - 5^2\right) \times 8 = 88\pi$ cu

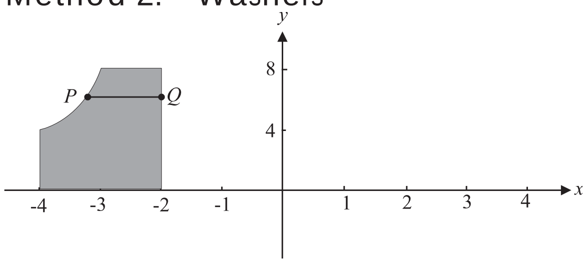
For $-4 \le x \le -3$: $\Delta V \doteqdot 2\pi r h \Delta x$

$$\therefore \Delta V \doteqdot 2\pi(3-x)\left(-\frac{8}{x+2}\right)\Delta x$$

$$\begin{aligned}
V &= \int_{-4}^{-3} 2\pi(3-x)\left(-\frac{8}{x+2}\right)dx \\
&= 16\pi \int_{-4}^{-3} \frac{x-3}{x+2}\,dx \\
&= 16\pi \int_{-4}^{-3} \frac{x+2-5}{x+2}\,dx \\
&= 16\pi \int_{-4}^{-3} \left(1 - \frac{5}{x+2}\right)dx \\
&= 16\pi\Big[x - 5\ln|x+2|\Big]_{-4}^{-3} \\
&= 16\pi\Big[(-3 - 5\ln 1) - (-4 - 5\ln 2)\Big] \\
&= 16\pi(1 + 5\ln 2) \\
&= 16\pi(1 + \ln 32)
\end{aligned}$$

$\therefore$ Total volume = $16\pi(1+\ln 32) + 88\pi = \pi(104 + 16\ln 32)$ cu

(b) Method 2: Washers



For $0 \le y \le 4$:

The volume of the annular cylinder or "pipe" created by the rotation is $4 \times \pi\left(7^2 - 5^2\right)$ i.e. 96π cu.

For $4 \le y \le 8$:

The area of the annulus formed by the rotation of PQ is given by

$$\begin{aligned}\text{Area } &= \pi\left(\text{radius of outer edge of annulus}\right)^2 - \left(\text{radius of inner edge of annulus}\right)^2\\ &= \pi\left[\left(3-x\right)^2 - 5^2\right]\\ &= \pi\left(9 - 6x + x^2 - 25\right)\\ &= \pi\left(x^2 - 6x - 16\right)\end{aligned}$$

Now $y = -\dfrac{8}{x+2} \Rightarrow x + 2 = -\dfrac{8}{y}$

$\therefore x = -\dfrac{8}{y} - 2 \Rightarrow x^2 = \dfrac{64}{y^2} + \dfrac{32}{y} + 4$

$$\begin{aligned}\text{Area } &= \pi\left(\frac{64}{y^2} + \frac{32}{y} + 4 + \frac{48}{y} + 12 - 16\right)\\ &= \pi\left(\frac{64}{y^2} + \frac{80}{y}\right)\\ &= 16\pi\left(\frac{4}{y^2} + \frac{5}{y}\right) \text{ sq. units}\end{aligned}$$

So required volume is $V = 16\pi\displaystyle\int_4^8 \left(\frac{4}{y^2} + \frac{5}{y}\right)dy + 96\pi$

$$\begin{aligned}\therefore V &= 16\pi\left[-4y^{-1} + 5\ln y\right]_4^8 + 96\pi\\ &= 16\pi\left\{\left(-\tfrac{1}{2} + 5\ln 8\right) - \left(-1 + 5\ln 4\right)\right\} + 96\pi\\ &= 16\pi\left(-\tfrac{1}{2} + 5\ln 8 + 1 - 5\ln 4\right) + 96\pi\\ &= 16\pi\left(\ln\frac{8^5}{4^5} + \tfrac{1}{2}\right) + 96\pi\\ &= 16\pi\left(\ln 32 + \tfrac{1}{2} + 6\right)\\ &= \pi\left(104 + 16\ln 32\right) \text{ cu}\end{aligned}$$

Question 14 (continued)

(c) Let $I_n = \int_0^1 x\left(x^2-1\right)^n\,dx$ for $n \ge 0$

(i) Use integration by parts to show that **3**

$$I_n = -\frac{n}{n+1} I_{n-1} \text{ for } n \ge 1$$

$$\begin{aligned}
I_n &= \int_0^1 x\left(x^2-1\right)^n dx \\
&= \int_0^1 \left(x^2-1\right)^n d\left(\tfrac{1}{2}x^2\right) \\
&= \left[\tfrac{1}{2}x^2\left(x^2-1\right)^n\right]_0^1 - \int_0^1 \tfrac{1}{2}x^2 d\left(x^2-1\right)^n \\
&= -\int_0^1 \tfrac{1}{2}x^2\left[2nx\left(x^2-1\right)^{n-1}\right]dx \\
&= -n\int_0^1 x\left[x^2\left(x^2-1\right)^{n-1}\right]dx \\
&= -n\int_0^1 x\left\{\left[\left(x^2-1\right)+1\right]\left(x^2-1\right)^{n-1}\right\}dx \\
&= -n\int_0^1 x\left[\left(x^2-1\right)^n+\left(x^2-1\right)^{n-1}\right]dx \\
&= -nI_n - nI_{n-1}
\end{aligned}$$

$$\begin{aligned}
&\therefore I_n = -nI_n - nI_{n-1} \\
&\therefore (n+1)I_n = -nI_{n-1} \\
&\therefore I_n = -\frac{n}{n+1}I_{n-1}
\end{aligned}$$

(ii) Hence find I_3 **2**

$$I_3 = -\frac{3}{4}I_2 = -\frac{3}{4}\left(\frac{2}{3}I_1\right) = -\frac{3}{4}\left(\frac{2}{3}\left(\frac{1}{2}I_0\right)\right) = -\frac{1}{4}I_0$$

$$I_0 = \int_0^1 x\left(x^2-1\right)^0 dx = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

$$I_3 = -\frac{1}{4}\times\frac{1}{2} = -\frac{1}{8}$$

Question 15

(a) (i) Prove for positive numbers x and y that $x+y \geq 2\sqrt{xy}$. **1**

$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$

$\therefore x+y-2\sqrt{xy} \geq 0$

$\therefore x+y \geq 2\sqrt{xy}$

(ii) Hence, or otherwise, prove $x^2+\dfrac{1}{x^2} \geq 2$, for all real x, $x \neq 0$. **2**

Also, state for what value(s) of x there is equality.

In (i), replace x with x^2 and y with $\dfrac{1}{x^2}$ (NB x^2, $\dfrac{1}{x^2} \geq 0$)

$$\therefore x^2+\frac{1}{x^2} \geq 2\sqrt{x^2 \times \frac{1}{x^2}}$$
$$= 2$$

Equality is when $x = \pm 1$

(b) In the diagram below, the points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ with $c, p > 0$, lie on different branches of the hyperbola, $\mathscr{H}$, with equation $xy = c^2$.

The tangent to $\mathscr{H}$ at P and the tangent to $\mathscr{H}$ at Q are parallel.

(i) Show that the equation of the tangent at P is $y = \dfrac{2c}{p} - \dfrac{x}{p^2}$. **2**

At P, $x = cp$ and $y = \dfrac{c}{p}$ and so $\dfrac{dx}{dp} = c$, $\dfrac{dy}{dp} = -\dfrac{c}{p^2}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$
$$= -\frac{1}{p^2}$$

Equation of tangent at P is $y - \dfrac{c}{p} = -\dfrac{1}{p^2}\left(x - cp\right)$

$\therefore y = \dfrac{2c}{p} - \dfrac{x}{p^2}$ as required

(ii) Show that $p = -q$. **1**

The gradient of the tangent at P is $-\dfrac{1}{p^2}$ from part (i).

So, the gradient of the tangent at Q is $-\dfrac{1}{q^2}$.

Since the gradients are equal: $-\dfrac{1}{p^2} = -\dfrac{1}{q^2}$.

$$\therefore q^2 = p^2$$

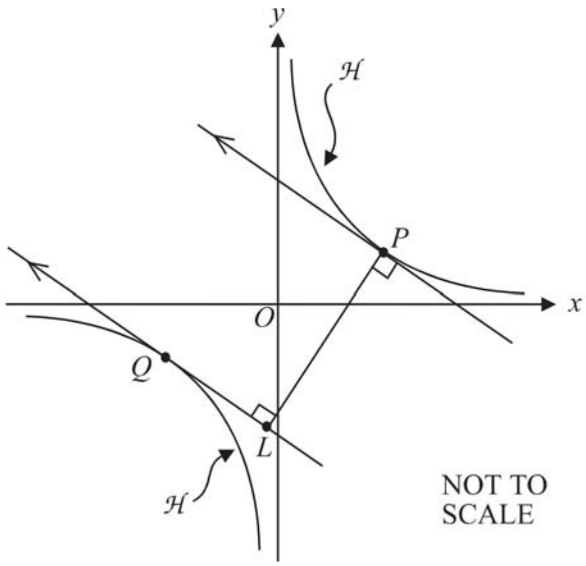
$\therefore \left(q-p\right)\left(q+p\right) = 0$

$\therefore q = p$ or $q = -p$

Since P and Q are distinct points, $q \neq p$ and so $q = -p$ as required.

(b) (iii) Show that the perpendicular distance from P to the tangent through Q is given by $\dfrac{4cp}{\sqrt{p^4+1}}$ **2**

The equation of the tangent through Q is $y = \dfrac{2c}{q} - \dfrac{x}{q^2} \Rightarrow x + q^2 y - 2cq = 0$



$$d = \left| \frac{cp + q^2\left(\frac{c}{p}\right) - 2cq}{\sqrt{\left(q^2\right)^2 + 1}} \right|$$

$$= \left| \frac{cp + p^2\left(\frac{c}{p}\right) + 2cp}{\sqrt{p^4 + 1}} \right| \qquad [p = -q]$$

$$= \left| \frac{4cp}{\sqrt{p^4+1}} \right|$$

$$= \frac{4cp}{\sqrt{p^4+1}} \qquad [c,\ p > 0]$$

(iv) Using (a), or otherwise, find the coordinates of the points P and Q when the perpendicular distance from P to the tangent through Q is a maximum. **3**

Method 1: "Hence"

$$\text{Let } D = \frac{4cp}{\sqrt{p^4+1}}$$

$$= \frac{4c}{\sqrt{p^2 + \frac{1}{p^2}}}$$

$$\le \frac{4c}{\sqrt{2}} \qquad \left[\begin{array}{l} p^2 + \dfrac{1}{p^2} \ge 2 \Rightarrow \dfrac{1}{p^2 + \frac{1}{p^2}} \le \dfrac{1}{2} \\[2ex] \dfrac{1}{\sqrt{p^2 + \frac{1}{p^2}}} \le \dfrac{1}{\sqrt{2}} \end{array} \right]$$

$$= 2\sqrt{2}c$$

So the maximum length of D is $2\sqrt{2}c$ and from (a)(ii) this maximum occurs when $p = 1$, as $p > 0$

$\therefore P(c, c)$ and $Q(-c, -c)$

Question 15 (continued)

(b) (iv) Method 2: "Otherwise"

Let D equal the perpendicular distance from P to the tangent through Q.

$$D = \frac{4cp}{\sqrt{p^4+1}} = 4c \times \frac{p}{\left(p^4+1\right)^{\frac{1}{2}}} \qquad \left[\text{ from part (iv)}\right]$$

$$\frac{dD}{dp} = 4c\left[\frac{\left(p^4+1\right)^{\frac{1}{2}} \times 1 - p \times \frac{1}{2}\left(p^4+1\right)^{-\frac{1}{2}} \times 4p^3}{p^4+1}\right]$$

$$= 4c\left[\frac{\left(p^4+1\right)^{-\frac{1}{2}}\left[\left(p^4+1\right)-2p^4\right]}{p^4+1}\right]$$

$$= 4c\left[\frac{1-p^4}{\sqrt{p^4+1}\left(p^4+1\right)}\right]$$

$$= \frac{4c\left(1-p^4\right)}{\sqrt{p^4+1}\left(p^4+1\right)}$$

Maximum occurs when $\dfrac{dD}{dp} = 0$

$$\therefore 4c - 4cp^4 = 0$$

$$p^4 = 1$$

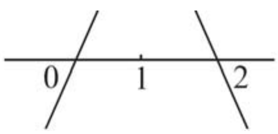
$$p = \pm 1 \qquad \left[p \in \mathbb{R}\right]$$

$$p = 1 \qquad \left[p > 0\right]$$

Test for max/min:

When $p = 0$, $\dfrac{dD}{dp} = 4c > 0$ $\qquad \left[c > 0\right]$

When $p = 2$, $\dfrac{dD}{dp} = \dfrac{4c - 64c}{17^{\frac{3}{2}}} = \dfrac{-60c}{17^{\frac{3}{2}}} < 0$



$\therefore$ a maximum when $p = 1$.

So the perpendicular distance from P to the tangent through Q is a maximum when $p = 1$.

This occurs when $P(c, c)$ and $Q(-c, -c)$

(c) (i) Using (a), or otherwise, prove for positive numbers x, y and z that **2**

$$\left(x+y\right)\left(y+z\right)\left(z+x\right)\geq 8xyz$$

From (a), $\left(x+y\right)\geq 2\sqrt{xy}$

$\therefore \left(y+z\right)\geq 2\sqrt{yz}$

$\left(z+x\right)\geq 2\sqrt{xz}$

$$\begin{aligned}\therefore \left(x+y\right)\left(y+z\right)\left(z+x\right) &\geq 2\sqrt{xy}\times 2\sqrt{yz}\times 2\sqrt{zx}\\ &=8\sqrt{x^2y^2z^2}\\ &=8xyz \qquad \left[x,\,y,\,z>0\right]\end{aligned}$$

(ii) By using part (i) above, or otherwise, if a, b and c are the sides of a triangle, prove that **2**

$$abc\geq\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right).$$

As a, b, c are the sides of a triangle then a, b, $c > 0$ and $a + b - c > 0$, $b + c - a > 0$, $c + a - b > 0$.

Let $a = x + y$, $b = y + z$ and $c = z + x$ for some positive real numbers x, y, z.

NB $a+b-c=x+y+y+z-\left(z+x\right)=2y>0$

Similarly, $b+c-a=2z>0$ and $c+a-b=2x>0$.

So the conditions of (a)(i) are valid i.e. $\left(x+y\right)\left(y+z\right)\left(z+x\right)\geq\left(2x\right)\left(2y\right)\left(2z\right)$

$\therefore abc\geq\left(c+a-b\right)\left(a+b-c\right)\left(b+c-a\right)$

$\therefore abc\geq\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right)$

ALTERNATIVE

From (i) $\left(x+y\right)\left(y+z\right)\left(z+x\right)\geq 8xyz$ (*)

Let $x = c + a - b$, $y = a + b - c$ and $z = b + c - a$.

NB x, y, $z > 0$ as a, b and c are the sides of a triangle.

Now $x + y = = c + a - b + a + b - c = 2a$.

Similarly, $y + z = 2b$ and $z + x = 2c$.

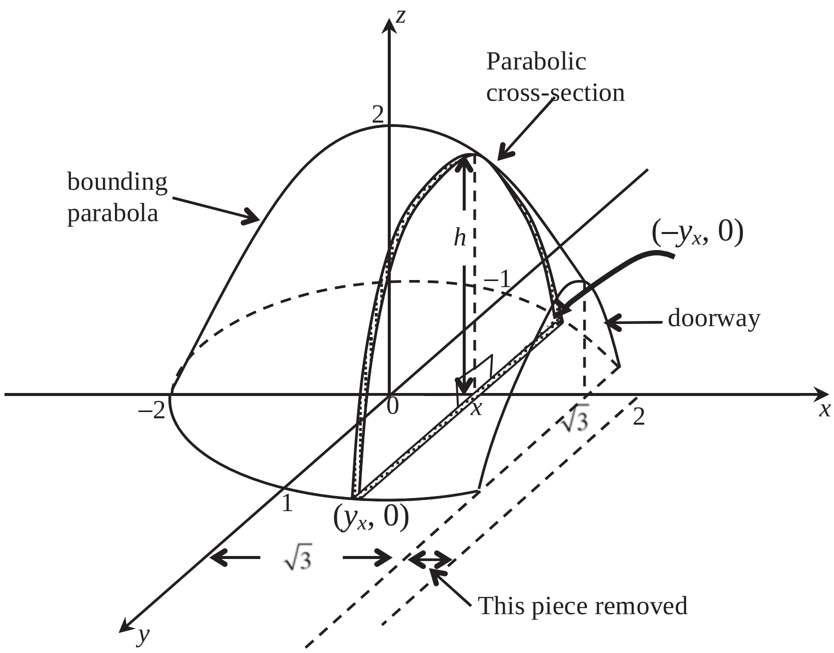
Substituting the above into (*) gets $\left(2a\right)\left(2b\right)\left(2c\right)\geq 8\left(c+a-b\right)\left(a+b-c\right)\left(b+c-a\right)$

$\therefore 8abc\geq 8\left(a+b-c\right)\left(a+b-c\right)\left(b+c-a\right)$

$\therefore abc\geq\left(a+b-c\right)\left(a+b-c\right)\left(b+c-a\right)$

Question 16

(a)



(i) Find the equation of the ellipse that bounds the floor of the interior of the igloo. **1**

$a = 2$ and $b = 1$

$\therefore \dfrac{x^2}{4} + y^2 = 1$

(ii) By finding the equation of the bounding parabola indicated in the diagram, show that the height h of the vertical cross-section, drawn at a distance x from the centre of the ellipse is given by $h = \frac{1}{2}\left(4 - x^2\right)$. **1**

The bounding parabola has equation $z = \frac{1}{2}\left(4 - x^2\right)$

The vertex of cross-sectional parabola has coordinates (x, h) on this bounding parabola.

$\therefore h = \frac{1}{2}\left(4 - x^2\right)$

(a) (iii) Using Simpson's rule, or otherwise, to find the area of the indicated cross-section of height h, show that the volume of air in the igloo is given by **3**

$$V = \frac{1}{3}\int_{-2}^{\sqrt{3}} \left(4 - x^2\right)^{\frac{3}{2}} \, dx$$

Simpson's rule gives the exact value of the area of a parabola
Using 3 function values:

y	z	w (weight)	zw
$-y_x$	0	1	0
0	h	4	$4h$
y_x	0	1	0
			$\Sigma\, zw = 4h$

$$y_x = \sqrt{1 - \frac{x^2}{4}}$$
$$= \frac{1}{2}\left(4 - x^2\right)^{\frac{1}{2}}$$

Area of cross-sectional slice $= \dfrac{y_x}{3}\left(4h\right)$

$$= \frac{4}{3} \times \frac{1}{2}\left(4 - x^2\right)^{\frac{1}{2}} \times \frac{1}{2}\left(4 - x^2\right)$$
$$= \frac{1}{3}\left(4 - x^2\right)^{\frac{3}{2}}$$

$$\therefore \Delta V \doteqdot \frac{1}{3}\left(4 - x^2\right)^{\frac{3}{2}} \Delta x$$

$$\therefore V = \frac{1}{3}\int_{-2}^{\sqrt{3}} \left(4 - x^2\right)^{\frac{3}{2}} \, dx$$

There are alternative solutions, but you need to know Simpson's Rule and how it works!

Question 16 (continued)

(a) (iii) Using $x = 2\sin\theta$, or otherwise, calculate the volume of air in the igloo. **4**

$$V = \frac{1}{3}\int_{-2}^{\sqrt{3}} \left(4 - x^2\right)^{\frac{3}{2}}\, dx$$

$$x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta$$

$$x = -2, \theta = -\frac{\pi}{2};\ x = \sqrt{3}, \theta = \frac{\pi}{3}$$

$$\begin{aligned}\left(4 - x^2\right)^{\frac{3}{2}} &= \left(4 - 4\sin^2\theta\right)^{\frac{3}{2}}\\ &= 8\left(1 - \sin^2\theta\right)^{\frac{3}{2}}\\ &= 8\left(\cos^2\theta\right)^{\frac{3}{2}}\\ &= 8\cos^3\theta\end{aligned}$$

$$V = \frac{1}{3}\int_{-2}^{\sqrt{3}} \left(4 - x^2\right)^{\frac{3}{2}}\, dx \qquad = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} 8\cos^3\theta \times 2\cos\theta\, d\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} 16\cos^4\theta\, d\theta \qquad = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(2\cos^2\theta\right)^2\, d\theta$$

$$= \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(1 + \cos 2\theta\right)^2\, d\theta \qquad = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta \qquad = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{4}{3}\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \frac{4}{3}\left[\left(\frac{3}{2} \times \frac{\pi}{3} + \sin\frac{2\pi}{3} + \frac{1}{8}\sin\frac{4\pi}{3}\right) - \frac{3}{2} \times \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{4}{3}\left[\left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{16}\right) + \frac{3\pi}{4}\right] = \frac{7}{4\sqrt{3}} + \frac{5\pi}{3}$$

$$\doteqdot 6.24635$$

$$V = \frac{7}{4\sqrt{3}} + \frac{5\pi}{3}$$

$$\doteqdot 6.24635$$

(b) A sequence of numbers U_n is such that $U_1 = 5$, $U_2 = 30$ and **3**

$U_n = 5U_{n-1} - 6U_{n-2}$ for $n \geq 3$

Prove by mathematical induction that $U_n = 2 \times 3^n + 3 \times 2^n$ for $n \geq 1$.

Test $n = 1$: LHS $= U_1 = 12$
RHS $= 2 \times 3^1 + 3 \times 2^1 = 12$
$\therefore$ true for $n = 1$

Test $n = 2$: LHS $= U_2 = 30$
RHS $= 2 \times 3^2 + 3 \times 2^2 = 30$
$\therefore$ true for $n = 1$

Assume true for $n = k - 2$ i.e. $U_{k-2} = 2 \times 3^{k-2} + 3 \times 2^{k-2}$

Assume true for $n = k - 1$ i.e. $U_{k-1} = 2 \times 3^{k-1} + 3 \times 2^{k-1}$

Need to prove true for $n = k$ i.e. $U_k = 2 \times 3^k + 3 \times 2^k$

$$\begin{aligned}
\text{LHS} &= U_k \\
&= 5U_{k-1} - 6U_{k-2} \\
&= 5\left(2 \times 3^{k-1} + 3 \times 2^{k-1}\right) - 6\left(2 \times 3^{k-2} + 3 \times 2^{k-2}\right) \\
&= 10 \times 3^{k-1} + 15 \times 2^{k-1} - 12 \times 3^{k-2} - 18 \times 2^{k-2} \\
&= 10 \times 3^{k-1} - 12 \times 3^{k-2} + 15 \times 2^{k-1} - 18 \times 2^{k-2} \\
&= 10 \times 3^{k-1} - 4 \times 3^{k-1} + 15 \times 2^{k-1} - 9 \times 2^{k-1} \\
&= 6 \times 3^{k-1} + 6 \times 2^{k-1} \\
&= 2 \times 3^k + 3 \times 2^k \\
&= \text{RHS}
\end{aligned}$$

So the formula is true for $n = k$ when it is true for $n = k - 2$ and $n = k - 1$.

By the principle of mathematical induction the formula is true for all $n \geq 1$.

(c) Let $z = (a+\cos\theta) + i(2a+\sin\theta)$, where a is a real number and $0 \le \theta \le 2\pi$. **3**

If $|z| \le 2$, then $|a| \le k$ for some real number k.

Find the value of k.

$$\begin{aligned}
|z|^2 &= (a+\cos\theta)^2 + (2a+\sin\theta)^2 \\
&= a^2 + 2a\cos\theta + \cos^2\theta + 4a^2 + 4a\sin\theta + \sin^2\theta \\
&= 5a^2 + 1 + 2a\cos\theta + 4a\sin\theta \\
&= 5a^2 + 1 + 2\sqrt{5}a\cos(\theta+\varepsilon) \quad \text{for some } \varepsilon \in {}^\circ
\end{aligned}$$

As $|z| \le 2$ then $|z|^2 \le 4$.

$\therefore 5a^2 + 1 + 2\sqrt{5}a\cos(\theta+\varepsilon) \le 4$

$\therefore 2\sqrt{5}a\cos(\theta+\varepsilon) \le 3 - 5a^2$

NB $2\sqrt{5}a\cos(\theta+\varepsilon) \le 2\sqrt{5}|a|$

$\therefore 2\sqrt{5}|a| \le 3 - 5a^2$

Now $a^2 = |a|^2$

$\therefore 2\sqrt{5}|a| \le 3 - 5|a|^2$

$\therefore 5|a|^2 + 2\sqrt{5}|a| - 3 \le 0$

$\therefore (\sqrt{5}|a| - 1)(\sqrt{5}|a| + 3) \le 0$

$\therefore -\dfrac{3}{\sqrt{5}} \le |a| \le \dfrac{1}{\sqrt{5}}$

But $|a| \ge 0$ and so $0 \le |a| \le \dfrac{1}{\sqrt{5}}$

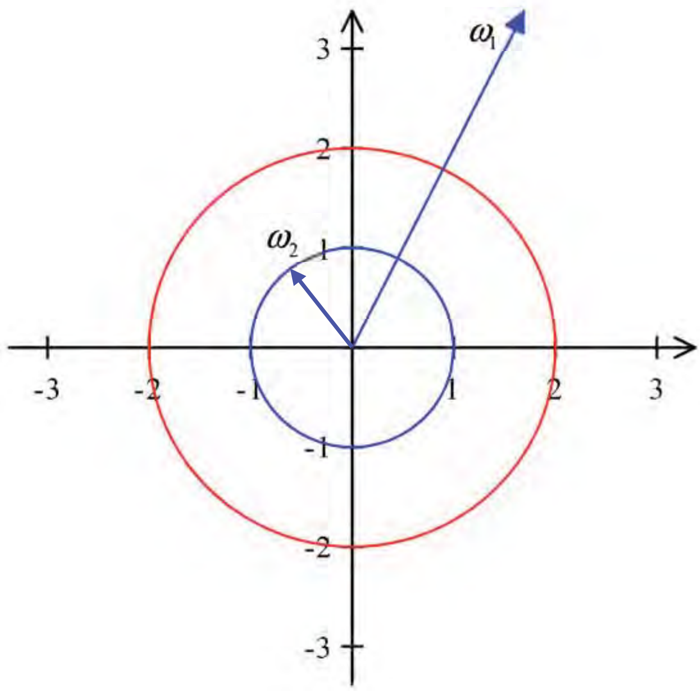
$\therefore |a| \le \dfrac{1}{\sqrt{5}}$

ALTERNATIVE Solution

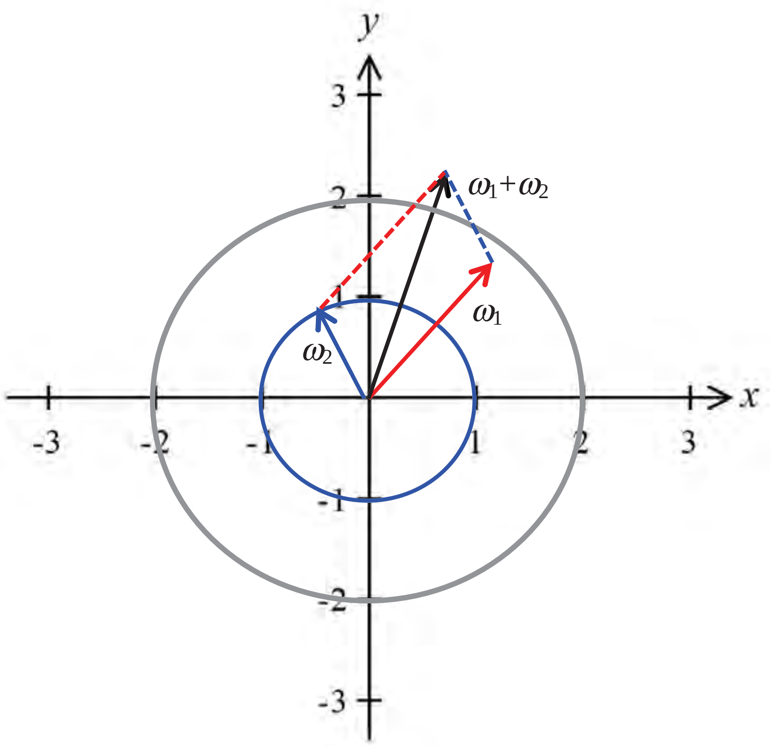
$$\begin{aligned} z &= (a+\cos\theta)+i(2a+\sin\theta) \\ &= (a+2ai)+(\cos\theta+i\sin\theta) \end{aligned}$$

Let $\omega_1 = a+2ai$ and $\omega_2 = \cos\theta+i\sin\theta$. Then, $z=\omega_1+\omega_2$.

$|\omega_1| = \sqrt{a^2+4a^2} = |a|\sqrt{5}$ and $\arg\omega_1 = \tan^{-1}2$ (which is a fixed angle in the 1st or 3rd quadrant).
$|\omega_2| = 1$ but $\arg\omega_2$ can be any angle between 0 and 2π.

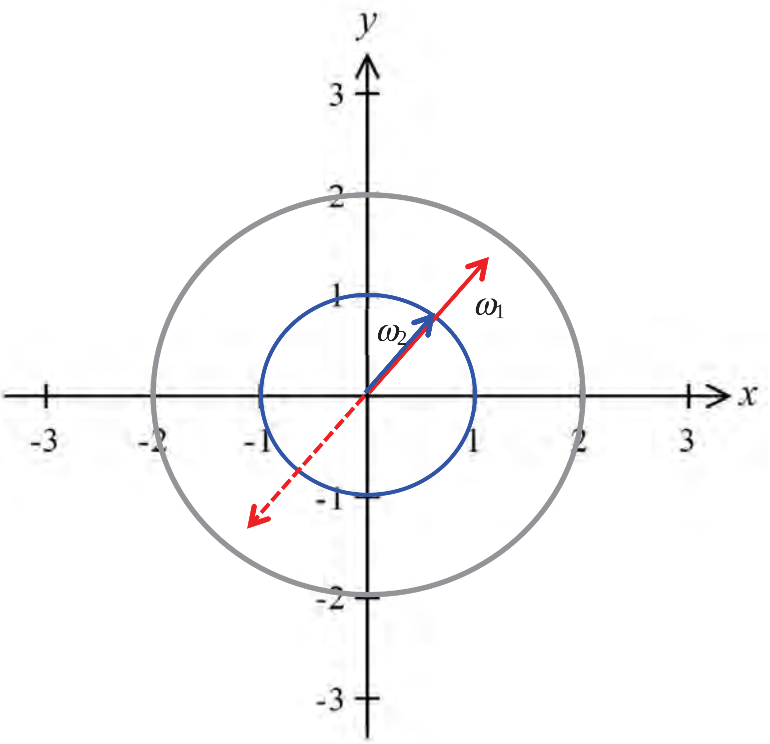


By the triangle inequality, $|z| \le |\omega_1| + |\omega_2|$.



With $|z| \le |\omega_1| + |\omega_2|$, equality is achieved when $\arg\omega_2 = \arg\omega_1$.

Therefore, if $|z| \le 2$, then $|a|\sqrt{5} + 1 \le 2$ or $|a| \le \dfrac{1}{\sqrt{5}}$.



End of solutions